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NORMAL PROJECTIVE DEGENERATIONS OF RATIONAL AND RULED SURFACES

BY

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This note announces some results concerning the structure of normal projective degenerations either of the rational surfaces with the second Betti number $\leqslant 10$, or of arbitrary ruled non-rational surfaces. The details will be published elsewhere.

In order to explain the results, let me first fix the set up. Let $f: \mathbf{X} \to \mathbf{T}$ be a projective flat morphism from the threefold $\mathbf{X}$ to the smooth affine curve $\mathbf{T}$, both defined over the field $\mathbf{C}$ of complex numbers, such that there is a distinguished point $o \in \mathbf{T}$ with the following properties: $\mathbf{X}_0 = f^{-1}(o)$ is a normal surface, while for every $t \in \mathbf{T}$, $t \neq o$, $\mathbf{X}_t = f^{-1}(t)$ is a smooth surface. From these assumptions it follows easily that $\mathbf{X}$ is a normal locally Cohen-Macaulay threefold, such that if $x \in \mathbf{X}$ is a singular point of $\mathbf{X}$, then $x \in \mathbf{X}_0$ and x is necessarily a singular point of $\mathbf{X}_0$.

In other words, f defines a flat projective family $\{\mathbf{X}_t\}_{t \in T}$ of surfaces with parameter space a smooth affine curve $\mathbf{T}$, such that $\mathbf{X}_t$ is smooth for $t \neq o$, and $\mathbf{X}_o$ is a normal surface. In the sequel the normal surface $\mathbf{X}_o$ will be referred to as a *normal projective degeneration* of the smooth family $\{\mathbf{X}_t\}_{t \in T - \{o\}}$ of surfaces. The problem I want to discuss in this paper is to describe all possible normal projective degenerations $\mathbf{X}_o$ when $\mathbf{X}_t$ is a rational (resp. ruled non-rational) surface for all $t \neq o$. If $\mathbf{X}_0$ is smooth and $\mathbf{X}_t$ rational (resp. ruled non-rational), then a result of I i t a k a (see [8]) tells that $\mathbf{X}_0$ is also rational (resp. ruled non-rational). Therefore one can assume that $\mathbf{X}_o$ has singularities.

Examples of normal projective degenerations of surfaces. 1) Let $\mathbf{Y}$ be a cubic surface in $\mathbf{P}^3$ having only isolated singularities. Then $\mathbf{Y}$ is normal, and in the space of all cubic surfaces in $\mathbf{P}^3$ (which can be identified to the projective space $\mathbf{P}^{19}$) we can pick an irreducible curve $\mathbf{T}$ passing simply through the point $o \in \mathbf{P}^{19}$ corresponding to the surface $\mathbf{Y}$, and such that the cubic surface $\mathbf{X}_t$ corresponding to every nearby point $t \in \mathbf{T}$, $t \neq o$, is smooth. Then $\mathbf{Y}$ appears as the degeneration of the family $\{\mathbf{X}_t\}_{t \in T'}$ (with $\mathbf{T}' = \mathbf{T} - \{o\}$, except possibly finitely many points) of smooth cubic surfaces in $\mathbf{P}^3$. The geometry of the cubic surfaces in $\mathbf{P}^3$ with isolated singularities is well known and classical. A modern reference for that is e. g. [7] (see also [14] for a different treatment). As far as the number and the nature of the possible singularities of such a cubic surface is concerned, one has two possi-

bilities: either $\mathbf{Y}$ is a cone over an elliptic curve (and hence $\mathbf{Y}$ has only one singularity which is simply-elliptic), or (otherwise) $\mathbf{Y}$ is a rational surface with only rational double points as singularities (and no more than 4), whose number and type A_n, D_n, E_n can be specificated (loc. cit.).

2) The method of constructing normal degenerations of surfaces that I am going to describe is classical and known in the modern literature as the „sweeping out the cone with hyperplane sections" method (see [9], page 46). Start with a smooth projective surface $\mathbf{F}$, a very ample line bundle $\mathbf{L}$ on $\mathbf{F}$ and a smooth curve $\mathbf{Y}$ belonging to the complete linear system $|\mathbf{L}|$. One assumes that $|\mathbf{L}|$ yields an arithmetically Cohen-Macaulay embedding $i = i_L: \mathbf{F} \to \mathbf{P}^N$ (with $N = \dim |\mathbf{L}|$). Then there is a hyperplane $\mathbf{H}$ of $\mathbf{P}^N$ such that $\mathbf{Y} = \mathbf{F} \cap \mathbf{H}$. Let $\mathbf{C}(\mathbf{F}, i)$ be the projective cone over $\mathbf{F}$ in $\mathbf{P}^{N+1}$ (i.e. with respect to the embedding i), and H' the hyperplane „at infinity" of $\mathbf{P}^{N+1}$. Then $\mathbf{H}$ is a 2-codimensional linear subspace of $\mathbf{P}^{N+1}$, which generates the pencil $\{\mathbf{H}_t\}_{t \in P^1}$ of all hyperplanes of P^{N+1} containing H. We may assume that the parametrization is taken in such a way that $\mathbf{H}_0$ is the hyperplane of this pencil passing through the vertex of $\mathbf{C}(\mathbf{F}, i)$ and $\mathbf{H}_\infty = \mathbf{H}'$. Then for every $t \in \mathbf{P}^1 - \{o\}$, $\mathbf{H}_t \cap \mathbf{C}(\mathbf{F}, i)$ is isomorphic to $\mathbf{F}$, while $\mathbf{H}_o \cap \mathbf{C}(\mathbf{F}, i)$ is just the cone $\mathbf{C}(\mathbf{Y}, i')$ over Y with respect to the embedding $i': \mathbf{Y} \to \mathbf{H} = \mathbf{P}^{N-1}$. Since $\mathbf{F}$ is arithmetically Cohen-Macaulay in $\mathbf{P}^N$, $\mathbf{C}(\mathbf{Y}, i')$ is a normal surface. In this way one gets a family $\{\mathbf{X}_t\}_{t \in P^1}$ of surfaces such that $\mathbf{X}_t = \mathbf{F}$ for every $t \neq o$, and $\mathbf{X}_o = \mathbf{C}(\mathbf{Y}, i')$. This family is actually an embedded family of surfaces in $\mathbf{P}^N$, i.e. the corresponding morphism $f: \mathbf{X} \to \mathbf{T} = \mathbf{P}^1$ fits into a commutative diagram of the form

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "f"'] & & T \times P^N \arrow[dl, "p_1"] \\ & T & \end{tikzcd}$$

with j a closed immersion and p_1 the projection onto the first factor.

3) If in the construction of 2) one takes $\mathbf{F} = \mathbf{P}^2$, $\mathbf{L} = O(s)$ with $s \geqslant 1$, and $\mathbf{Y}$ a smooth plane curve of degree s, one gets an embedded family $\{\mathbf{X}_t\}_{t \in P^1}$ of surfaces of degree s^2 in $\mathbf{P}^N$ (with $N = s(s+3))/2$, such that $\mathbf{X}_t \cong \mathbf{P}^2$ for every $t \neq o$, and $\mathbf{X}_0 \cong \mathbf{C}(\mathbf{Y}, v_s)$, where v_s is the restriction to $\mathbf{Y}$ of the s-fold Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^N$. If $s = 1$, $\mathbf{X}_0$ is of course (isomorphic to) $\mathbf{P}^2$ itself.

In order to state the results that I was able to prove, let me introduce a few definitions. Let $\mathbf{Z}$ be a normal projective surface. Then it makes sense to speak about Grothendieck's dualizing sheaf ω_Z, which is a reflexive sheaf of rank one over O_Z, and which coincides to the sheaf $\Omega^2_{Z/C}$ of 2-differentials on $\mathbf{Z}$, if $\mathbf{Z}$ is smooth. In particular, ω_Z corresponds to a class of linearly equivalent Weil divisors. Such a divisor $\mathbf{K}_Z$ will be referred to as a canonical divisor. The sheaf ω_Z is invertible if and only if $\mathbf{Z}$ is (locally) Gorenstein. If $n > 0$ is a positive integer, $\omega_Z^{[n]}$ will denote the double dual of $\omega_Z^{\otimes n}$; $\omega_Z^{(n)}$ corresponds to the class of the Weil divisor nK, and hence one can write $\omega_Z^{(n)} = O_Z(nK_Z)$.

Now one can define: the irregularity $q(\mathbf{Z})$ of $\mathbf{Z}$ by $q(\mathbf{Z}) = \dim_C \mathrm{H}^1(\mathbf{Z}, \mathrm{O}_Z)$, the geometric genus $p_g(\mathbf{Z})$ of $\mathbf{Z}$ by $p_g(\mathbf{Z}) = \dim_C \mathrm{H}^2(\mathbf{Z}, \mathrm{O}_Z)$, and for every $n \geq 1$ the n-genus $p_n(\mathbf{Z})$ of $\mathbf{Z}$ by $p_n(\mathbf{Z}) = \dim_C \mathrm{H}^0(\mathbf{Z}, \omega_Z^{(n)})$. By duality on $\mathbf{Z}$, $p_g(\mathbf{Z}) = p_1(\mathbf{Z})$.

Theorem 1. *In the set up of the beginning, assume that X_t is a rational surface with $b_2(X_t) \leq 10$ for every $t \neq o$, where $b_i(Z)$ denotes the i-th Betti number of a normal projective surface Z. Then the following statements hold:*

i) *$p_n(X_o)) = o$ for every $n \geq 1$, $b_2(X_o) \leq b_2(X_t)$ and $b_1(X_0) = q(X_0) = 0$.*

ii) *If $u: X_0' \to X_0$ is the minimal desingularization of X_0, then X_0' is a ruled surface, $\dim_Z R^1 u_*(O_{X_0'}) = q$, and $b_3(X_0) = 2q$, where q is the irregularity of X_0'. In particular, if X_0' is rational, then all the singularities of X_0 are rational in the sense of* [1].

iii) *If X_0' is ruled non-rational, then X_0 has precisely one non-rational singularity x, and (possibly) finitely many rational singularities. The irreducible components of the fibre $u^{-1}(x)$ are: a section of the canonical ruled fibration $\pi: X_0' \to B$ (with B a smooth projective curve of genus q) plus (possibly) some components of the degenerated fibres of π. The exceptional fibre of u over every rational singularity of X_0 (if any) is contained in a degenerated fibre of π.*

iv) *$b_2(X_0') - b_2(X_0)$ is equal to the number of irreducible components of all exceptional fibres of u.*

One of the main points of the proof of theorem 1 is to show that $p_n(X_0) = 0$ for every $n \geq 1$. The hypothesis „$b_2(X_t) \leq 10$ for $t \neq o$" was necessary only to prove these vanishings. I suspect however that this hypothesis is not really necessary. For example, this is the case at least if X_0 is assumed to be Gorenstein (see [12]). As soon as we know that $p_n(X_0) = 0$ for every $n \geq 1$, we easily deduce that $p_n(X_0') = 0$ for every $n \geq 1$ (because in general one has $p_n(X_0') \leq p_n(X_0)$ for every $n \geq 1$), and therefore X_0' is ruled by Enriques' criterion of ruledness (see [4], or [5]). In general the differences $p_n(X_0) - p_n(X_0')$ ($n \geq 1$), roughly speaking, give a measure of the complexity of the singularities of X_0, and since these are all zero, the singularities of X_0 are not very complicated. The statements concerning the Betti numbers follow by using the methods of [6] together with an argument of [3].

Theorem 2. *In the set up of the beginning, assume that X_t is a ruled non-rational surface for one (and hence for all) $t \neq o$. Then the following statements hold:*

i) *$p_n(X_0) = 0$ for every $n \geq 1$.*

ii) *If $q(X_t) = q$ and $u: X_0' \to X_0$ is the minimal desingularization of X_0, then X_0' is a ruled surface of irregularity q.*

iii) *X_0 has at most rational singularities and their exceptional fibres are contained in the degenerated fibres of the canonical ruled fibration $\pi: X_0' \to B$.*

Theorem 3. *In the set up of the beginning, assume that X_0 is Gorenstein and the general fibre X_t is a rational surface. Then X_0 is either rational (i.e. X_0' is rational) with at most rational double points as singularities, or X_0' is ruled of irregularity one and X_0 has precisely one simple elliptic singularity x and (possibly) finitely many rational double points. In the latter case the support of the fibre $u^{-1}(x)$ (the notations being the same as in theorem 1) is a section of the ruled fibration π, and the fibres of u over the rational double points of X_0 are contained in the degenerated fibres of π.*

As I have already remarked, in the Gorenstein case, Wilson proved that $p_n(\mathbf{X}_0) = 0$ for every $n \geq 1$ without assuming anything about the second Betti number of the general surface $\mathbf{X}_t$. Theorem 3 can then be deduced either from the classification of normal projective Gorenstein surfaces given by S a k a i in [11], or directly from theorem 1 above using a standard argument in order to prove that $u^{-1}(x)$ has only one irreducible component, which is an elliptic curve.

Theorem 4. *In the set up of the beginning, assume that $\mathbf{X}_t$ is (isomorphic to) $\mathbf{P}^2$ for one (and hence for all) $t \neq o$. Then the conclusions i)—iv) of theorem 1 hold, the anti-Kodaira dimension of $\mathbf{X}_0'$ is 2, and $\mathbf{X}_0'$ is the anticanonical model of $\mathbf{X}_0$ (in the sense of [10] if $\mathbf{X}_0'$ is rational, and of [2] if $\mathbf{X}_0'$ is ruled non-rational). Moreover, there is a positive integer $a \geq 1$ such that $-a\mathbf{K}_{X_0}$ is an ample Cartier divisor (with $\mathbf{K}_{X_0}$ a canonical (Weil) divisor of $\mathbf{X}_0$), or equivalently (see [2]), the anticanonical ring of $\mathbf{X}_0$, $\bigoplus_{n=0}^{\infty} \mathbf{H}^0(\mathbf{X}'_0, \mathbf{O}_{X_0}(-n\mathbf{K}_{X_0'}))$ is a finitely generated $\mathbf{C}$-algebra (the latter algebra is also isomorphic to $\bigoplus_{n=0}^{\infty} \mathbf{H}^0(\mathbf{X}_0, \mathbf{O}_{X_0}(-n\mathbf{K}_{X_0}))$). If we assume moreover that $\mathbf{X}_0$ is Gorenstein, then $\mathbf{X}_0$ is isomorphic either to $\mathbf{P}^2$, or to an elliptic cone of degree 9 in $\mathbf{P}^9$ (of type $\mathbf{C}(\mathbf{Y}, v_3)$, with $\mathbf{Y}$ a smooth cubic plane curve, see example 3) above).*

I suspect that much more than the statement of theorem 4 is true, namely that the only normal projective degenerations of $\mathbf{P}^2$ are essentially given by the example 3) above. Theorem 4 tells us that at least in the Gorenstein case this is true.

I end this paper by explaining briefly what the anticanonical model $\mathbf{Y}$ of a smooth projective surface $\mathbf{X}$ with anti-Kodaira dimension 2 is (see [10], [2]). The fact that the anti-Kodaira dimension of $\mathbf{X}$ is 2 means for example that the function $n \to \dim_C \mathbf{H}_0(X, \mathbf{O}_X(-n\mathbf{K}_X))$ behaves quadratically with n. Let then $-\mathbf{K}_X = \mathbf{P} + \mathbf{N}$ be the Zariski decomposition of an anticanonical divisor of $\mathbf{H}$ (see [13], [10]), with $\mathbf{P}$ a numerically effective Q-divisor, and either $\mathbf{N} = 0$, or $N > 0$ and in the latter case the intersection matrix of Supp (N) is negatively defined. Moreover, we have that $\mathbf{P} \,.\, \mathbf{E} = 0$ for every component $\mathbf{E}$ of Supp (N). The fact that the anti-Kodaira dimension of $\mathbf{X}$ is 2 is equivalent to $\mathbf{P}^2 > 0$, and this implies that $\mathbf{X}$ is ruled. Let then $\mathbf{A}$ be the set of all irreducible curves $\mathbf{E}$ on $\mathbf{X}$ such that $\mathbf{P} \,.\, \mathbf{E} = 0$. Since $\mathbf{P}^2 > 0$, the Hodge index theorem implies that $\mathbf{A}$ is a finite set and the intersection matrix of it is negatively defined. Therefore the connected components of $\mathbf{A}$ can be blown down to a finite number of singular points of a (unique) normal compact analytic surface $\mathbf{Y}$ (by a criterion of Grauert), and the point is that $\mathbf{Y}$ turns out to be actually a projective surface. This surface is called the anticanonical model of $\mathbf{X}$. If $\mathbf{X}$ is rational, $\mathbf{Y}$ has only rational singularities and $-a\mathbf{K}_Y$ is an ample Cartier divisor for some $a > 0$ (see [10]). If X is ruled non-rational, the singularities of $\mathbf{Y}$ are described in [2], and there is a positive integer $a > 0$ such that $-a\mathbf{K}_Y$ is an ample Cartier divisor if and only if the anticanonical ring $\bigoplus_{n=0}^{\infty} \mathbf{H}_0(\mathbf{X}, \mathbf{O}_X(-n\mathbf{K}_Y))$ is a finitely generated $\mathbf{C}$-algebra. It may happen that in the non-rational case the latter algebra is not finitely generated over $\mathbf{C}$ (unlike in the rational case).

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ARTIN APPROXIMATION THEORY AND APPLICATIONS

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§1. **Artin approximation properties.** Let A be a noetherian ring, $\underline{a}\subset A$ an ideal and $\hat{A}$ the completion of A in the $\underline{a}$-adic topology. We consider the following statements:

(AP) Every finite system of polynomial equations over A has its set of solutions in A dense with respect to the $\underline{a}$-adic topology in the set of its solutions in $\hat{A}$, i.e. for every solution $\hat{Y}$ from $\hat{A}$ and every positive integer c there exists a solution Y_c in A such that $Y_c\equiv\hat{Y} \bmod \underline{a}^c\hat{A}$.

(SAP) For every finite system of polynomial equations f over A there exists a function $\nu : N\to N$ with the following property:

For every positive integer c and for every system of elements $\hat{Y}$ from A which form a solution of f modulo $\underline{a}^{\nu(c)}$, there exists a solution Y_c of f in A such that $Y_c\equiv\hat{Y} \bmod \underline{a}^c$.

We call $(A, \underline{a})$ an *AP-couple* (resp. *SAP-couple*) if it satisfies (AP) (resp. (SAP)). A local ring $(A, \underline{m})$ is called a *ring with the property of Artin approximation* or an *AP-ring* if the couple $(A, \underline{m})$ is an AP-couple.

The study of these properties started with the following result due to M. Artin [A_1], [A_2]:

(1.0) **Theorem** (Artin). *The convergent power series rings in $X=(X_1, \ldots, X_n)$ over a valued field of characteristic zero, and the henselization of a local ring which is essentially of finite type over a field or an excellent Dedekind ring are AP-rings.*

A couple $(A, \underline{a})$ is called *quasi-excellent* if A is noetherian and the completion map $A\to\hat{A}$ is regular, $\hat{A}$ being the completion of A in the $\underline{a}$-adic topology. A recent result (see [P_2] Theorem (1.3)) says that

(1.1) **Theorem.** *Every quasi-excellent henselian couple is an AP-couple. In particular every excellent henselian local ring is an AP-ring.*

The above theorem gives a positive answer to Conjecture (1.3) from [A_3]. The converse of Theorem (1.1) is not completely solved. If $(A, \underline{m})$ is an AP-ring then A is henselian and the formal fibers of the canonical map $A\to\hat{A}$ are geometrically normal (see [Br] or [CP]). In particular a two dimensional local ring A is AP iff it is excellent henselian (an AP-ring is catenarian by [K], Ch. V). Recently C. Rotthaus [R] showed that an AP-ring A is excellent if $A[[X]]$ is also an AP-ring. If A is an AP-ring then the hen-