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TERNARY FORMATIONS

BY

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The ternary formations or triads [12] are classes of new objects generated by three objects which belong to three different sets (classes). We extend here, for ternary formations, some properties of binary forma tions.

1. **Ternary relations.** The ternary relations on the sets M_1, M_2, M_3 are the subsets of the cartesian product $M_1 \times M_2 \times M_3$. They form a Boolean algebra. In the case $M_1 = M_2 = M_3 = M$, we clasify the ternary relations in a manner that recalls binary relations : a). *Reflexivity relations* R_l, R_s, R_r, R_t, respectively on the left side, on both sides, on the right side, total, i.e. for any $x, y \in M$, $(x, x, y) \in R_l$, $(x, y, x) \in R_s$, $(x, y, y) \in R_r$, $(x, x, x) \in R_t$. Clearly $R_t \subseteq R_l \cap R_s \cap R_r$. b) *Symmetry relations* S_l, S_s, S_r, S_c, S_t (with the sense as above, S_c means cyclic), if for any $x, y, z \in M$, we have : $(x, y, z) \in S_l$ implies $(y, x, z) \in S_l$ and so on. c). *Transitivity relations* T_l, T_s, T_r, T (the last being of absorbtion), if (x, y, z), $(y, u, z) \in T_l \Rightarrow (x, u, z) \in T_l$; (x, y, z), $(z, y, u) \in T_s \Rightarrow (x, y, u) \in T_s$; (x, y, z), $(x, z, u) \in T_r \Rightarrow (x, y, u) \in T_r$; (x, y, z), (y, z, u), $(z, u, v) \in T \Rightarrow (u, v, x)$, $(v, x, y) \in T$.

A ternary preordering on the left side (resp. on sides or on the right side) is a left reflexive, transitive, ternary relation (resp. on sides or on the right). A left ternary equivalence is a left reflexive, transitive, symmetric ternary relation. A left ternary preordering having the property : (x, y, z), $(y, x, z) \in 0 \Rightarrow x = y$ is called a partial ordering on the left side.

2. **Ternary operations.** The ternary operations on the set M are mappings $\tau : M \times M \times M \to M$, $(x, y, z)_\tau = t \in M$. The pair (M, τ) is called a *ternary groupoid*. There exist classes of ternary groupoids which extend some properties of binary groupoids. So there exist ternary groupoids with left (resp. right or two-sided) commutativity, if for all $x, y, z \in M$, we have $(x, y, z)_\tau = (y, x, z)$ (resp. $(x, y, z)_\tau = (x, z, y)_\tau$ or $(x, y, z)_\tau = (z, y, x)_\tau$; there exist ternary groupoids with cyclic commutativity : $(x, y, z)_\tau = (y, z, x)_\tau$ or with total commutativity, when the three factors which form the ternary product can be changed one another.

For the ternary groupoids, different manners to introduce the associativity have been studied. By extending the associativity from the binary operations to the ternary ones, we give the definitions :

i) The operation τ is left associative, if $((x, y, z)_\tau, u, v)_\tau = (x, (y, z, u)_\tau, v)_\tau$, for any $x, y, z, u, v \in M$; then (M, τ) is a *left ternary semigroup*.

ii) τ is right associative, if $(x, (y, z, u)_\tau, v)_\tau = (x, y, (z, u, v)_\tau)_\tau$; (M, τ) is a *right ternary semigroup*.

iii) The two-sided associativity of τ is : $((x, y, z)_\tau, u, v)_\tau = (x, y, (z, u, v)_\tau)_\tau$, for any elements of M ; (M, τ) is a *two-sided ternary semigroup*.

iv) If two of the above properties are fulfiled, then the third is also fulfiled and we say that τ is totally associative and (M, τ) is a *ternary semigroup*. ([5, 6]).

v) S.A. Kiss and G. Birkhoff [1] proposed in 1947 for ternary structures in lattices the associativity : $((x, y, z)_\tau, u, v)_\tau = ((x, u, v)_\tau, y, (z, u, v)_\tau)_\tau$.

vi) V.V. Vagner [4], in 1952, used the associativity : $((x, y, z)_\tau, u, v)_\tau = (x, (u, z, y)_\tau, v)_\tau = (x, y, (z, u, v)_\tau)_\tau$.

vii) At the National Conference of Geometry at Piatra Neamţ (1983), we introduced the associativity : $((x, y, z)_\tau, u, v)_\tau = ((x, z, v)_\tau, y, u)_\tau$ (see [23]).

viii) Yoshiake Taniguchi [21] proposed for ternary algebras the associativity : $(u, v, (x, y, z)_\tau)_\tau = ((u, v, x)_\tau, y, z)_\tau + (x, (u, v, y)_\tau, z)_\tau + (x, y, (u, v, z)_\tau)_\tau$.

3. **Ternary groupoids with special elements.** Let (M, τ) be a ternary groupoid. If $a, b \in M$, a is called a *left biunity* for b, if $(a, a, b)_\tau = b$, while b is called a *right zero* of a (see [13]).

If there exists $u \in M$, such that $(u, u, x)_\tau = x$ for any $x \in M$, u is called a *left biunity* of M. u is a *right* (*two-sided*) *biunity*, if $(x, u, u)_\tau = x$ (resp. $(u, x, u)_\tau = x$) for any $x \in M$. A *left* (*right*, *middle*) *unity* of M is an element $u \in M$, for any $x \in M$, $(u, x, x)_\tau = x$ (resp. $(x, x, u)_\tau = x$ and $(x, u, x)_\tau = x$).

In [13] the definitions of *left*, *middle* and *right zeros* ($(0, x, y)_\tau = 0$, $(x, 0, y)_\tau = 0$, resp. $(x, y, 0)_\tau = 0$, for any $x, y \in M$), *left*, *middle* and *right biannulators* ($(a, a, x)_\tau = a$, $(a, x, a)_\tau = a$, $(x, a, a)_\tau = a$, for any $x \in M$) and *left*, *middle* and *right annulators* ($(a, x, x)_\tau = a$, resp. $(x, a, x)_\tau = a$ and $(x, x, a)_\tau = a$) were given. We remark that :

1°. Any zero is also a biannulator of the same type.

2°. The unities, the biunities, the zeros, the biannulators and the annulators — when they exist — are idempotent.

3°. When there exist left biunities and right unities (resp. right biunities and left unities), then they are equal and unique.

4°. In a two-sided ternary semigroup, right unities are also right biunities. The same property on the left.

5°. In a left ternary semigroup, if e is a middle unity and a two-sided biunity, then e is also a right unity ; an analogous property holds for a right ternary semigroup. In a ternary semigroup any unity is total and unique, hence this semigroup is called a *ternary monoid* ; all the elements of a ternary monoid are idempotent.

6°. In a ternary monoid, one can introduce a binary operation $*$: $x * y = (e, x, y)_\tau$, $(M, *)$ is a binary monoid with e as identity.

7°. Let A be the set of left annulators in a two-sided ternary semigroup ; A is an ideal of M since $(M, A, A)_\tau \subseteq A$ and $(M, M, A)_\tau \subseteq A$. The same statement holds for the right annulators A' : $(A', A', M)_\tau \subseteq A'$ and $(A', M, M)_\tau \subseteq A'$.

8°. In a ternary groupoid, there exists at most a zero, which is on the left, on the right and in the middle in the same time.

4. **Ternary groupoids with cancellation, ternary groupoids with division. Ternary groups.** Let (T, τ) be a ternary groupoid. We say that T is *with left cancellation*, if $(u, v, x) = (u, v, y)$ implies $x = y$, with cancellation

on both sides, if $(u, x, v)_\tau = (u, y, v)_\tau$ implies $x = y$, with right cancellation, if $(x, u, v)_\tau = (y, u, v)_\tau$ implies $x = y$.

Remarks. 1). In a two-sided ternary groupoid with left (right) cancellation any left (right) annulator is a right (left) biunity and all the elements are idempotent. 2). In a ternary groupoid with left (right) cancellation there exists at most a right (left) unity; in a ternary groupoid with cancellation on sides there exists at most a middle unity.

In the ternary groupoid (T, τ), consider the equations: $(a, b, x)_\tau = c$, $(a, x, b)_\tau = c$, $(x, a, b)_\tau = c$. If one of these equations has solutions in T, then we say that c admits right (resp. middle, left) division with respect to a and b. If this is the case for any $a, b, c \in T$, then T has *right (middle, left) division of first type*.

Remarks. 1). If T is with cancellation, the solutions of the above equations, when they exist, are unique. 2). The unities of T divide on the same side any element of T.

We consider now ternary equations with two variables: $(a, x, y)_\tau = b$, $(x, a, y)_\tau = b$, $(x, y, a)_\tau = b$. If for any $a, b \in T$ there exist solutions for one of these equations, then we say that the groupoid T is *with division of second type*, resp. *on the right*, *on both sides* and *on the left*. Obviously, if T is with first type division, then T is with second type division. If T is with right (middle, left) first type division, and u is a right (middle, left) unity, then there exists for any $a \in T$, an element $a'(a'', a''')$ such that $(a, a, a')_\tau = u$, $((a, a'', a)_\tau = u$, $(a''', a, a)_\tau = u)$ which is a quasi-inverse of a with respect to u. If T is with right first type division, for any $a \in T$, there exists $a' \in T$ for which $(a, a, a')_\tau = a$; the same for middle and left.

A ternary groupoid with the properties: i) Total associativity; ii). Right, middle and left first type division; iii) Left, right and two-sided cancellation, is called a *ternary group* and we denote it by G_t (see [10]).

Remarks. 1) A ternary group with more than an element has no unities. Indeed, if e is a right unity, then $((x, x, x)_\tau, e, e)_\tau = (x, (x, x, e)_\tau, e)_\tau = (x, x, (x, e, e)_\tau)_\tau$ implies $(x, x, e)_\tau = (x, x, x)_\tau$, hence $x = e$. 2). There are other definitions for ternary groups: In [22], a ternary group is a ternary groupoid T with: i) total associativity; ii) two-sided commutativity; iii) any $x \in T$ is a left biunity for any $y \in T$. In [5], Al. Ghika defined the ternary group by the following conditions: i) total associativity; ii) $\exists e \in T$, $\forall x \in T$, $(x, e, e)_\tau = (e, e, x)_\tau$; iii) for any $x \in T$, there exist $\bar{x}, \tilde{x} \in T$ such that $(x, \bar{x}, e)_\tau = (\bar{x}, x, e)_\tau = (\tilde{x}, e, x)_\tau = (x, e, \tilde{x})_\tau = e$ and these elements are unique. Moreover $\bar{\bar{x}} = x$, $\tilde{x} = (e, \bar{x}, e)$.

In [17], M. Ştefănescu has defined two types of ternary groups:

I; i) $\exists 0 \in G$, $(x, 0, (y, z, t)_\tau)_\tau = ((x, 0, y)_\tau, z, t)_\tau$; ii) $(x, 0, 0)_\tau = x$;

iii) $(x, x, 0)_\tau = 0$; iv). $(x, y, z)_\tau = (z, y, x)_\tau$, for any $x, y, z, t \in G$.

II: i) $\exists 0 \in G$, $((x, y, 0)_\tau, z, 0)_\tau = (x, (y, z, 0)_\tau\, 0)_\tau$; ii) $(0, x, 0)_\tau = x$;

iii) $((0, 0, x)_\tau, x, 0)_\tau = 0$, for any $x, y, z \in G$.

5. **Ternary rings. Ternary algebras.** The extension of the concept of ring to the ternary formations was made in various manners ([11], [18]).

The simplest way to define a ternary ring is by giving a ternary associative operation on a binary abelian group, satisfying a distributivity law.

But some other extensions were already studied or are studying now: a). $(M, +, \tau)$, such that $(M, +)$ is a group, (M, τ) is a ternary semigroup, τ is distributive with respect to $+$; b). $(M, +, \cdot, \tau)$, where $(M, +, \cdot)$ is a ring, and τ is distributive with respect to $+$, (M, τ) being a ternary semigroup; c). (M, τ_1, τ_2), where τ_1 and τ_2 are ternary operations related by a distributivity law; d). A ternary groupoid on a module.

Consider a ternary ring given by the axioms: i). $(I_t, +)$ is an Abelian group; ii). (I_t, τ) is a ternary semigroup; iii). τ is distributive with respect to $+$ (see [13]). We may define: a) Ternary subrings J, $J - J \subseteq J$, $(J, J, J)_\tau \subseteq J$; b) First kind of ideals J, $J - J \subseteq J$, $(J, J, I_t)_\tau \subseteq J$ — right ideals $(J, I_t, J)_\tau \subseteq J$ — two-sided ideals, or $(I_t, J, J)_\tau \subseteq J$ — left ideals); c) Second kind of ideals J, $J - J \subseteq J$, $(J, I_t, I_t)_\tau \subseteq J$ ((resp. $(I_t, J, I_t)_\tau \subseteq J$ and $(I_t, I_t, J)_\tau \subseteq J$). The concepts of ideals may be defined also in ternary semigroups or in ternary algebras.

A ternary ring was defined in [19] by the axioms which are fulfiled by the ternary operation τ: 1. $(x, y, z)_\tau = (y, x, z)_\tau$, for any $x, y, z \in I_t$; 2. $\exists 0, 1 \in I_t$, $\forall x \in I_t$, $(x, 0, 1)_\tau = x$ and $(1, 0, x)_\tau = x$; 3. $\forall x \in I_t (0, 0, x)_\tau = 0$; 4. $\forall x, y, u, v \in I_t$, $(u, x, y)_\tau = (v, x, y)_\tau \Rightarrow u = v$; 5. $\forall x, y, z \in I_t$, $((x, y, 1)_\tau, z, 1)_\tau = (x, (y, z, 1)_\tau, 1)_\tau$; 6. $\forall a, b \in I_t$, $\exists x \in I_t$, $(a, x, b)_\tau = 0$; 7. $\forall x, y, z, u \in I_t$, $(x, y, (z, 0, u)_\tau)_\tau = ((x, y, z)_\tau, 0, u)_\tau$; 8. $\forall x, y, z \in I_t$, $((x, y, z)_\tau = (x, 0, z)_\tau (y, 0, z)_\tau, 1)_\tau$.

V. Tamaş [20] introduced the concept of *coordinatizing ternary rings*.

In a vector space V over a field K with the characteristic $p > 0$, we introduce a ternary operation τ, and V becomes a ternary algebra A_t. Consider here only the ternary algebras given by the axioms: 1. A_t is a finite dimensional vector space over the field K; 2. A_t satisfies the condition: $\forall x, y, z \in A_t$, $\forall \alpha, \beta, \gamma \in K$, $(\alpha x, \beta y, \gamma z)_\tau = \alpha\beta\gamma(x, y, z)_\tau$; 3. τ is distributive with respect to $+$, i.e. $(\sum_i x_i, \sum_j y_j, \sum_k z_k)_\tau = \sum_{i,j,k} (x_i, y_j, z_k)_\tau$. Given a base $\{e_1, e_2, \ldots, e_n\}$ in A_t, we form the ternary products $(e_i, e_j, e_k)_\tau = \sum_{h=1}^{n} \alpha_{ijk}^h e_h$. The coefficients α_{ijk}^h determine a tensor and they are caled the *structure constants* of the ternary algebra A_t. We have: i) Commutative ternary algebras, if the tensor $\{\alpha_{ijk}^h\}$ is symmetric covariant; ii) Associative ternary algebras, if the tensor satisfies certain structure equations.

We note that in finite dimensional ternary algebras over K, by using the structure tensor, we can find analitically unities, biunities, annulators, biannulators, and so on. The zero is that of the vector space.

Let A_t and B_t be two ternary algebras. A mapping $\varphi: A_t \to B_t$ is called a *homomorphism of ternary algebras*, if $\varphi(x+y) = \varphi(x) + \varphi(y)$, $\varphi(\alpha x) = \alpha\varphi(x)$, $\varphi((x, y, z)_\tau) = (\varphi(x), \varphi(y), \varphi(z))_\tau$, for any $x, y, z \in A_t$ and any $\alpha \in K$. If we choose the bases $\{e_1, e_2, \ldots, e_n\}$ in A_t and $\{e'_1, e'_2, \ldots, e'_m\}$ in B_t, to determine the homomorphisms from A_t to B_t, one must solve the equations:

$\sum_{s=1}^{n} \alpha_{pqr}^s \beta_s^h = \sum_{i,j,k} \alpha_{ijk}^h \beta_p^i \beta_q^j \beta_r^k$, $p, q, r, s = 1, 2, \ldots, n$ and $i, j, k, h = 1, 2, \ldots, m$, where

$\varphi(e_p) = \sum_{i=1}^{m} \beta_p^i e'_i$, the nm variables being β_p^i.

The following examples of ternary algebras were considered :

a) F being a symmetric bilinear form over the field K, the ternary operation is defined by : $(x, y, z)_\tau = F(x, z)y - F(y, z)x$ and it satisfies the conditions : $(x, y, z)_\tau + (y, z, x)_\tau + (z, x, y)_\tau = 0$ and $(x, x, y)_\tau = 0$ and $(u, v, (x, y, z)_\tau)_\tau = ((u, v, x)_\tau, y, z)_\tau + (x, (u, v, y)_\tau, z)_\tau + (x, y, (u, v, z)_\tau)_\tau$, for any x, y, z, u, v, hence it is a ternary algebra from the family proposed by Yoshiaki Taniguchi [21].

b) B_1, B_2, B_3 being bilinear forms in V, the ternary operation is given by $(x, y, z)_\tau = B_1(y, z)x + B_2(z, x)y + B_3(x, y)z$ and the ternary algebra obtained in this way has interesting properties.

c) In the family of algebras given above, we remark the case $B_1 = -B_2 = B_3 = B$, $B(x, y) = L(x)L(y)$, where L is a linear form over K, when the ternary algebra has some properties of a ternary field (see [13], [18]).

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PROPERTIES OF A NON-ASSOCIATIVE TERNARY STRUCTURE

BY

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Recently, the algebraic structures with n-ary operations, $n \geq 3$, appear more often in specialized papers. Remarkable results have been obtained in the field known under the name of the n-group and n-semigroup theory. All the structures investigated satisfy in one or another form the n-ary associativity.

In [5], a ternary structure has been introduced which is useful for the coordinatization of some projective planes. We shall give here some properties of this algebraic structure.

Definition 1. *The non-empty set M on which the ternary operation (x, y, z) is defined is called the coordinatizing ternary ring, if in M there are two different elements denoted* 0 *and* 1 *and the following axioms are fulfilled :*

1. $\forall x, y, z \in M, \quad (0, y, z) = z, \ (x, 0, z) = z, \ (x, 1, 0) = x$;
2. $\forall x, y, z \in M, \quad ((x, 1, y), 1, z) = (x, 1, (y, 1, z))$;
3. $\forall x, y \in M, \quad (x, 1, y) = (y, 1, x) = (1, x, y)$;
4. $\forall x, y, z_1, z_2 \in M, \quad (x, y, z_1) = (x, y, z_2) \Rightarrow z_1 = z_2$

and analogously for the other places ;

5. $\forall a \in M, \ \exists x \in M$ *so that* $(a, 1, x) = 0$,
6. $\forall x, y, z \in M, \ (x, (y, z, 0), 0) = ((x, y, 0), z, 0)$,
7. $\forall x, y, z \in M, \ ((x, 1, y), z, 0) = ((x, z, 0), 1, (y, z, 0))$ *and*

$$(x, (y, 1, z), 0) = ((x, y, 0), 1, (x, z, 0)).$$

Remarks. 1 The solution of the equation in axiom 5 is unique. We denote this solution by $-a$ and we call this element the *opposite* of a. 2) If for any $x, y \in M$, $(x, y, 0) = (y, x, 0)$, the coordinatizing ternary ring is called *commutative*. 3) If the equation $(a, x, 0) = 1$ has a solution in M we shall say that a is *invertible* and in this case $a \neq 0$.

Definition 2. *If for any $a \in M$, $a \neq 0$, is invertible, then we say that the coordinatizing ring is a field.*