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ON RINGS FOR WHICH SOME TYPES OF PREPRADICAL IDEALS ARE TRIVIAL

BY

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In this paper, we study the unital rings R having the property that $\sigma(R)=0$ for any preradical σ of a determined type, which verifies the condition $\sigma(R)<I$, where I is a fixed two-sided ideal of R. Here σ may be a left exact preradical, idempotent preradical, left exact radical, cotorsion radical, left exact cotorsion radical. For the definitions and basic properties of preradicals and torsion theories we refer to [3], [4], [11]; a classification of the types of preradicals may be found in [6].

Denote by R-mod the category of left unital R-modules. For a left R-module M, we define an idempotent preradical t_M and a radical K_M by taking

$$t_M(X)=\sum_{\alpha\in \mathrm{Hom}_R(M,X)} \mathrm{Im}\,\alpha, \qquad K_M(X)=\bigcap_{\alpha\in\mathrm{Hom}_R(X,M)} \mathrm{Ker}\,\alpha,$$

for every $X\in R$-mod.

We give now a lemma necessary for the rest of the paper.

1. **Lemma.** *Let τ-and σ be a preradical and, respectively, an epiradical. If $\tau(R)=\sigma(R)$, then $\sigma\leqslant\tau$.*

Proof. Indeed, if M is a left R-module and $x\in M$, then $\tau(R)x\subseteq\tau(M)$, therefore $\tau(R)\,M\subseteq\tau(M)$, and $\sigma(M)=\sigma(R)M=\tau(R)M\subseteq\tau(M)$, hence $\sigma\leqslant\tau$.

We shall use the following notations and conventions: $N\leqslant M(<N M)-N$ is a submodule (proper submodule) of M; $I\leqslant R(I<R)-I$ is a left ideal (proper left ideal) of R; $I\nsubseteq J(I\subsetneq J)-I$ is not (strictly) included in J; all the preradicals and radicals are in the category R-mod.

Let I be a two-sided ideal of the ring R.

2. **Definition.** *The ring R is called I-simple if I is a minimal two-sided ideal of R.*

3. **Theorem.** *For a ring R the following properties are equivalent:*

(1) $\sigma(R)=0$, *for every preradical σ which verifies the condition $\sigma(R)<I$;*

(2) $\sigma(R)=0$, *for every radical σ which verifies the condition $\sigma(R)<I$;*

(3) $\sigma(R)=0$, *for every epiradical which verifies the condition $\sigma(R)<I$;*

(4) *If σ is an epiradical having the property that $\sigma(R)<I$, then $\sigma=0$;*

(5) *R is an I-simple ring;*

(6) *Every left R-module (cyclic) M with $\mathrm{Ann}_R(M)<I$ is faithful;*

(7) *For every nonzero right ideal K of R, included in I, $R\,.\,K=I$.*

Proof. Obviously $(1)\Rightarrow(2)\Rightarrow(3)$ and $(4)\Rightarrow(5)$.
From lemma 1 it results that $(3)\Rightarrow(4)$.
$(5)\Rightarrow(6)$: Since $\mathrm{Ann}_R(M)$ is a two-sided ideal of R, it results that $\mathrm{Ann}_R(M)=0$.
$(6)\Rightarrow(7)$: Assume $R\,.\,K<I$. Then $\mathrm{Ann}_R(R/R\,.\,K)=RK$ and, since $K\subseteq R\,.\,K$, we have $R\,.\,K=K=0$.
$(7)\Rightarrow(1)$: If σ is a preradical with $\sigma(R)\neq0$ and $\sigma(R)\leqslant I$, then $\sigma(R)=R\,.\,\sigma(R)=I$.

4. *Remarks.* (i) R is a simple ring iff R is an R-simple ring.

(ii) If R is an I-simple ring, then $I^2=I$ or $I^2=0$.

(iii) If R is an I_1-simple ring and an I_2-simple ring, with $I_1\neq I_2$, then $I_1\,.\,I_2=I_1\cap I_2=0$.

5. **Theorem.** *For a ring R the following properties are equivalent:*
(1) $\sigma(R)=0$, *for every idempotent radical σ with $\sigma(R)<I$;*
(2) $\mathrm{Hom}_R(J\,;\,R/J)\neq0$, *for every nonzero left ideal J of R, $J<I$;*
(3) $\mathrm{Hom}_R(J\,;\,R/J)\neq0$, *for every nonzero two-sided ideal J of R, $J<I$.*

Proof: $(1)\Rightarrow(2)$: Let $(\mathcal{T}\,;\mathcal{F})$ be the torsion theory cogenerated by R/J and let σ be the idempotent radical coresponding to it. Because $\sigma(R)\in\mathcal{T}$ and $R/J\in\mathcal{F}$, it results that $\mathrm{Hom}_R(\sigma(R);\ R/J)=0$. Taking the homomorphisms $i:\sigma(R)\to R$ and $\pi:R\to R/J$, it results that $\pi\circ i=0$, therefore $\sigma(R)\leqslant J$. Then $\sigma(R)<I$ and thus $\sigma(R)=0$. It follows that $R\in\mathcal{F}$, $J\in\mathcal{F}$ and $\mathrm{Hom}_R(J\,;\,R/J)\neq0$.
$(2)\Rightarrow(3)$: Evidently.
$(3)\Rightarrow(1)$: Let σ an idempotent radical for which $\sigma(R)<I$. Because $\sigma(R)$ is a two-sided ideal of R and $\mathrm{Hom}_R(\sigma(R);\ R/\sigma(R))=0$, it results that $\sigma(R)=0$.

6. **Definition.** *A ring which satisfies one of the equivalent conditions of theorem 5 is called a left I-ring.*

After H. Katayama [8], for every $x\in R$, we denote by $\sigma_x=\wedge\{\sigma\mid\sigma$ is a left exact preradical with $x\in\sigma(R)\}$ and $\rho_x=\wedge\{\rho\mid\rho$ is a left exact radical with $x\in\rho(R)\}$. Then σ_x is a left exact preradical and ρ_x is a left exact radical.

Let $L_x=\{J\leqslant R\mid(\exists)F\subseteq R$, F finite such that $\mathrm{Ann}_R(Fx)\subseteq L\}$ and $G_x=\{I\leqslant R\mid(\forall)J<R$ with $I\leqslant J$, $(\exists)s\in R-J$ such that $J:s\in L_x\}$.

Then L_x is the left linear topology of R corresponding to σ_x and G_x is the left Gabriel topology corresponding to ρ_x([8], Lemma 2, Lemma 3 and Corollary 4).

7. **Theorem.** *The following properties of a ring R are equivalent:* (1) $\sigma(R)=0$ *for every left exact preradical σ with $\sigma(R)<I$;* (2) $(\forall)x\in I$, $x\neq0$, $\sigma_x(R)\nless I$; (3) $(\forall)x\in I$, $x\neq0$, *one of the following conditions holds:*

(*) $y\in I\Leftrightarrow(\exists)F\subseteq R$, *$F$ finite, such that* $\mathrm{Ann}_R(Fx)\subseteq\mathrm{Ann}_R(y)$;

(**) $(\exists)z\in R-I$ *and* $(\exists)$ $F\subseteq R$, *F finite, such that* $\mathrm{Ann}_R(Fx)\subseteq\mathrm{Ann}_R(z)$.

Proof: $(1)\Rightarrow(2)$: $x\in\sigma_x(R)$ so $\sigma_x(R)\neq0$ and then $\sigma_x(R)\nless I$.
$(2)\Rightarrow(3)$: If $\sigma_x(R)\nless I$ then $\sigma_x(R)=I$ or $(\exists)z\in R$ such that $z\in\sigma_x(R)$ and $z\notin I$. The condition (*) is equivalent to the relation $\sigma_x(R)=I$ and the condition (**) is equivalent to the fact that $(\exists)z\in R-I$ such that $z\in\sigma_x(R)$.
$(3)\Rightarrow(1)$: Let σ be a left exact preradical, with $\sigma(R)\neq0$ and $\sigma(R)<I$. Let $x\in\sigma(R)$, $x\neq0$. Because $\sigma_x\leqslant\sigma$ it results that $\sigma(R)\nless I$.

8. **Definition.** *A ring which satisfies one of the equivalent conditions of theorem 7 is called a left I SP-ring.*

9. **Theorem.** *The following properties are equivalent for a ring R: (1) $\rho(R)=0$, for every left exact radical ρ which verifies the condition $\rho(R)<I$; (2) $(\forall)x\in I$, $x\neq 0$, $\rho_x(R)\not< I$; (3) $(\forall)x\in I$, $x\neq 0$ one of the following conditions holds:*

(*) $x\in I\Leftrightarrow((\forall)J<R$ with $\mathrm{Ann}_R(x)\leqslant J$, $(\exists)s\in R-J$ and $(\exists)F\subseteq R$, F finite, such that, $\mathrm{Ann}_R(Fx)\subseteq J:s)$;

(**) $(\exists)z\in R-I$ such $(\forall)J<R$, with $\mathrm{Ann}_R(z)\leqslant J$, $(\exists)s\in R-J$ and $(\exists)F\subseteq\subseteq R$, F finite such that $\mathrm{Ann}_R(Fx)\subseteq J:s$; (4) Every injective left R-module Q for which $K_Q(R)<I$ is faithful.

The proof is analogous with the proof of the theorem 7.

10. **Remark.** $K_Q(R)<R\Leftrightarrow Q\neq 0$, if Q is an injective left R-module.

11. **Definition.** *A ring which verifies one of the equivalent conditions of theorem 9 is called a left ICTF-ring.*

12. **Theorem.** *The following properties are equivalent for a ring R: (1) $\sigma(R)=0$, for every idempotent preradical σ, which verifies the condition $\sigma(R)<I$; (2) Every left R-module M with nonzero dual verifies the condition $t_M(R)\not< I$; (3) Every nonzero torsionless left R-module M verifies the condition $t_M(R)\not< I$; (4) Every nonzero submodule N of a projective left R-module verifies the condition $t_N(R)\not< I$; (5) Every nonzero left ideal J of the ring R, $J<I$, verifies the condition $t_J(R)\not< I$; (6) Every nonzero two-sided ideal J of the ring R, $J<I$, verifies the condition $t_M(R)\not< I$.*

Proof.: (1)$\Rightarrow$(2): Let M be a left R-module with $M^*\neq 0$. Because $t_M(R)\neq 0$ it results that $t_M(R)\not< I$.
(2)$\Rightarrow$(3). If M is a nonzero torsionless left R-module then $M^*\neq 0$ and therefore $t_M(R)\not< I$.
(3)$\Rightarrow$(4): Every submodule of a projective module is torsionless and so $t_N(R)\not< I$.
(4)$\Rightarrow$(5)$\Rightarrow$(6): Clear.
(6)$\Rightarrow$(1): Let σ be an idempotent preradical with $\sigma(R)\neq 0$. Then, $t_{\sigma(R)}[\sigma(R)]\not< I$. Because $t_{\sigma(R)}\leqslant\sigma$, we obtain $\sigma(R)\not< I$.

13. **Remark.** $t_M(R)\not< R\Leftrightarrow M$ is a generator for R-mod.

14. **Definition.** *A ring which satisfies one of the equivalent conditions of theorem 12 is called a left IG-ring.*

15. **Theorem.** *The following properties are equivalent for a ring R: (1) $\sigma(R)=0$ for every cotorsion radical which verifies the condition $\sigma(R)<I$; (2) If σ is a cotorsion radical and if $\sigma(R)<I$ then $\sigma=0$; (3) The ring R has no non-trivial idempotent ideals J with $J<I$.*

Proof. There exists a bijection between the set of the idempotent ideals of the ring R and the set of the cotorsion radicals of the category R-mod and hence (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3).

It results from [1], Proposition 5 and from [9]:

16. **Proposition.**: *The following properties are equivalent for a two-sided ideal I of the ring R: (1) R/I is flat as a right R-module; (2) $I\cap J=I\,.\,J$ for every left ideal J of R; (3) $J=I\,.\,J$ for every left ideal J of R, $J\leqslant I$; (4) $a\in I\,.\,a$, $(\forall)a\in I$; (5) $I\,.\,M\cap N=I\,.\,N$ for every left R-module M and every submodule N of M;*
(6) There exists a left exact cotorsion radical σ such that $\sigma(R)=I$.

17. **Definition.** *A two-sided ideal I of the ring R which satisfies one of the equivalent conditions of proposition 16 is called a left strongly idempotent ideal.*

18. **Theorem.** *The following properties are equivalent for a ring R: (1) $\sigma(R)=0$ for every left exact cotorsion radical σ which verifies the condition $\sigma(R)<I$; (2) If σ is a left exact cotorsion radical with $\sigma(R)<I$, then $\sigma=0$; (3) There are no non-trivial two-sided (idempotent) ideals J, $J<I$ such that R/J are flat as left R-modules; (4) There are no non-trivial left strongly idempotent ideals J, $J<I$.*

19. **Definition.** *A ring which satisfies one of the equivalent conditions of theorem 18 is called a left IE2-ring.*

If $I=R$ then, theorem 3, 5, 7, 9, 12, 18 we obtain respectively Proposition 2.1 from [6], Proposition 1.10 from [2], Theorem 2.1 from [12] (Theorem 1.7 from [10], [6] p. 2), Theorem 2.4 from [2] ([6] p. 91), Theorem 3.1, Proposition 5.1 and Theorem 5.4 from [6].

Examples of rings in the sense of the above definitions may be found in [2], [5], [6], [7], [8], [10], [12].

REFERENCES

1. Azumaya G.—*Some properties of TTF-classes*, Proc. of the Conf. on orders, group rings and related topics, Springer-Verlag, 1973.
2. Bican L., Jambor P., Kepka T. and Nemec P.—*On rings with trivial torsion parts*, Bull. Austral. Math. Soc. 9 (1973), 275—290.
3. Dickson S. E.,—*A torsion theory for abelian categories*, Trans. Amer. Math. Soc. 121 (1966), 223—235.
4. Goel V. K.—*A study in cotorsion theory*, Ring theory, Proc. of the Ohio Univ. Conf., 1976.
5. Goodearl K. R., Handelman D. and Lawrence J.—*Strongly prime and completely torsionfree rings*, Carleton Mathematical, Series 109, Carleton University (1974).
6. Katayama H.—*A study of rings with trivial preradical ideals*, Tsukuba J. Math., 5 (1981), 67—83.
7. Katayama H.,—*On rings for which various types of nonzero preradicals are cofaithful*, Hokkaido Math. J., 12 (1983), 49—63.
8. Katayama H.—*On completely torsionfree modules*, Math. J. Okayama Univ., (to appear).
9. Ramamurthi V. S.—*On splitting cotorsion radical*, Proc. Amer. Math. Soc. 39 (1973), 457—461.
10. Rubin R. A.,—*Absolutely torsion-free rings*, Pacific J. Math., 46 (1973), 503—514.
11. Stenström B.—*Rings of quotients*, Springer Verlag, Berlin Heidelberg, New York (1975).
12. Viola-Prioli J.—*On absolutely torsion-free rings*, Pacific J. Math., 56 (1975), 275—283.

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