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# RELATIONS BETWEEN THE HOMOMORPHISMS OF SEMIGROUPS AND HOMOMORPHISMS OF THEIR $n$-ARY EXTENSIONS

BY

I. COROVEI and I. PURDEA

Let $(S,.)$ be a semigroup, $a \in S$, $\alpha : S \to S$ an endomorphism and $\varphi : S^n \to S$ a $n$-*ary* operation defined by $\varphi(x_1,\dots, x_n) = x_1\alpha(x_2)\alpha^2(x_3)\dots\alpha^{n-1}(x_n)a$. $(S, \varphi)$ is called the $n$-ary extension of $(S,.,\alpha, a)$, it is denoted by $\mathrm{Ext}^n_{\alpha,a}\, S = (S, \varphi)$ and $\mathrm{Ext}^n_a\, S = \mathrm{Ext}^n_{1_S,a}\, S$.

M. S. Pop [2] has proved that $\mathrm{Ext}^n_{\alpha,a}S$ is an $n$-semigroup if

$$(*) \qquad \forall x \in S, \quad \alpha^n(x)\alpha(a) = a\alpha(x).$$

We have proved that if $\mathrm{Ext}^n_{\alpha,a}S$ is an $n$-semigroup then (*) is true.

In the sequel we suppose that the semigroups $(S,.)$ and $(S',.)$ are with the identities denoted by 1, respectively $e$. Let $\alpha \in \mathrm{Aut}\,(S,.)$, $\beta \in \mathrm{Aut}\,(S',.)$, $a \in S$ and $b \in S'$, verifying $(*)$ and $b$ is a regular element.

**Theorem 1.** *A mapping $f : S \to S'$ with $f(1)$ a regular element is a homomorphism of the n-semigroup $\mathrm{Ext}^n_{\alpha,a}S$ into $\mathrm{Ext}^n_{\beta,b}S'$ iff it exists a homomorphism of semigroups $g : S \to S'$ such that $f(x) = f(1)g(x)$ and $g(\alpha(x))\beta(f(1)) = \beta(f(1))\beta(g(x))$, for any $x \in G$, and $g(a) = \beta(f(1))\beta^2(f(1))\dots\beta^{n-1}(f(1))b$.*

**Corollary.** *A mapping $f : S \to S'$ with $f(1)$ a regular element is a homomorphism of n-semigroups from $\mathrm{Ext}^n_a\, S$ into $\mathrm{Ext}^n_b S'$ iff it exists a homomorphism of semigroups $g : S \to S'$ such that $\forall x \in S$, $f(x) = f(1)g(x)$ and $f(1)$ belongs to the centralizer of $f(S)$, and $g(a) = [f(1)]^{n-1}b$.*

Let $(S,.)$ *and* $(S,*)$ be two semigroups with the same support set and with identities denoted by 1 respectively e and 1 is a regular element in $(S, *)$.

**Theorem 2.** *If $a, b \in S$, $\alpha \in$ Aut $(S,.)$, $\beta \in$ Aut $(S, *)$, and $(a, \alpha,.)$, $(b, \beta, *)$ verify (*) respectively, and $b$ is a regular element in $(S, *)$, then $\mathrm{Ext}^n_{\alpha,a}(S,.) = \mathrm{Ext}^n_{\beta,b}(S, *)$ iff: for $\forall x_1, x_2 \in S$, $e\,.\,x_1\,.\,x_2 = (e\,.\,x_1) * (e\,.\,x_2)$ ; $\forall x \in S$, $\alpha(x) * \beta(1) = 1 * \beta(x)$* and $a = 1 * \beta(1) * \beta^2(1) * \dots * \beta^{n-1}(1) * b$.

**Corollary.** *We have, for $\forall x_1,\dots, x_n \in S$, $x_1\,.\,x_2\dots x_n = x_1 * x_2 * \dots * x_n$* i.e. $\mathrm{Ext}^n_1(S,.) = \mathrm{Ext}^n_e(S, *)$ *iff for $\forall x_1, x_2 \in S$, $1\,.\,x_1\,.\,x_2 = (e\,.\,x_1) * (e\,.\,x_2)$, and 1 belongs to the center of $(S, *)$, and the order of 1 in $(S, *)$ is a divisor of $n-1$.*

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# HOMOMORPHIC RELATIONS ON $n$-GROUPS

BY

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1. Let $(A, \Omega)$, $(B, \Omega')$ be two universal algebras of the same type and $\theta$, the similarity-map. The relation $\rho \subseteq A \times B$ is called $(\omega, \theta(\omega))$-*homomorphic* if $x_i \rho y_i$, $i = 1,\dots,n \Rightarrow \omega(x_1,\dots,x_n)\rho(\theta(\omega)(y_1,\dots,y_n))$. If $\rho$ is $(\omega, \theta(\omega))$-homomorphic for all $\omega \in \Omega_1 \subseteq \Omega$ then $\rho$ is called $(\Omega_1, \theta(\Omega_1))$-*homomorphic*. In the case $(A, \Omega) = (B, \Omega')$, $\theta = 1_A$, $\rho$ is called $\Omega_1$-*compatible*.

M. Hosszú proved the following *Theorem*:

*$(G, \varphi)$ is a n-group iff there exist a constant a in G, a group operation „$\cdot$" on G and an automorphism $\alpha$ of $(G, \cdot)$ so that: $\varphi(x_1,\dots, x_n) = x_1 \cdot \alpha(x_2), \dots, \alpha^{n-1}(x_n) \cdot a$, $\alpha(a) = a$* and $\alpha^{n-1}(x) = a \cdot x \cdot a^{-1}\ \forall x \in G$.

The structure $(G, \cdot, \alpha, a)$ is called a *reduce* of $(G, \varphi)$ and $(G, \varphi)$ is the *n-ary extension* of $(G, \cdot\, \alpha, a)$. For each $x \in G$ there is an element $\bar{x} \in G$ which is the solution of the equation $\varphi(x, x,\dots, x, \bar{x}) = x$.

**Proposition 1.** *Let $(G, \varphi)$ and $(G', \psi)$ be two n-groups. Each $((\varphi, -), (\psi, -))$—homomorphic relation is a difunctional.*

2. We study the relationship between the homomorphic relations on $n$-groups and these on their reduces.

**Proposition 2.** (a). *If $\rho$ is $((.\,, \alpha, a), (*, \beta, b))$-homomorphic then $\rho$ is $(\varphi, \psi)$-homomorphic too.*

(b). *If $\rho$ is $((\varphi, 1, a^{-1}), (\psi, 1', b^{-1}))$- homomorphic then $\rho$ is $((.\,, \alpha, a), (*, \beta, b))$-homomorphic.*

3. Consider now the structures $(G, \varphi, -)$, $(G', \psi, -)$.

**Proposition 3.** (a). *If $\rho$ is $((.\,,\alpha, a^{-1}), (*, \beta, b^{-1}))$-homomorphic then $\rho$ is $((\varphi, -)\ (\psi, -))$-homomorphic too.*

(b). *If $\rho$ is $((\varphi, -1), (\psi, -, 1'))$-homomorphic then is $((.\,, \alpha, a^{-1}), (*, \beta, b^{-1}))$-homomorphic too.*

**Corollaries. 1.** *If $(G, \varphi) = (G', \psi)$ then a reflexive relation is $(.\,, \alpha, a^{-1})$-compatible iff is $(\varphi, -))$compatible.*

2. $C(G, \varphi) = C(G, \varphi, -) = C(G, .\,, \alpha, a^{-1}) = C(G, .\,, \alpha)$, *where* $C(G, \varphi)$ *denotes the congruences on* $(G, \varphi)$.

3. *The lattice $C(G, \varphi, -)$ is isomorphic to the lattice of the $\alpha$-invariant normal-subgroups of $(G, .)$.*

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