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ON RELATIVELY PSEUDOCOMPLEMENTED POSETS AND HILBERT ALGEBRAS

BY

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An important and much developed chapter of lattice theory deals with residuated lattices, pseudocomplemented lattices and relatively pseudocomplemented lattices; see e.g. Birkhoff [4] ch. IX §12, ch. XI §5, ch. XIII §§1–3, Balbes and Dwinger [2] ch. IX, Rasiowa [13] ch. IV and Grätzer [7] ch. II §6 ff. As a matter of fact the concept of pseudocomplement and the more general concept of relative pseudocomplement apply exactly as they stand to meet semilattices and the theory has been extended to relatively pseudocomplemented semilattices and even to pseudocomplemented semilattices; see e.g. Frink [6], Nemitz [12], E. T. Schmidt [15], Katriňák [9], [11], Varlet [18], Venkatanarasimhan [20], J. Schmidt and Tsinakis [17], J. Schmidt [16] and the litterature quoted in these papers.

It is a natural idea to extend further the basic concepts of the theory of (relative) pseudocomplementation from semilattices to posets. The pioneering work in this line was done by de Barros [3], Katriňák [10], Venkatanarasimhan [19] and J. Schmidt and Tsinakis [17], who dealt mainly with the Glivenko-Frink theorem, the connection with ideal theory and the connection with join-extensions and meet-retractions.

Our long range aim in this and a series of subsequent papers is to recapture as much as possible of the theory of relative pseudocomplementation for semilattices within the framework of Hilbert algebras with least element. In view of the theorem proved in this paper, the expected theory will apply in particular to relatively pseudocomplemented posets with least element.

To be specific, we recall first that Hilbert algebras were introduced by Henkin [8] in dual form, as an algebraic counterpart of Hilbert's positive implicative propositional calculus (cf. Rasiowa [13] ch. II) and they were intensively studied by Diego [5]. A Hilbert algebra is an algebra $<A,\rightarrow,1>$ of type (2,0) satisfying certain axioms which will be reproduced below. Every Hilbert algebra is made into a poset $<A,\leqslant,1>$ with greatest element 1 under the ordering $x\leqslant y\Leftrightarrow x\rightarrow y=1$. Conversely, every poset $<P,\leqslant,1>$ with greatest element 1 becomes a Hilbert algebra $<P,\overset{H}{\rightarrow},1>$, where $x\overset{H}{\rightarrow}y$ is defined to be 1 if $x\leqslant y$ and y otherwise. Moreover, the order structure induced in the above way by the Hilbert algebra structure $<P,\overset{H}{\rightarrow},1>$

coincides with the original poset $<P,\leqslant,1>$; let us refer to this property as the *consistency* of the Hilbert algebra with the original poset.

In this paper we show that for every relatively pseudocomplemented poset $<P,\leqslant>$ there is a second Hilbert algebra structure consistent with the original ordering, namely $<P,\rightarrow,1>$, where $\rightarrow$ is the operation of taking relative pseudocomplements and 1 is the greatest element of P. Thus the class of relatively pseudocomplemented posets is a proper subclass of the class of Hilbert algebras; moreover, it is also distinct from the subclass of Abbott and Kleindorfer's implicative algebras; see e.g. Abbott [1], and Rasiowa [13] ch. II.

In a series of subsequent papers we shall introduce and study a new „almost Boolean" algebraic structure for Hilbert algebras with least element, as an attempt to recapture much of the theory of relative pseudocomplementation for lattices and semilattices; cf. [14]. As Hilbert algebras with least element essentially coincide with bounded posets, our theory will also refer to the latter and in view of the theorem proved in this paper it will apply even better to relatively pseudocomplemented posets with least element (to the effect that the operation of relative pseudocomplementation $\rightarrow$ seems to be more natural than $\overset{H}{\rightarrow}$).

Let $<P,\leqslant>$ be a poset (partially ordered set) and let $a,b\in P$. The *relative pseudocomplement of a with respect to* b, denoted by $a\rightarrow b$, is an element of P such that for every $x\in Q$ the following equivalence holds:

(1) $$(\forall z(z\leqslant a\,\&\,z\leqslant x\Rightarrow z\leqslant b))\Leftrightarrow x\leqslant a\rightarrow b.$$

The above definition is due to Katriňák [10]; in the case of a meet semilattice it reduces to the usual definition $a\wedge x\leqslant b\Leftrightarrow x\leqslant a\rightarrow b$. As in the case of semilattices, $a\rightarrow b$ is uniquely determined by a and b (because it is the greatest element of the set of those x which fulfil $\forall z(z\leqslant a\,\&\,z\leqslant x\Rightarrow z\leqslant b)$). If $a\rightarrow b$ exists for every $a,b\in P$ we say that P is a *relatively pseudocomplemented poset*. Again as for semilattices, one sees that every relatively pseudocomplemented poset P has greatest element $1=a\rightarrow a$ for any $a\in P$ (because the property $\forall z(z\leqslant a\,\&\,z\leqslant x\Rightarrow z\leqslant a)$ is true for every x, hence $x\leqslant a\rightarrow a$). As noted by J. Schmidt and Tsinakis [17] and following an idea of Venkatanarasimhan [19], the definition (1) can be given the following shorter formulation:

(1') $$(a]\cap(x]\subseteq(b]\Leftrightarrow(x]\subseteq(a\rightarrow b],$$

where $(c]$ stands for the *principal ideal generated by* c, i.e. $(c]=\{z\in P\mid z\leqslant c\}$.

Recall also [5] that a *Hilbert algebra* is an algebra $<A,\rightarrow,1>$ of type (2,0) satisfying the following axioms: for every $a,b,c,\in A$,

(2) $$a\rightarrow 1=1,$$

(3) $$a\rightarrow(b\rightarrow a)=1,$$

(4) $$(a\rightarrow(b\rightarrow c))\rightarrow((a\rightarrow b)\rightarrow(a\rightarrow c))=1,$$

(5) $$a\rightarrow b=b\rightarrow a=1\Rightarrow a=b.$$

The Hilbert algebra structure induces a partial order $\leqslant$ on A, defined by

(6) $$a \leqslant b \Leftrightarrow a \to b = 1$$

and 1 is the greatest element of the induced poset $<A, \leqslant>$. Conversely, every poset with greatest element $<P, \leqslant, 1>$ can be made into a Hilbert algebra $<P, \overset{H}{\to}, 1>$, where the operation $\overset{H}{\to}$ is defined by

$$a \overset{H}{\to} b = \begin{cases} 1, & \text{if } a \leqslant b, \\ b, & \text{otherwise.} \end{cases}$$

We are now in a position to establish the main result :

Theorem. *Let $<P, \leqslant>$ be a relatively pseudocomplemented poset. Denote by $\to$ the operation of taking relative pseudocomplements and by 1 the greatest element of P. Then $<P, \to, 1>$ is a Hilbert algebra and its induced order* (6) *coincides with the original order* $\leqslant$.

Proof. We have to check (2)—(6) using (1).

To prove (6) notice that if $a \leqslant b$ then $\forall x \forall z (z \leqslant a \,\&\, z \leqslant x \Rightarrow z \leqslant b)$, hence $\forall x\, x \leqslant a \to b$ by (1), therefore $a \to b = 1$. Conversely, if $a \to b = 1$, then $a \leqslant a \to b$, hence by applying (1) with $x = a$ we obtain $\forall z (z \leqslant a \Rightarrow z \leqslant b)$; taking further $z = a$, we get $a \leqslant b$.

Now we are going to use (6) in the remainder of the proof. Thus (2) and (5) are obtained as mere translations of $a \leqslant 1$ and antisymmetry, respectively. Further notice that $\forall z (z \leqslant b \,\&\, z \leqslant a \Rightarrow z \leqslant a)$ hence $a \leqslant b \to a$ by (1), thus proving (3).

To prove (4) note first that

(7) $$\forall z (z \leqslant a \,\&\, z \leqslant a \to b \Rightarrow z \leqslant b)$$

by (1) with $x = a \to b$. On the other hand, (4) is translated as

(8) $$a \to (b \to c) \leqslant (a \to b) \to (a \to c)$$

by (6), while (8) is equivalent to

(9) $$\forall z (z \leqslant a \to b \,\&\, z \leqslant a \to (b \to c) \Rightarrow z \leqslant a \to c)$$

by (1). To prove (9) take z such that $z \leqslant a \to b \,\&\, z \leqslant a \to (b \to c)$. Take further t such that $t \leqslant a \,\&\, t \leqslant z$. Then $t \leqslant a \to b$ and $t \leqslant a \to (b \to c)$, hence $t \leqslant b$ and $t \leqslant b \to c$ by a double application of (7), therefore $t \leqslant c$ again by (7). We have thus proved that $\forall t (t \leqslant a \,\&\, t \leqslant z \Rightarrow t \leqslant c)$, hence $z \leqslant a \to c$ by (1). So (9) is established, completing the whole proof.

Examples. The poset in Fig. 1 is the simplest relatively pseudocomplemented poset that is not a meet semilattice. The operation $\to$ coincides with $\overset{H}{\to}$ and is given in Table 1. It is easy to check that $(x \to y) \to y = (y \to x) \to x$ for every x, y, therefore this Hilbert algebra is also an implication algebra by [1] Theorem 26.

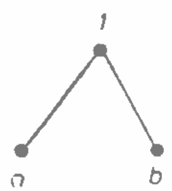

Fig. 1

$\to$	a	b	1
a	1	b	1
b	a	1	1
1	a	b	1

Table 1

The relatively pseudocomplemented poset in Fig. 2 is neither a meet-nor a join-semilattice. The operations $\to$ and $\overset{H}{\to}$ are given in Tables 2 and 3, respectively ; we see that $x \to y = x \overset{H}{\to} y = 1$ whenever $x \leqslant y$, as expected. Since $(a \to c) \to c = c$ while $(c \to a) \to a = 1$, this Hilbert algebra is not an implication algebra.

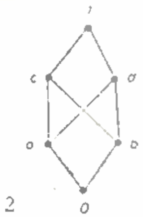

Fig. 2

$\to$	0	a	b	c	d	1
	0	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	d	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Table 2

H	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	1	1	1
b	0	a	1	1	1	1
c	0	a	b	1	d	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Table 3

Open problem (A. Monteiro). Decide whether or not relatively pseudocomplemented posets are equationally definable. (It is known that Hilbert algebras are so [5]. See also [14] for new computational possibilities).

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