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LATTICIAL ASPECTS OF SOME RELATIVE CHAIN CONDITIONS FOR MODULES

BY

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The concept of (ascending) Loewy series of an upper continuous and modular lattice appears explicitly for the first time in N ă s t ă s e s c u [2]. The aim of this paper is to show how this notion can be efficaciously used to obtain simplified proofs of some results concerning chain conditions for modules relative to Gabriel topologies, by placing these conditions in a latticial setting.

Section 0 of this paper presents mainly the notion of Loewy series of a lattice. The next section gives a necessary and sufficient condition for an artinian, upper continuous and modular lattice to be noetherian. From this in a very simple manner a latticial as well as a module-theoretical proof of the T e p l y-M i l l e r theorem may be derived. In section 2 the definition of the notion of composition series of a module with respect to a set of Gabriel topologies is presented and is outlined a short proof of a result due to T e p l y [1], [2] concerning the invariance of the type of such a composition series, by using two simple lemmas on lattices.

The complete proofs of the results presented in this paper are given in A l b u [1], [2].

0. Preliminaries. Throughout this paper **L** will denote an upper continuous and modular lattice with a least element 0 and with a greatest element 1. The notation and terminology will follow S t e n s t r ö m [1].

Recall that a nonzero element $a \in \mathbf{L}$ is an *atom* if whenever $b \in \mathbf{L}$ and $b<a$, then $b=0$, i.e., the interval $[0, a]$ has exactly two elements, 0 and a. If $x, y \in \mathbf{L}$ and $x<y$, then the interval $[x, y]$ is said to be *simple* if y is an atom in the sublattice $[x, y]$ of **L**. The lattice **L** is called *semi-atomic* if 1 is a join of atoms, and **L** is called *semi-artinian* if for every $x \in \mathbf{L}$, $x \neq 1$, the sublattice $[x, 1]$ of **L** contains an atom.

The (*ascending*) *Loewy series* of **L**, $s_0(\mathbf{L})<s_1(\mathbf{L})<\ldots<s_{\lambda(L)}(\mathbf{L})$, is defined inductively as follows: $s_0(\mathbf{L})=0$, $s_1(\mathbf{L})$ is the *socle* $\mathrm{So}(\mathbf{L})$ of **L** (i.e., the join of all atoms of **L**), and if the elements $s_\beta(\mathbf{L})$ of **L** have been defined for all ordinals $\beta<\alpha$, then $s_\alpha(\mathbf{L})=\bigvee_{\beta<\alpha} s_\beta(\mathbf{L})$ if α is a limit ordinal, and $s_\alpha(\mathbf{L})=\mathrm{So}([s_\gamma(\mathbf{L}), 1])$ if $\gamma=\alpha+1$; $\lambda(\mathbf{L})$ is the least ordinal λ such that $s_\lambda(\mathbf{L})=s_{\lambda+1}(\mathbf{L})$, and is called the *Loewy length* of **L**.

Recall that a (*Jordan-Hölder*) *composition series* of **L** is a chain

$$(*) \qquad 0=a_0<a_1<\ldots<a_n=1,$$

such that each interval $[a_{i-1}, a_i]$, $1 \leqslant i \leqslant n$ is a simple one. The lattice **L** is said to be of *finite length* if **L** has a composition series; in this case, if (*) is a composition series of **L**, then n is called the length of L and will be denoted in the sequel by $l(\mathbf{L})$. It is well-known that **L** is of finite length if and only if it is both artinian and noetherian.

0.1. **Proposition.** *The following properties of* **L** *are equivalent:*
(1) **L** *is a lattice of finite length.* (2) **L** *is an artinian lattice with* $\lambda(\mathbf{L})$ *finite.* (3) **L** *is a noetherian and semi-artinian lattice.*

1. Loewy series of a lattice and the relative Hopkins-Levitzki theorem.

1.1. **Lemma.** *If* **A** *is an artinian and modular lattice with* 1, *then there exists an element* $a^* \in \mathbf{A}$ *which is the least element of* **A** *such that the sublattice* $[a^*, 1]$ *of* **A** *is of finite length.*

If A is a lattice as in the above Lemma, then $l([a^*, 1])$ will be called the *reduced length* of A, and will be denoted by $l^*(A)$.

1.2. **Theorem.** *Let* **L** *be an upper continuous and modular lattice satisfying the following condition:*

(l^*) *For each* $y<x$ *in* **L** *there exists an element* $a_{yx} \in \mathbf{L}$ *such that* $a_{yx} \leqslant x$, $a_{yx} \leqslant y$ *and* $[0, a_{yx}]$ *is artinian; in addition, there exists a natural number* t *such that* $l^*([0, a_{yx}])<t$ *for all* $y<x$ *in* **L**. *Then* **L** *is semi-artinian and has finite Loewy length.*

1.3. **Remark.** The condition (l^*) may be reformulated as follows, if we denote $I=\{(y, x) \mid x, y \in \mathbf{L}, y<x\}$ and $a_i=a_{yx}$ for $i=(y, x) \in I$:

(l^*) *There exists a nonempty family* $(a_i)_{i \in I}$ *of elements of* **L** *and a natural number* t *such that* $[0, a_i]$ *is artinian for each* $i \in I$, *each element of* **L** *is a join of a subfamily of* $(a_i)_{i \in I}$, *and* $l^*([0, a_i])<t$ *for each* $i \in I$.

By using 0.1, from the above Theorem one deduces immediately:

1.4. **Corollary.** *If the lattice* **L** *satisfies the condition* (l^*), *then* **L** *is artinian if and only if* **L** *is noetherian.*

Note that the condition (l^*) about an artinian, upper continuous and modular lattice **L** is necessary for **L** to be noetherian. Indeed, in this case, for each $y<x$ in **L** choose $a_{yx}=x$; then $l^*([0, a_{yx}])=l([0, a_{yx}]) \leqslant l(\mathbf{L})$.

We shall apply now 1.4 to get a simple noncategorical proof of the relative Hopkins-Levitzki Theorem. Let R be an associative and unitary ring, and Mod-R the category of unitary right R-modules. Let F be a right Gabriel topology on R, $(\mathcal{T}_F, \mathcal{F}_F)$ the corresponding hereditary torsion theory on Mod-R, and t_F the torsion radical associated to $(\mathcal{T}_F, \mathcal{F}_F)$. If $M \in$ Mod-R, we shall denote by $C_F(M)$ the set of all submodules N of M such that $M/N \in \mathcal{F}_F$. It is well-known that $C_F(M)$ is an upper continuous modular lattice with respect to inclusion, having the least element $t_F(M)$ and the greatest element M.

Recall that $M \in$ Mod-R is said to be F-*noetherian* (resp. F-*artinian*) if $C_F(M)$ is a noetherian (resp. artinian) lattice. The ring R is said to be F-noetherian (resp. F-artinian) if the R-module R_R is F-noetherian (resp. F-artinian).

1.5. **Theorem** (M i l l e r and T e p l y [1]). *Let* F *be a right Gabriel topology on the ring* R *such that* R *is* F-*artinian. Then, a right* R-*module* M *is* F-*artinian if and only if* M *is* F-*noetherian.*

Proof. According to 1.4, it will suffice to check that the lattice $C_F(M)$ satisfies the condition ($\mathcal{L}^*$).

1.6. **Remarks**. (1) When the proof of 1.4 is carried out on the particular lattice $C_F(M)$, F being a right Gabriel topology on R such that R is F-artinian, one obtains a short module-theoretical proof of the above Theorem, which is the relative form of the Hopkins-Levitzki Theorem, and which is quoted in Faith [1] as the Teply-Miller Theorem.

(2) The above Theorem is proved in Miller and Teply [1] in a complicated manner; another module-theoretical proof of this Theorem, also long and complicated, is available in Faith [1]. A short categorical proof of this Theorem is due to Năstăsescu [1] and it led us to place the Hopkins-Levitzki Theorem in a latticial setting. A discussion on the various forms of the Hopkins-Levitzki Theorem and the connection between them may be found in Albu and Năstăsescu [1].

2. **Loewy series of a lattice and relative $\mathcal{X}$-composition series of a module**. As in the previous sections, $\mathbf{L}$ will denote an upper continuous modular lattice.

2.1. **Lemma.** *Let $(x_i)_{0\leqslant i\leqslant n}$ be a finite family of elements of* $\mathbf{L}$ *such that* $0=x_0<x_1<\ldots<x_n=1$. *If $y\in\mathbf{L}$, $y\neq 0$, then there exist j, $1\leqslant j\leqslant n$, $z\in\,]x_{j-1}, x_j]$, and $u\in\,]0, y]$ such that the intervals $[0, u]$ and $[x_{j-1}, z]$ are similar.*

2.2. **Lemma.** *Let $(x_i)_{0\leqslant i\leqslant n}$ be a finite family of elements of* $\mathbf{L}$ *such that* $0=x_0<x_1<\ldots<x_n$ *and $[x_{i-1}, x_i]$ are simple intervals for each i, $1\leqslant i\leqslant n$. If the interval $[x_n, 1]$ contains no atom, then $\lambda(\mathbf{L})\leqslant n$ and $x_n=s_{\lambda(\mathbf{L})}(\mathbf{L})$.*

Let F be a right Gabriel topology on the ring R. Recall that $M\in\text{Mod-}R$ is said to be *F-cocritical* if $M\neq 0$, $M\in\mathcal{F}_F$ and $M/M'\in\mathcal{T}_F$ for every nonzero submodule M' of M, or equivalently, if $M\neq 0$ and $C_F(M)=\{0, M\}$.

An *F-composition series* of $M\in\text{Mod-}R$ is a chain $t_F(M)=M_0<M_1<\ldots<M_n=M$ of submodules of M such that M_i/M_{i-1} is an F-cocritical module for each i, $0<i\leqslant n$.

2.3. **Lemma.** *Let $M\in\text{Mod-}R$ and*

$$(*)\qquad M_0<M_1<\ldots<M_n=M$$

ie a chain of submodules of M. Then (*) is an F-composition series of M if and only if (*) is a composition series of the lattice $C_F(M)$.

Definition. (Teply [1], [2]). *Let $\mathcal{X}$ be a nonempty set of right Gabriel topologies on the ring R. A right R-module M is said to have an $\mathcal{X}$-composition series if there exists a chain (*) of submodules of M such that*

(1) each M_i/M_{i-1} is F_i-cocritical for some $F_i\in\mathcal{X}$; (2) if M_i/M_{i-1} contains an F-cocritical submodule for some $F\in\mathcal{X}$, then $F=F_i$; (3) $F_1\leqslant F_2\leqslant\ldots\leqslant\ldots\leqslant F_n$; (4) $M_0=t_{F_1}(M)$.

If M has an $\mathcal{X}$-composition series (*), then by renumbering the Gabriel topologies F_i one obtains for M a so called $\mathcal{X}$-composition series of type $(F_1, n_1;\ F_2, n_2;\ldots;F_k, n_k)$, where $k\geqslant 1$, $n_1, n_2, \ldots, n_k$ are natural numbers $\geqslant 1$, and $F_1, F_2, \ldots, F_k\in\mathcal{X}$ such that (i) $F_1<F_2<\ldots<F_k$. (ii) The first n_1 factors of the series (*) are F_1-cocritical, the next n_2 factors are F_2-cocritical, and so on, the last n_k factors of (*) are F_k-cocritical.

By using the previous three Lemmas one proves very easy, in a slight generalized context, a result established by Teply in a more complicated manner:

2.4. **Theorem** (*Teply* [1], [2]) *Let be $M\in\text{Mod-}R$ and $\mathcal{X}$ a nonempty set of right topologies on R. If M has an $\mathcal{X}$-composition series $M_0<M_1<\ldots<M_n=M$ of type $(F_1, n_1;\ F_2, n_2;\ldots;F_k, n_k)$, then any other $\mathcal{X}$-composition series of M beginning with M_0 is of the same type, and the injective hulls of actors M_i/M_{i-1} are unique up to order and isomorphism. Moreover, $t_{F_{j+1}}(M)=M_{n_1+n_2+\ldots+n_j}$ for each j, $1\leqslant j\leqslant k-1$.*

2.5. **Remark**. The notion of an *F-semicritical module*, as it appears in Teply [2] can be reformulated in a latticial setting as follows: an R-module M is F-semicritical iff $M\in\mathcal{F}_F$ and $C_F(M)$ is a semi-atomic lattice of finite length. Note also that the *F-semicritical socle series* of M is exactly the Loewy series of the lattice $C_F(M)$.

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