which satisfies the ascending chain condition for R-left and S-right graded subbimodules, then M satisfies the ascending chain condition for R-left and S-right sub-bimodules. As a consequence, we deduce :

Corollary 4. *If $R=\bigoplus_{\sigma\in G} R_\sigma$ is a graded ring of type G(R is abelian finitely generated) which is left and right gr-noetherian, then R satisfies the ascending chain condition for twosided ideals.*

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MUTUALLY NON-ISOMORPHIC UNITARY $\mathbf{F}_p$-ALGEBRAS OF ORDER p^4

BY

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Introduction. In this paper we determine all unitary $\mathbf{F}_p$-algebras (i.e. unitary rings of characteristic p) of order p^4, where p is a prime natural number. Since a finite unitary ring is uniquely expressible as a direct sum of unitary rings of prime power order, it is important to study unitary rings R such that for some prime p and natural numbers m and n, $0<m\leqslant n$, the characteristic c_R of R is p^m and the order $|R|$ of R is p^n. Note that in this case the Jacobson radical of R, $J=J(R)$, has order p^e, $0\leqslant e<n$ and J is the largest nilpotent ideal of R.

§1. **Preliminaries.** There are four non-isomorphic unitary rings of p^2 elements and, depending on whether p is even or odd, there are respectively eleven or twelve mutually non-isomorphic unitary rings of order p^3 (see [1], Exercise XIX. 9). Because in our enumeration of $\mathbf{F}_p$-algebras of order p^4 an important tool is the cardinality of J, we state this result as:

Proposition 1. *Let R be a ring with unity of p^2 elements.*

(i) *If $c_R=p$ and $J=0$, then R is isomorphic to $\mathbf{F}_{p^2}$ or $\mathbf{F}_p\times\mathbf{F}_p$.*

(ii) *If $c_R=p$ and $|I|=p$, then R is isomorphic to $\mathbf{F}_p[X]/(X^2)$.*

(iii) *If $c_R=p^2$, then R is isomorphic to $\mathbf{Z}_{p^2}$.*

Proposition 2. *Let R be a ring with unity of p^3 elements, $p>2$.*

(i) *If $c_R=p$ and $J=0$, then R is isomorphic to $\mathbf{F}_{p^3}$, $\mathbf{F}_p\times\mathbf{F}_{p^2}$, or $\mathbf{F}_p\times\times\mathbf{F}_p\times\mathbf{F}_p$.*

(ii) *If $c_R=p$ and $|J|=p$, then R is isomorphic to $\mathbf{F}_p\times\mathbf{F}_p[X]/(X^2)$ or $\begin{pmatrix}\mathbf{F}_p & \mathbf{F}_p\\ 0 & \mathbf{F}_p\end{pmatrix}$.*

(iii) *If $c_R=p$ and $|J|=p^2$, then R is isomorphic to $\mathbf{F}_p[X, Y]/(Y^2, Y^2, XY)$ or $\mathbf{F}_p[X]/(X^3)$.*

(iv) *If $c_R=p^2$, then $J\neq 0$ and R is isomorphic to $\mathbf{Z}_{p^2}\times\mathbf{F}_p$ when $|J|=p$, to $\mathbf{Z}_{p^2}[X]/(X^2, pX)$ when $|J|=p^2$ and $J^2=0$, to $\mathbf{Z}_{p^2}[X]/(X^3, X^2-p, pX)$ or to $\mathbf{Z}_{p^2}[X]/(X^3, X^2-dX, pX)$ with $d\in\mathbf{F}_p^*\setminus\mathbf{F}_p^{*2}$ when $|J|=p^2$ and $J^2\neq 0$. The rings $\mathbf{Z}_{p^2}[X]/(X^3, X^2-dX, pX)$, $d\in\mathbf{F}_p^*\setminus\mathbf{F}_p^{*2}$ are mutually isomorphic.*

(v) *If $c_R=p^3$, then R is isomorphic to $\mathbf{Z}_{p^3}$.*

If $p=2$, the last type in (iv) does not appear.

§2. **Unitary $\mathbf{F}_p$-algebras of order p^4.** The enumeration of all types of unitary rings of order p^4 seem to be a difficult problem. We restrict our attention to the case of characteristic p; we shall prove that in this case there are twenty mutually non-isomorphic unitary rings. More explicitly, we have :

Theorem. *Let R be an algebra with unity over $\mathbf{F}_p$ of order p^4.*

(i) *If $J=0$, then the algebra R is isomorphic to one of following types:* $M_2(\mathbf{F}_p)$, $\mathbf{F}_{p^4}$, $\mathbf{F}_{p^3}\times\mathbf{F}_p$, $\mathbf{F}_{p^2}\times\mathbf{F}_{p^2}$, $\mathbf{F}_{p^2}\times\mathbf{F}_p\times\mathbf{F}_p$, $\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p$.

(ii) *If $|J|=p$, then R is isomorphic to* $\mathbf{F}_{p^2}\times\mathbf{F}_p[X]/(X^2)$, $\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p[X]/(X^2)$ or $\mathbf{F}_p\times\begin{pmatrix}\mathbf{F}_p & \mathbf{F}_p\\ 0 & \mathbf{F}_p\end{pmatrix}$.

(iii) *If $|J|=p^2$ and $R/J\simeq\mathbf{F}_{p^2}$ then R is isomorphic to $\mathbf{F}_{p^2}[X]/(X^2)$ or $\mathbf{F}_{p^2}[X,\sigma]/(X^2)$ where $\mathbf{F}_{p^2}[X_p,\sigma]$ is the skew-polynomial ring over $\mathbf{F}_{p^2}$ with automorphism $\sigma:\mathbf{F}_{p^2}\to\mathbf{F}_{p^2}$, $\sigma(\alpha)=\alpha^p$,*

(iv) *If R is commutative, $|J|=p^2$ and $R/J\simeq\mathbf{F}_p\times\mathbf{F}_p$, then R is isomorphic to $\mathbf{F}_p\times\mathbf{F}_p[X]/(X^3)$, $\mathbf{F}_p\times\mathbf{F}_p[X,Y]/(X^2,Y^2,XY)$ or $\mathbf{F}_p[X]/(X^2)\times\mathbf{F}_p[\mathbf{X}]/(\mathbf{X}^2)$*

(v) *If R is non-commutative, $|J|=p^2$ and $R/J\simeq\mathbf{F}_p\times\mathbf{F}_p$, then R is isomorphic to*

$$\begin{pmatrix}\mathbf{F}_p & V\\ 0 & \mathbf{F}_p\end{pmatrix},\ \begin{pmatrix}S & I\\ 0 & \mathbf{F}_p\end{pmatrix},\ \begin{pmatrix}\mathbf{F}_p & I\\ 0 & S\end{pmatrix}\ \text{or}\ \begin{pmatrix}\mathbf{F}_p & I\\ I & \mathbf{F}_p\end{pmatrix}$$

where V is a 2-dimensional vector space over $\mathbf{F}_p$, $S=\mathbf{F}_p[X]/(X^2)$ and I is the Jacobson radical of S.

(vi) *If $|J|=p^3$, then R is isomorphic to $\mathbf{F}_p[X]/(X^4)$ or $\mathbf{F}_p[X,Y,Z]/(X^2,Y^2,Z^2,XY,YZ,ZX)$.*

The statement (i) is an immediate consequence of the Wedderburn-Artin theorem. For statements (ii)—(vi), we have an elementary and very long proof. In the sequel we shall present a succinct proof for statements (ii) and (v).

Proof of the statement (ii). By Proposition 1, R/J may be isomorphic to $\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p$, $\mathbf{F}_{p^2}\times\mathbf{F}_p$ or $\mathbf{F}_{p^3}$. Because J is nilpotent, we have $J^2\neq J$; since $|J|=p$ we deduce that $J^2=0$. It follows that J is a module over R/J; if $R/J\simeq\mathbf{F}_{p^3}$, then $|p|=|J|\geq|\mathbf{F}_{p^3}|=p^3$, which is a contradiction. Hence the case $R/J\simeq\mathbf{F}_{p^3}$ is not possible.

Let a be in J, $a\neq0$. Since J is a vector space over $\mathbf{F}_p$ and $|J|=p$, we have $J=\{ta \mid t\in\mathbf{F}_p\}$.

Case I: $R/J\simeq\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p$. Since idempotents modulo J may be lifted, we have $1=e_1+e_2+e_3$, where the e_i are mutually orthogonal idempotents in R with $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ in R/J. The set $\{e_1,e_2,e_3,a\}$ is linearly independent over $\mathbf{F}_p$ and since $|R|=p^4$, we have $R=e_1\mathbf{F}_p\oplus e_2\mathbf{F}_p\oplus e_3\mathbf{F}_p\oplus a\mathbf{F}_p$. Clearly e_ia, $ae_i\in J=\{ta \mid t\in\mathbf{F}_p\}$, $1\leqslant i\leqslant3$; in fact we have e_ia, $ae_i\in\{0,a\}$, $a\leqslant i\leqslant3$. If R is non-commutative, then by rearranging the e_i, if necessary, we have $e_1a=a$, $e_2a=e_3a=0$, $ae_1=0$, $ae_2=ae_3=0$, $a^2=0$.

If we choose in $\mathbf{F}_p\times\begin{pmatrix}\mathbf{F}_p & \mathbf{F}_p\\ 0 & \mathbf{F}_p\end{pmatrix}$ the following elements

$$E_1=\left(0,\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right),\quad E_2=\left(0,\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right),\quad E_3=\left(1,\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right),\quad A=\left(0,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$$

then there exists an algebra isomorphism $\varphi: R\to\mathbf{F}_p\times\begin{pmatrix}\mathbf{F}_p & \mathbf{F}_p\\ 0 & \mathbf{F}_p\end{pmatrix}$ such that $\varphi(e_i)=E_i$, $1\leqslant i\leqslant3$ and $\varphi(a)=A$.

Suppose now that R is commutative. Since $R/J\simeq\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p$, R is not a local ring. But R is artinian and then $R\simeq R_1\times\ldots\times R_n$, where the R_i are commutative local rings, $n>1$ (see [2]). Because $|R|=p^4$, $n>1$, $J=J(R)=J(R_1)\times\ldots\times J(R_n)$ and $|J|=p$, by Propositions 1 and 2 we conclude that $R\simeq\mathbf{F}_p\times\mathbf{F}_p\times\mathbf{F}_p[X]/(X^2)$.

Case II: $R/J\simeq\mathbf{F}_{p^2}\times\mathbf{F}_p$. We have $1=e+f$, where e and f are orthogonal idempotents in R with $e=(1,0)$ and $f=(0,1)$ in R/J. Let $\varphi: R\to\mathbf{F}_{p^2}$ be the composite of canonical maps $R\to R/J$, $R/J\simeq\mathbf{F}_{p^2}\times\mathbf{F}_p$ and $\mathbf{F}_{p^2}\times\mathbf{F}_p\to\mathbf{F}_{p^2}$. Clearly φ is a surjective algebra homomorphism and $|U|=p^2$, where $U=\operatorname{Ker}\varphi$. The set $\{f,a\}$ is linearly independent over $\mathbf{F}_p$; since $a,f\in U$ and $|U|=p^2$ we conclude that $U=f\mathbf{F}_p\oplus a\mathbf{F}_p$. We have $f^2=f$, $fa=af=a$, $a^2=0$. It follows that U is a subring of R with identity f. By Proposition 1, $U\simeq\mathbf{F}_p[X]/(X^2)$.

We can also prove that $R=eRe\oplus U$ and since $eRe\simeq R/U\simeq R/\operatorname{Ker}\varphi\simeq\mathbf{F}_{p^2}$, we conclude that $R\simeq\mathbf{F}_{p^2}\times\mathbf{F}_p[X]/(X^2)$.

Proof of the statement (v). We have $1=e+f$, where e and f are orthogonal idempotents in R with $e=(1,0)$ and $f=(0,1)$ in $R/J\simeq\mathbf{F}_p\times\mathbf{F}_p$. First we prove that $x^2=0$ for any $x\in J$. Suppose that for some $a\in J$ we have $a^2\neq0$. Then $\{a,a^2\}$ is a $\mathbf{F}_p$-basis for J and it is easy to show that ea, $ae\in\{0,a\}$ and hence fa, $af\in\{0,a\}$. Suppose that $ea=a$ and hence $fa=0$. Then $\{e,f,a,a^2\}$ is a $\mathbf{F}_p$-basis for R. Consequently $R=e\mathbf{F}_p\oplus f\mathbf{F}_p\oplus a\mathbf{F}_p\oplus a^2\mathbf{F}_p$. If $ae=a$ and hence $af=0$, then R is commutative, which is a contradiction. Thus we have $ea=a$ and $ae=0$; it follows that $aea=a(ea)=a^2\neq0$ and $aea=(ae)a=0$, which is a contradiction. We conclude that $x^2=0$ for any $x\in J$.

Now we can prove that $J^2=0$. Let $\{a,b\}$ be a $\mathbf{F}_p$-basis of J. We have $ab=ba=0$. Indeed, suppose that $ab\neq0$, $ab=ia+jb$, where $i,j\in\mathbf{F}_p$. Since $0=a^2b=ia^2+jab=jab$, $0=ab^2=iab$ and $ab\neq0$, we deduce $i=j=0$, hence $ab=0$ which is a contradiction. Similarly $ba=0$. It follows $xy=0$ for any $x,y\in J$, hence $J^2=0$. Clearly $J=eJ\oplus fJ=Je\oplus Jf$. Since $\dim_{\mathbf{F}_p}J=2$ it follows that eJ, fJ, Je and Jf are subspaces of J with dimensions 0, 1 or 2.

We examine only the case:

$$(\dim_{\mathbf{F}_p}eJ,\ \dim_{\mathbf{F}_p}fJ)=(2,0)\ \text{and}\ (\dim_{\mathbf{F}_p}Je,\ \dim_{\mathbf{F}_p}Jf)=(0,2).$$

Let $\{a,b\}$ be an $\mathbf{F}_p$-basis for J. We have $a^2=b^2=ab=ba=0$ and $\{e,f,a,b\}$ is a $\mathbf{F}_p$-basis for R. Clearly $ea=a$, $eb=b$, $ae=0$, $be=0$ and hence $fa=0$, $fb=0$, $af=a$ and $bf=b$.

Let V be a 2-dimensional vector space over $\mathbf{F}_p$ and $\{u,v\}$ a $\mathbf{F}_p$-basis for V. Then there exists an algebra isomorphism $\varphi: R\to\begin{pmatrix}\mathbf{F}_p & V\\ 0 & \mathbf{F}_p\end{pmatrix}$ such that

$$\varphi(e)=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \varphi(f)=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad \varphi(a)=\begin{pmatrix}0 & u\\ 0 & 0\end{pmatrix},\quad \varphi(b)=\begin{pmatrix}0 & v\\ 0 & 0\end{pmatrix}\quad\text{and}\ R\simeq\begin{pmatrix}\mathbf{F}_p & V\\ 0 & \mathbf{F}_p\end{pmatrix}.$$

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