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THE LATTICE OF DEDUCTIVE SYSTEMS OF A HILBERT ALGEBRA

BY

DUMITRU BUSNEAG

For a Hilbert algebra A [3], we denote by $Ds(A)$ the lattice of all deductive systems of A and by $Irc(A)$ the set of completely irreducible deductive systems of A.

If A is a bounded Hilbert algebra and $x \in A$, we denote by $x^* = x \to 0$.

Lemma 1 [2]. *If A is a bounded Hilbert algebra, then the following assertions are equivalent*: (i) *A is a Boolean algebra (relatively to natural ordering)*; (ii) *For every $x \in A$, $x^{**} = x$.*

Corollary 2 [2]. *If A is a bounded Hilbert algebra, then A is a Boolean algebra iff for any elements $x, y \in A$, $(x \to y) \to x = x$.*

For a Hilbert algebra A and $x_1, x_2, \ldots, x_n \in A$, we define

$$(x_1, x_2, \ldots, x_{n-1}; x_n) = \begin{cases} x_n & , \text{ if } n = 1 \\ x_1 \to (x_2, \ldots, x_{n-1}; x_n), & \text{ if } n \geq 2. \end{cases}$$

Remark 3. [2]. (c_1) If σ is a permutation of the set $\{1, 2, \ldots, n-1\}$, $n \geq 2$, then $(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}; x_n) = (x_1, x_2, \ldots, x_{n-1}; x_n)$;

(c_2) If $x \in A$, then $x \to (x_1, \ldots, x_{n-1}; x_n) = (x, x_1, \ldots, x_{n-1}; x_n) = \ldots = (x_1, \ldots, x_{n-1}, x; x_n)$;

(c_3) If $x, y \in A$, then $(x_1, x_2, \ldots, x_n; x \to y) = (x_1, \ldots, x_n; x) \to (x_1, \ldots, x_n; y)$;

(c_4) If A is a Heyting algebra, then $(x_1, \ldots, x_{n-1}; x_n) = (x_1 \wedge x_2 \wedge \ldots \wedge x_{n-1}) \to x_n$.

If $X \subseteq A$, we denote by $(X) = \cap \{D \in Ds(A) : X \subseteq D\}$ ((X) is called the *deductive system generated by X*). If $X = \{x_1, x_2, \ldots, x_n\}$, we denote by $(x_1, x_2, \ldots, x_n) = (\{x_1, x_2, \ldots, x_n\})$; the deductive system generated by one element $a \in A$ will be denoted by $[a)$ and is easy to verify that $[a) = \{x \in A : a \leq x\}$.

For $D_1, D_2 \in Ds(A)$, we put $D_1 \wedge D_2 = D_1 \cap D_2$, $D_1 \vee D_2 = (D_1 \cup D_2)$, $D_1 \to D_2 = \{a \in A : [a) \cap D_1 \subseteq D_2\}$.

Then $(Ds(A), \wedge, \vee, \to)$ is a Heyting algebra [2], where 0 is $\{1\}$ and 1 is A.

Lemma 4 [2]. *If A is a Hilbert algebra and $D_1, D_2 \in Ds(A)$, then $D_1 \vee D_2 = \{x \in A$: there exist $x_1, \ldots, x_n \in D_1$ such that $(x_1, \ldots, x_n; x) \in D_2\}$.*

Corollary 5. *If A is a Hilbert algebra, $D \in Ds(A)$, $a, x_1, \ldots, x_n \in A$, then*:

(c_5) $[a) \vee D) = \{x \in A : a \to x \in D\}$; (c_6) $(x_1, \ldots, x_n) = \{x \in A : (x_1, \ldots, x_n; x) = 1\}$.

Remark 6. If $a \in A$ and $D \in Ds(A)$, then $D^* = D \to \{1\} = \{x \in A : [x) \cap D = 1\}$ and so $[a)^* = \{x \in A : [x) \cap [a) = \{1\}\}$. If A is a Heyting algebra, then $D^* = \{x \in A : x \vee y = 1$, for every $y \in D\}$.

Theorem 6. *If A is a bounded Hilbert algebra, then the following assertions are equivalent*:

(i) $(Ds(A), \wedge, \vee, {}^*)$ *is a Boolean algebra*; (ii) *A is a finite Boolean algebra (relatively to the natural ordering).*

Proof. (i)$\Rightarrow$(ii). Let $x \in A$; since $Ds(A)$ is Boolean algebra, then $[x)^* \vee [x) = A$. By remark 6, $[x)^* \vee [x) = \{y \in A : x \to y \in [x)^*\} = \{y \in A : [x \to y) \cap [x) = \{1\}\}$, so for every $y \in A$, $[x \to y) \cap [x) = \{1\}$. Since $x \to y \leq ((x \to y) \to x) \to x$ and $x \leq ((x \to y) \to x) \to x$ it follows that $[((x \to y) \to x) \to x) \subseteq [x \to y) \cap [x) = \{1\}$ and so $((x \to y) \to x) \to x) = 1$.

Therefore $((x \to y) \to x) \to x = 1$.

Since $x \leq (x \to y) \to x$, by (1) we deduce that $(x \to y) \to x = y$; since x and y are arbitrary, by corollary 2, it follows that A is a Boolean algebra.

Let us prove now that every filter of A is principal. If $D \in Ds(A)$, then $D \vee D^* = A$ (because $Ds(A)$ is supposed a Boolean algebra) and thus $0 \in D \vee D^*$, and so, by lemma 4 above, there exist $x_1, x_2, \ldots, x_n \in D$ such that $(x_1, \ldots, x_n; 0) \in D^*$ and so, by remark 6, for every $y \in D$, $(x_1, \ldots, x_n; 0) \vee y = 1$.

By (c_4) we obtain succesively $((x_1 \wedge x_2 \wedge \ldots \wedge x_n) \to 0) \vee y = 1 \Leftrightarrow (x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee y = 1 \Leftrightarrow (x_1 \wedge x_2 \wedge \ldots \wedge x_n) \wedge y^* = 0 \Leftrightarrow x_1 \wedge x_2 \wedge \ldots \wedge x_n \leq y$ and so $D = [a)$, where $a = x_1 \wedge x_2 \wedge \ldots \wedge x_n \in D$.

Because every filter of A is principal, then A is finite [4].

(ii)$\Rightarrow$(i). Suppose that A is a finite Boolean algebra; because $Ds(A)$ is a Heyting algebra, to prove $Ds(A)$ is a Boolean algebra, we must show [1] $D^* = \{1\}$ only for $D = A$. Since A is a finite Boolean algebra, every filter of A is principal and so $D = [a)$, $a \in A$. By remark 6, $D^* = [a)^* = \{x \in A : x \vee y = 1$ for every $a \leq y\}$. Since, for $y \geq a$, $a^* \vee y \geq a^* \vee a = 1$ it results that $a^* \vee y = 1$ and so $a^* \in [a)^* = \{1\}$, therefore $a^* = 1$. It results $a = 0$ and so $D = [a) = [0) = A$.

Lemma 7 [2]. *If A is a bounded Hilbert algebra, then for every $D \in Ds(A)$*

$$D = \cap \{M \in Irc(A) : D \subseteq M\}.$$

In the same manner, we prove:

Theorem 8. *If A is a bounded Hilbert algebra, then the following assertions are equivalent*: (i) *Every $D \in Ds(A)$ has a unique representation as an intersection of elements of $Irc(A)$.* (ii) *A is a finite Boolean (relatively to natural ordering).*

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Department of Mathematics
University of Craiova