Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, 1985 (supliment)

ON MARTINDALE RINGS OF QUOTIENTS

BY

ŞTEFAN GHINEA

The purpose of this paper is to give the behaviour of the Martindale ring of quotients under the direct products of semiprime rings. The ring $Q_0(R)$ was defined for prime rings by M a r t i n d a l e [3] and extended for semiprime rings by A m i t s u r [1]. In [5, 4] this ring appears as a strong technique for the analysis of the liberal or normalizing extensions of rings.

This Note essentially splits into two parts. Section 1 is devoted to prove RC is centrally closed for R a semiprime ring and C the extended centroid of this ring. In Section 2 we obtain the main result about Martindale ring of quotients under the direct product of rings; then we prove that the notion of centrally closed ring is an invariant under the direct products.

1. **Centrally closed rings.** Let R be a semiprime ring with identity, let $\mathcal{F}$ be the collection of all nonzero twosided ideals of R which are essential as left R-modules (i.e. rt ann $_R U=$ lt ann $_R U=(0)$). Then the Martindale ring of quotients $Q_0(R)$ is defined to be the ring of left quotients with respect to $\mathcal{F}$ that is $Q_0(R)=\varinjlim_{A\in\mathcal{F}} \mathrm{Hom}_R({}_RA, {}_RR)$, where $\mathrm{Hom}_R({}_RA, {}_RR)$ denotes the set of left R-module homomorphisms from A to R. A more explicite description of $Q_0(R)$ is given in [2]. Note that we can view R as a subring of $Q_0(R)$ with the same 1.

The center $C=C(R)$ of $Q_0(R)$ is called the *extended centroid* of the ring R. The subring RC of $Q_0(R)$ is called the *central closure of* R and R is *centrally closed* if $R=RC$. One can easily check that R is centrally closed iff $C\subseteq R$. The following lemmas contain some basic properties of $Q_0(R)$ and $C(R)$ (see [1], Theorems 3, 5, Lemma 4; [2], Lemma 1).

Lemma 1.1. *Let R be a semiprime ring, let $Q_0(R)$ be its Martindale ring of quotients and let C be the extended centroid of R. (i) For each $q\in Q_0(R)$, there exists an ideal $A\in\mathcal{F}$ with $Aq\subseteq R$. (ii). If $q\in Q_0(R)$ satisfies $Aq=0$ for an ideal $A\in\mathcal{F}$, then $q=0$. (iii). $Q_0(R)$ is a semiprime ring. (iv) C is a field iff R is a prime ring. (v) Every nonzero (R, R)-subbimodule of $Q_0(R)$ intersects R nontrivially.*

Lemma 1.2. *Let R be a semiprime ring, $Q_0(R)$ and C as above. Then we have: $C=\{q\in Q_0(R)\,|\,q=\hat{f},\ f:{}_RA\to{}_RR,\ A\in\mathcal{F},\ f$ — (R, R)-bimodule homomorphism$\}=\{q\in Q_0(R)\,|\,qr=rq, \forall r\in R\}=C_{Q_0(R)}R$.*

Some examples of rings of quotients in Martindale sense are: 1. If R is a semiprime simple ring with identity, then the filter $\mathcal{F}$ is $\{R\}$ and $Q_0(R)=$

$=R$. Thus $RC=R$ and R is centrally closed, C being the usual center of R. In particular any division ring is centrally closed.

2. If R is a principal commutative domain, then any nonzero ideal of R belongs to $\mathcal{F}$ and $\mathrm{Hom}_R(Ra, R)\simeq R$, hence $R=Q_0(R)$, $RC=R$ and R is centrally closed.

The following result gives a new class of centrally closed rings.

Theorem 1.3. *If R is a semiprime ring then the ring RC is centrally closed, C being the extended centroid of R.*

Proof. If C_1 is the extended centroid of the ring $Q_0(RC)$, then we have to prove that $C_1\subseteq RC$. It is enough to prove that $C_1\subseteq C$, since $C\subseteq R$, as the diagram shows:

$$\begin{tikzcd}
R \arrow[r] \arrow[dd] & & RC \arrow[r] & Q_0(R) \\
& C \arrow[ur] & & \\
RC \arrow[rr] & & C_1RC \arrow[r] & Q_0(RC) \\
& C_1 \arrow[uu, dashed] \arrow[ur] & &
\end{tikzcd}$$

Consider $c\in C$, $c=\hat{f}$, $f\in\mathrm{Hom}_R(I, R)$ and $I\in\mathcal{F}$. By Lemma 1.2, $f\in\mathrm{Hom}_{R-R}(I, R)$ and let us denote $I_1=\{r\in I\cap R\,|\,f(r)\in R\}$. We can show $I_1\in\mathcal{F}$. Again by 1.2, $f\in\mathrm{Hom}_{RC-RC}(I, RC)$. Considering $f_1: I_1\to R$, $f_1(x)=f(x)$, for any $x\in I_1$, we obtain that the map $f\to f_1$ is a homomorphism of rings which is injective.

2. **Martindale ring of quotients for a direct product of semiprime rings.** First we give some properties of the direct product of rings. Let R be the direct product $\prod_{i\in I} R_i$ and J_i a twosided ideal of R_i, $i\in I$, then $\prod_{i\in I} J_i$ is a twosided ideal of R which is essential when J_i are essential. We have also that if J is an (essential) ideal of R, then $p_i(J)$ are twosided (essential) ideals in R_i, $i\in I$.

Theorem 2.1. *If $(R_i)_{i\in I}$ is an arbitrary collection of semiprime rings, then $\prod_{i\in I} Q_0(R_i)\simeq Q_0(\prod_{i\in I} R_i)$ is an isomorphism of rings.*

Proof. We define $\theta:\prod_{i\in I} Q_0(R_i)\to Q_0(\prod_{i\in I} R_i)$, by $\theta((q_i)_{i\in I})=\hat{f}$, where $q_i\in Q_0(R_i)$, $q_i=\hat{f_i}$, $i\in I$ and $f_i\in\mathrm{Hom}_{R_i}(I_i, R_i)$, which is an isomorphism of rings.

Using the properties of the product of rings and Theorem 2.1, we obtain the following corollaries:

Corollary 2.2. *If $(R_i)_{i\in\overline{1,n}}$ is a finite set of semiprime rings, then $Q_0(\bigoplus_{i=1}^{n} R_i)\simeq\bigoplus_{i=1}^{n} Q_0(R_i)$.*

Corollary 2.3. *If $(R_i)_{i\in I}$ is a set of semiprime rings, C is the extended centroid of $\prod_{i\in I} R_i$ and C_i the extended centroid of R_i, $i\in I$, then $\prod_{i\in I} C_i\simeq C$.*

Corollary 2.4. *If $(R_i)_{i\in I}$ is an arbitrary collection of semiprime centrally closed rings, then $\prod_{i\in I} R_i$ is a semiprime centrally closed ring.*

Corollary 2.5. *If $(R_i)_{i\in I}$ is an arbitrary collection of semiprime rings, then the ring $\prod_{i\in I}(R_iC_i)$ is centrally closed, where C_i is the extended centroid of R_i.*

Corollary 2.6. *Let $(R_i)_{i=\overline{1,n}}$ be a finite set of semiprime rings, $R=\bigoplus_{i=1}^{n} R_i$, C the extended centroid of R and C_i the extended centroid of R_i, $i=\overline{1, n}$, then : i). $\bigoplus_{i=1}^{n} C_i\simeq C$; ii). If R_i is centrally closed for each $i=\overline{1, n}$, then R is centrally closed ; iii). $\bigoplus_{i=1}^{n} R_iC_i$ is centrally closed.*

REFERENCES

1. Amitsur, S.A. — *On rings of quotients*. Symposia Math., 8 (1972), 149—164.
2. Lorenz, M. — *Finite normalizing extension of rings*. Math. Zeit., 176 (1981), 447—484.
3. Martindale, W.S. 3 rd — *Prime rings satisfying a generalized polynomial identity*. J. Algebra, 12 (1969), 576—584.
4. Robson, J.C., Heinicke, A.G. — *Normalizing extensions: Prime ideals and incomparability*. J. Algebra, 72 (1981), 237—268.
5. Robson, J.C., Small, L.W. — *Liberal extensions*. Proc. London Math. Soc., 42 (1981), 87—103.

Faculty of Mathematics
University of Bucharest

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, 1985 (supliment)

EXTENSIONS OF LEFT NEAR-RINGS

BY

MIRELA ŞTEFĂNESCU

In the theory of groups, the problem of constructing a group E such that it has a normal subgroup A' isomorphic to a given group A and its factor-goup E/A' is isomorphic to another given group B has known a nice development, since, in 1926, Schreier [12] had written his paper. Schreier's extensions of groups inspired the same constructions for rings (Hall [5], Everett [3], Szendrei [14], Rédei [11]), for universal algebras (Zitomirskii [17]) and monoids (Strecker [13], van Buggenhout [16], Coudron [2], Hancock [6], Inasaridze [7]). If A is an abelian group, then the extensions of A by B can be described in terms of the cohomology theory and this solution, although not exhaustive, is more illuminating. Lausch [8, 9] used the cohomology theory in order to build the extensions of a ring by a distributively generated near-ring.

In this paper, we include a brief outline of the general extension problem for zero-symmetric left near-rings by using Schreier's idea. Recall some definitions and notations in the near-ring theory. For other informations, see Pilz [10].

Definition 1. *A* **left near-ring** *is a triple $(N, +,\cdot)$, where N is a nonvoid set, „+" and „·" are binary operations such that : 1. $(N, +)$ is a group (not necessarily abelian); 2. $(N, \cdot)$ is a semigroup; 3. for all $x, y, z\in N$, $x\cdot(y+z)= =x\cdot y+x\cdot z$.*

We consider only the left near-rings for which $0\cdot x=0$, for all $x\in N$, i.e. the *zero-symmetric left near-rings*. Such a near-ring is, for example, the set of affine transformations of a linear space, with respect to the pointwise addition and the mappings composition.

Definition 2. *A nonvoid subset I of a left near-ring N is called an ideal of N if: $(I,+)$ is a normal subgroup of $(N, +)$, $N. I\subseteq I$ and for all $x, y\in N$ and $i\in I$, $(i+x)\cdot y-x\cdot y\in I$.*

We note that each ideal I is also a subnear-ring.

Definition 3. *Let N and N' be two left near-rings. A mapping $\varphi: N\to N'$ is a near-ring morphism, if $\varphi(x+y)=\varphi(x)+\varphi(y)$, $\varphi(x\,.\,y)=\varphi(x)\,.\,\varphi(y)$, for all $x, y\in N$.*

The kernel of φ, $\operatorname{Ker}\varphi=\{x\in N/\varphi(x)=0'\in N'\}$ is an ideal of N. Given the ideal I of N, we may talk about the factor near-ring N/I. The notion of exact sequence is similar with the same notion for groups.