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EXTERNAL HOMOGENIZATION FOR FINITELY GENERATED ABELIAN GROUPS

BY

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One of the general problems of Graded Ring Theory is to see whether a graded module having a certain property has a similar property when regarded without grading. The main difficulty in proving a statement from the above mentioned class is that from the properties of a small class of objects (e.g. graded submodules, graded ideals) one must deduce similar properties for a larger class of objects (submodules, ideals not necessarily graded).

In [3], using External Homogenization, we have proved the following result: *if* $R=\bigoplus_{i\in Z} R_i$ *is a graded ring of type* Z, H *is a graded filter on* R [4], $F=\{I$ *left ideal of* $R \mid \exists J\in H,\ J\subseteq I\}$ *is the Gabriel topology* [6] *generated by* H, *and* M *is a graded* R*-module which is gr-H-noetherian, then* M, *regarded without grading is an* F*-noetherian* R*-module*. In this note, we extend External Homogenization in order to prove the above result for graded modules of type G, where G is a finitely generated abelian group, as well as a similar result for graded bimodules.

All rings considered in this note will be unitary. If R is a ring, by an R-module we will mean a left R-module. Missing proofs and unexplained notation concerning Graded Ring Theory (resp. Torsion Theory) may be found in [4] (resp. [6]).

Let G be a finitely generated abelian group and $R=\bigoplus_{\sigma\in G} R_\sigma$ a graded ring of type G. Suppose that G is generated by $\theta_1, \ldots, \theta_n$. The polynomial ring $R[X_1, \ldots, X_m]$, $1\leqslant m\leqslant n$, becomes a graded ring of type G if we put $\deg X_i=\theta_i$, $i=1, \ldots, m$, and $R[X_1, \ldots, X_m]_\sigma=\sum_{\substack{\alpha\in N^m\\ \tau\theta^\alpha=\sigma}} R_\tau X^\alpha$ where $\alpha=(a_1, \ldots, a_m)$, $X^\alpha=X_1^{a_1}\ldots X_m^{a_m}$ and $\theta^\alpha=\theta_1^{a_1}\ldots\theta_m^{a_m}$. The same construction, applied to a graded R-module M, yields a graded $R[X_1, \ldots, X_m]$-module, whose underlying module is $M[X_1, \ldots, X_m]=R[X_1, \ldots, X_m]\otimes_R M$.

Let $x\in M$ and write $x=x_{\sigma_1}+\ldots+x_{\sigma_p}$ as a sum of homogeneous elements, where

$$\sigma_i=\theta_1^{a_{i1}}\ldots\theta_n^{a_{in}},\quad 1\leqslant i\leqslant p. \tag{1}$$

Let $t_j=\max\{a_{1j}, \ldots, a_{pj}\}$, $1\leqslant j\leqslant n$, $\alpha=(t_1, \ldots, t_n)$ and $\alpha_i=(a_{i1}, \ldots, a_{in})$, $1\leqslant i\leqslant p$. Then we may associate to $x\in M$ a homogeneous (in the above introduced

grading) element $x^*\in M[X_1, \ldots, X_n]$ by putting

$$x^*=x_{\sigma_1}X^{\alpha-\alpha_1}+\ldots+x_{\sigma_p}X^{\alpha-\alpha_p}\in M[X_1, \ldots, X_n]_{\theta^\alpha}.$$

where $X^{\alpha-\alpha_i}=X_1^{t_1-a_{i1}}\ldots X_n^{t_n-a_{in}}$, $1\leqslant i\leqslant p$ and $\theta^\alpha=\theta_1^{t_1}\ldots\theta_n^{t_n}$. Note that for a given $x\in M$, x^* depends on the relations (1).

If N is an R-submodule of M, we will let N^* to denote the graded $R[X_1, \ldots, X_n]$-submodule of $M[X_1, \ldots, X_n]$ generated by all x^*, $x\in N$.

If u is a homogeneous element of $M[X_1, \ldots, X_n]$, we will let u_* denote the element of M obtained from u by replacing each of the $X_1, \ldots, X_n$ by 1. If L is y graded submodule of $M[X_1, \ldots, X_n]$ we will put $L_*=\{u_* \mid u\in L$ and u is homogeneous$\}$, which is an R-submodule of M.

With the above notation we have the following:

Lemma 1. (i) $(x^*)_*=x$; (ii) $(N^*)_*=N$; (iii) $N\subseteq P\Rightarrow N^*\subseteq P^*$.

Let R and S be graded rings and $\varphi : R\to S$ a graded morphism. If H is a graded filter on R, then $\varphi(H)$ is a graded filter on S [1 ; 1.1], where $\varphi(H)=\{I$ left graded ideal of $S \mid \varphi^{-1}(I : y)\in H,\ \forall y\in S,\ y$ homogeneous$\}$.

We recall that a graded R-module M is said to be *gr-H-noetherian* [3] (H is a graded filter on R) if M satisfies the ascending chain condition for graded submodules N with the following property: if $x\in M$ is homogeneous and $(N : x)\in H$, then $x\in N$.

Proposition 2. *If* $R=\bigoplus_{\sigma\in G} R_\sigma$ *is a graded ring of type* G, *where* G *is a finitely generated abelian group,* H *is a graded filter on* R, $i : R\to R[X_1, \ldots, X_n]$ *is the inclusion and* $M=\bigoplus_{\sigma\in G} M_\sigma$ *is a gr-H-noetherian* R*-module, then* $M[X_1, \ldots, X_n]$ *is gr-*$i(H)$*-noetherian.*

Proof. By induction on n. The case $n=1$, as well as [3 ; 2.1], is a mere transcription of the ungraded case [1 ; 1.12]. Let now $2\leqslant m\leqslant n$. If G is generated by $\theta_1, \ldots, \theta_n$ and $\tau\in G$, then $M[X_1, \ldots, X_m]_\tau=\sum_{\substack{k_1, \ldots, k_m\\ \sigma\theta_1\ldots\theta_m=\tau}} M_\sigma X_1^{k_1}\ldots X_m^{k_m}=$

$$=\sum_{\substack{k_m\\ \varepsilon\theta_m=\tau\\ \varepsilon\in G}}\Big(\sum_{\substack{k_1\ldots k_{m-1}\\ \sigma\theta_1\ldots\theta_{m-1}=\varepsilon\\ \sigma\in G}} M_\sigma X_1^{k_1}\ldots X_{m-1}^{k_{m-1}}\Big)X_m^{k_m}=\sum_{\substack{k_m\\ \varepsilon\theta_m=\tau\\ \varepsilon\in G}} M[X_1, \ldots, X_{m-1}]_\varepsilon X_m^{k_m}$$

which makes the inductive step work.

Theorem 3. *Let* $R=\bigoplus_{\sigma\in G} R_\sigma$ *be a graded ring of type* G *(*G *is finitely generated and abelian),* H *a graded filter on* R *and* M *a graded* R*-module which is gr-H-noetherian. Then, if* F *is the smallest Gabriel topology containing* H, M *is an* F*-noetherian* R*-module.*

Remark. The proof of Theorem 3 is the same as the one of [3 ; 2.2] and it provides an affirmative answer to Problem 2.6 of [5].

Let now R and S be two graded rings of type G, where G is a finitely generated abelian group. As extended above from the case $G=\mathbf{Z}$ to the case of the gradation over finitely generated abelian groups External Homogenization may be easily applied to the case of bimodules as well. Proceeding as above, one can show that if M is an R-left and S-right bimodule which satisfies the ascending chain condition for R-left and S-right graded subbimodules, then M satisfies the ascending chain condition for R-left and S-right sub-bimodules. As a consequence, we deduce :

Corollary 4. *If $R=\bigoplus_{\sigma\in G} R_\sigma$ is a graded ring of type G(R is abelian finitely generated) which is left and right gr-noetherian, then R satisfies the ascending chain condition for twosided ideals.*

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MUTUALLY NON-ISOMORPHIC UNITARY $\mathbf{F}_p$-ALGEBRAS OF ORDER p^4

BY

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Introduction. In this paper we determine all unitary $\mathbf{F}_p$-algebras (i.e. unitary rings of characteristic p) of order p^4, where p is a prime natural number. Since a finite unitary ring is uniquely expressible as a direct sum of unitary rings of prime power order, it is important to study unitary rings R such that for some prime p and natural numbers m and n, $0<m\leqslant n$, the characteristic c_R of R is p^m and the order $|R|$ of R is p^n. Note that in this case the Jacobson radical of R, $J=J(R)$, has order p^e, $0\leqslant e<n$ and J is the largest nilpotent ideal of R.

§1. **Preliminaries.** There are four non-isomorphic unitary rings of p^2 elements and, depending on whether p is even or odd, there are respectively eleven or twelve mutually non-isomorphic unitary rings of order p^3 (see [1], Exercise XIX. 9). Because in our enumeration of $\mathbf{F}_p$-algebras of order p^4 an important tool is the cardinality of J, we state this result as:

Proposition 1. *Let R be a ring with unity of p^2 elements.*

(i) *If $c_R=p$ and $J=0$, then R is isomorphic to $\mathbf{F}_{p^2}$ or $\mathbf{F}_p\times\mathbf{F}_p$.*

(ii) *If $c_R=p$ and $|I|=p$,* then *R is isomorphic to $\mathbf{F}_p[X]/(X^2)$.*

(iii) *If $c_R=p^2$, then R is isomorphic to $\mathbf{Z}_{p^2}$.*

Proposition 2. *Let R be a ring with unity of p^3 elements, $p>2$.*

(i) *If $c_R=p$ and $J=0$, then R is isomorphic to $\mathbf{F}_{p^3}$, $\mathbf{F}_p\times\mathbf{F}_{p^2}$, or $\mathbf{F}_p\times\times\mathbf{F}_p\times\mathbf{F}_p$.*

(ii) *If $c_R=p$ and $|J|=p$, then R is isomorphic to $\mathbf{F}_p\times\mathbf{F}_p[X]/(X^2)$ or $\begin{pmatrix}\mathbf{F}_p & \mathbf{F}_p\\ 0 & \mathbf{F}_p\end{pmatrix}$.*

(iii) *If $c_R=p$ and $|J|=p^2$, then R is isomorphic to $\mathbf{F}_p[X, Y]/(Y^2, Y^2, XY)$ or $\mathbf{F}_p[X]/(X^3)$.*

(iv) *If $c_R=p^2$, then $J\neq 0$ and R is isomorphic to $\mathbf{Z}_{p^2}\times\mathbf{F}_p$ when $|J|=p$, to $\mathbf{Z}_{p^2}[X]/(X^2, pX)$ when $|J|=p^2$ and $J^2=0$, to $\mathbf{Z}_{p^2}[X]/(X^3, X^2-p, pX)$ or to $\mathbf{Z}_{p^2}[X]/(X^3, X^2-dX, pX)$ with $d\in\mathbf{F}_p^*\setminus\mathbf{F}_p^{*2}$ when $|J|=p^2$ and $J^2\neq 0$. The rings $\mathbf{Z}_{p^2}[X]/(X^3, X^2-dX, pX)$, $d\in\mathbf{F}_p^*\setminus\mathbf{F}_p^{*2}$ are mutually isomorphic.*

(v) *If $c_R=p^3$, then R is isomorphic to $\mathbf{Z}_{p^3}$.*

If *$p=2$, the last type in (iv) does not appear.*

§2. **Unitary $\mathbf{F}_p$-algebras of order p^4.** The enumeration of all types of unitary rings of order p^4 seem to be a difficult problem. We restrict our attention to the case of characteristic p ; we shall prove that in this case