REFERENCES

1. M. Artin — *On isolated rational singularities of surfaces*, Amer. J. Math. 88 (1966), 129—136.
2. L. Bădescu — *Anticanonical models of ruled surfaces*, Annali Univ. Ferrara 29 (1983), 165—177.
3. L. Bădescu — *Hyperplane sections and deformations*, Springer Lecture Notes Math. Vol. 1056 (1984), 1—33.
4. L. Bădescu — *Suprafețe algebrice*, Editura Academiei R. S. R., București, 1981
5. E. Bombieri, D. Husemoller — *Classification and embeddings of surfaces*, Proc. Symp. Pure Math. 29 (1975), 329—420.
6. L. Brenton — *Some algebraicity criteria for singular surfaces*, Inv. Math. 41 (1977), 129—147.
7. J. W. Bruce, C. T. C. Wall — *On the classification of cubic surfaces*, J. London Math. Soc. 19 (1979).
8. S. Iitaka — *Deformations of compact complex surfaces*. II, J. Math. Soc. Japan 22 (1970) 247—261.
9. H. Pinkham — *Deformations of algebraic varieties with G_m — action*, Astérisque 20, Société Math. France (1974).
10. F. Sakai — *Anticanonical models of rational surfaces*, Preprint, 1982.
11. F. Sakai — *Enriques' classification of normal Gorenstein surfaces*, Amer. J. Math. 104 (1981) 1233—1241.
12. P. M. H. Wilson — *The arithmetic plurigenera of surfaces*, Math. Proc. Cambr. Phil. Soc. 85 (1979) 25—31.
13. O. Zariski — *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. Math. 76 (1962) 560—615.
14. O. Păsărescu — *Asupra clasificării singularităților suprafețelor normale Gorenstein cu divizorul anticanonic amplu*, Studii Cerc. Matem. 36 (1984) 227—243.

Institute of Mathematics
University of Bucharest

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXXI, s. I a, 1985 (supliment)

ARTIN APPROXIMATION THEORY AND APPLICATIONS

BY

DORIN POPESCU

§1. **Artin approximation properties.** Let A be a noetherian ring, $\underline{a}\subset A$ an ideal and $\hat{A}$ the completion of A in the $\underline{a}$-adic topology. We consider the following statements:

(AP) Every finite system of polynomial equations over A has its set of solutions in A dense with respect to the $\underline{a}$-adic topology in the set of its solutions in $\hat{A}$, i.e. for every solution $\hat{Y}$ from $\hat{A}$ and every positive integer c there exists a solution Y_c in A such that $Y_c\equiv\hat{Y}$ mod $\underline{a}^c\hat{A}$.

(SAP) For every finite system of polynomial equations f over A there exists a function $\nu: \mathbf{N}\to\mathbf{N}$ with the following property:

For every positive integer c and for every system of elements $\hat{Y}$ from A which form a solution of f modulo $\underline{a}^{\nu(c)}$, there exists a solution Y_c of f in A such that $Y_c\equiv\hat{Y}$ mod $\underline{a}^c$.

We call $(A, \underline{a})$ an *AP-couple* (resp. *SAP-couple*) if it satisfies (AP) (resp. (SAP)). A local ring $(A, \underline{m})$ is called a *ring with the property of Artin approximation* or an *AP-ring* if the couple $(A, \underline{m})$ is an AP-couple.

The study of these properties started with the following result due to M. Artin [A_1], [A_2]:

(1.0) **Theorem** (Artin). *The convergent power series rings in $X=(X_1, \ldots, X_n)$ over a valued field of characteristic zero, and the henselization of a local ring which is essentially of finite type over a field or an excellent Dedekind ring are AP-rings.*

A couple $(A, \underline{a})$ is called *quasi-excellent* if A is noetherian and the completion map $A\to\hat{A}$ is regular, $\hat{A}$ being the completion of A in the $\underline{a}$-adic topology. A recent result (see [P_2] Theorem (1.3)) says that

(1.1) **Theorem.** *Every quasi-excellent henselian couple is an AP-couple. In particular every excellent henselian local ring is an AP-ring.*

The above theorem gives a positive answer to Conjecture (1.3) from [A_3]. The converse of Theorem (1.1) is not completely solved. If $(A, \underline{m})$ is an AP-ring then A is henselian and the formal fibers of the canonical map $A\to\hat{A}$ are geometrically normal (see [Br] or [CP]). In particular a two dimensional local ring A is AP iff it is excellent henselian (an AP-ring is catenarian by [K], Ch. V). Recently C. Rotthaus [R] showed that an AP-ring A is excellent if $A[[X]]$ is also an AP-ring. If A is an AP-ring then the henselization $A<X>$ of $A[X]$ at (X) is still an AP-ring (see [P_2] Corollary (3.5)) but this does not help.

(1.2) **Theorem** (Pfister-Popescu). *A noetherian complete local ring (A, m) has (SAP). Moreover a noetherian local ring (A, m) is an AP-ring iff (A, m) has (SAP).*

This is a particular form of [PP] Theorem (2.5); its proof was given in an improved form in [K] Ch. II Theorem (1.4). Other proofs involving ultrapowers are given in [DL] and [P_1]. It remains open the general question if an AP-couple has (SAP) or equivalently if a noetherian complete couple has (SAP).

The next theorem solves completely the question of so called approximation on nested subrings.

(1.3) **Theorem.** *Let (A, m) be an AP-ring, $\hat{A}$ its completion, c a positive integer, $A<X>$ the algebraic power series ring in $X=(X_1, \dots, X_r)$ over A, f a finite system of polynomial equations over $A<X>$ and $\hat{y}=(\hat{y}_1, \dots, \hat{y}_n) \in \hat{A}[[X]]^n$ a formal solution of f such that $\hat{y}_i \in \hat{A}[[X_1, \dots, X_{s_i}]]$, $1 \leq i \leq n$ for some positive integers $s_i \leq r$. Then there exists a solution $y=(y_1, \dots, y_n)$ of f in $A<X>$ such that $Y_i \in A<X_1, \dots, X_{s_i}>$, $1 \leq i \leq n$ and $Y \equiv \hat{Y}$* mod $(m, X)^c$.

This Theorem follows from Theorem (1.1) (see [P_2] Theorem (3.7) and Remark (3.7.1)). A similar question in the analytic case was solved negatively in [G]. If in the Theorem (1.3) $\hat{y}_i \in \hat{A}[[X^{(i)}]]$ for some subsets $X^{(i)} \subset X$, $i=1, \dots, n$ and $X^{(k)} \not\subset X^{(j)}$, $X^{(j)} \not\subset X^{(k)}$ for some $j \neq k$ then it is possible to find no solution y of f in $A<X>$ such that $y_i \in A<X^{(i)}>$, $1 \leq i \leq n$ as shows the following example adapted after [B].

(1.4) *Example.* By computation, one shows that there exists a unique formal power series $\hat{u}$ in $\mathbf{C}[[X]]$ such that $\hat{u}(X+X^2)=2\hat{u}-X$. Note that $\hat{u}$ is divergent. By Taylor's formula there exist some formal power series $\hat{s}, \hat{t}$ from $\mathbf{C}[[X, Y]]$ such that $\hat{u}(Y)=\hat{u}+\hat{s}.(Y-X)$, and $\hat{u}(Y)=\hat{u}(X+X^2)-\hat{t}.(Y-X-X^2)$.

Then $\hat{u}, \hat{s}, \hat{t}$ and $\hat{v}:=\hat{u}(Y)$ form a solution in $\mathbf{C}[[X, Y]]$ of the following system of equations

$$(*) \qquad V=U+S(Y-X), \quad V+T(Y-X-X^2)=2U-X.$$

If there exists a solution u, v, s, t of $(*)$ in $\mathbf{C}<X, Y>$ such that $u \in \mathbf{C}<X>$, $v \in \mathbf{C}<Y>$, then we get $v(X)=u$, $v(X+X^2)=2u-X$ and so u is a convergent power series such that $u(X+X^2)=2u-X$. By unicity $u=\hat{u}$. Contradiction!

§2. Some applications.

(2.1) **Theorem** ([P_2]. *Corollary* (3.4)). *The completion of an excellent factorial henselian local ring is factorial too.*

For two dimensional local rings whose residue fields are algebrically closed the above result is the subject of [L] Theorem (17.4), p. 237.

Proof of Theorem (2.1). Let (A, m) be an excellent factorial henselian local ring and $\hat{A}$ its completion.

$\hat{A}$ is a domain. Indeed if it holds $\hat{y}\hat{x}=0$ for some nonzero elements $\hat{x}, \hat{y} \in \hat{A}$ then choose a positive integer $c \in \mathbf{N}$ such that $\hat{x}, \hat{y} \notin m^c\hat{A}$. By Theorem (1.1), A is an AP-ring and so there exist $x, y \in A$ such that $xy=0$ and $x \equiv \hat{x}$, $y \equiv \hat{y}$ mod $m^c\hat{A}$. Then $x, y \notin m^c\hat{A} \cap A=m^c$ and so $\{x, y\}$ are nonzero elements, hence A is not a domain. Contradiction!

$\hat{A}$ is factorial. It is enough to show that an irreducible element $\hat{x} \in \hat{A}$ is also prime. Otherwise there exist $\hat{u}, \hat{v}, \hat{w} \in \hat{A}$ such that $\hat{u}\hat{v}=\hat{x}\hat{w}$ but $\hat{u} \notin \hat{x}\hat{A}$, $\hat{v} \notin \hat{y}\hat{A}$. Choose a system of generators $a=(a_1, \dots, a_n)$ for m and consider the following polynomials $f:=UV-XW$, $g:=U-XZ$, $g':=V-XZ'$ $h:=\sum_{i,j=1}^{n} a_i a_j T_i T'_j - X$, from $A[U, V, X, W, Z, Z', T, T']$, $T=(T_1, \dots, T_n)$, $T'=(T'_1, \dots, T'_n)$. Then $(\hat{u}, \hat{v}, \hat{x}, \hat{w})$ is a solution of f such that each of the following equation: $g(\hat{u}, \hat{x}, Z)=0$, $g'(\hat{v}, \hat{x}, Z')=0$, $h(\hat{x}, T, T')=0$ has no solutions in $\hat{A}$ (the incompatibility of the latter one says that $\hat{x}$ is not a product of some nonunit elements from $\hat{A}$, i.e. $\hat{x}$ is an irreducible element in $\hat{A}$). Applying the following Lemma there exists a solution (u, v, x, w) of f such that each of the following equations $g(u, x, Z)=0$, $g'(v, x, Z')=0$, $h(x, T, T')=0$, has no solutions in A and then A is not factorial, the element x being irreducible but not prime. Contradiction!.

(2.1.1) **Lemma.** *Let F be a polynomial in $X=(X_1, \dots, X_r)$ over an AP-ring (C, m) and $G_1, \dots, G_s$ some polynomials in X, $Y=(Y_1, \dots, Y_n)$ over C. Then the following statements are equivalent: i) there exists a solution x of F in C such that each equation $G_i(x, Y)=0$ has no solutions in C, $1 \leq i \leq s$; ii) there exists a solution $\hat{x}$ of F in $\hat{C}$ such that each equation $G_i(\hat{x}, Y)=0$ has no solutions in $\hat{C}$, $1 \leq i \leq s$.*

Proof. i)$\Rightarrow$ii) Take $\hat{x}:=x$. If $G_i(\hat{x}, Y)=0$ has a solution in $\hat{C}$, then it has also one in C by (AP). Contradiction!

ii)$\Rightarrow$i) Note that $\hat{C}$ has (SAP) by Theorem (1.2). Let $\nu_i: \mathbf{N} \to \mathbf{N}$ be the function associated to $G_i(\hat{x}, Y)$, $1 \leq i \leq s$ and denote $c := \max_{i=1,\dots,s} \nu_i(1)$.

Using (AP) there exists a solution x of F in C such that $x \equiv \hat{x}$ mod $m^c\hat{C}$. If $G_i(x, Y)=0$ has a solution in C, then it has one in $C/m^c \cong \hat{C}/m^c\hat{C}$. Since $x \equiv \hat{x}$ mod $m^c\hat{C}$, the equation $G_i(\hat{x}, Y)=0$ has a solution in $\hat{C}/m^c\hat{C}$ and so in $\hat{C}$ (see the property of ν from (SAP)). Contradiction!

The following application was suggested to us by A. Constantinescu.

(2.2) **Proposition.** *Let (R, m) be an excellent henselian local ring, A, B the henselizations of $R[Y]$, $Y=(Y_1, \dots, Y_N)$ resp. $R[Z]$, $Z=(Z_1, \dots, Z_r)$ in (m, Y) resp. (m, Z), $b=(b_1, \dots, b_k)$ some elements from B, $f: A \to B/bB$ a* *local R-morphism and* $\hat{f}: \hat{A} \to \hat{B}/b\hat{B}$ *the map induced by f. Then f is surjective whenever* $\hat{f}$ *is surjective.*

Proof. Let f be given by $Y \to h = (h_1, \ldots, h_N) \in (\underline{m}, Z)B^N$. Let $\hat{f}$ be surjective. Then there exists some formal power series $\hat{t} = (\hat{t}_1, \ldots, \hat{t}_r)$ from $\hat{A}$ such that $\hat{t}_i(h) \equiv Z_i \bmod b\hat{B}$; $i = 1, \ldots, r$. By Taylor's formula there exist some formal power series $\hat{s}_{ij}$ from $\hat{R}[[Y, Z]]$ such that $-\hat{t}_i + \hat{t}_i(h) = \sum_{j=1}^{N} \hat{s}_{ij}(Y_j - h_j)$, $i = 1, \ldots, r$. Thus $\hat{f}$ is surjective iff the system of polynomial equations

$$(**) \qquad T_i + \sum_{j=1}^{N} S_{ij}(Y_j - h_j) = Z_i + \sum_{i=1}^{k} b_i U_k, \quad i = 1, \ldots, r,$$

has in $\hat{R}[[Y, Z]]$ the formal solution $(\hat{t}, \hat{s}, \hat{u})$ where $\hat{t}_i \in \hat{R}[[Y]]$, $\hat{s}_{ij}$, $\hat{u}_k \in \hat{R}[[Y, Z]]$. By Theorem (1.3) there exists a solution (t, s, u) of $(**)$ in the henselization $R\langle Y, Z\rangle$ of $R[Y, Z]$ at $(\underline{m}, Y, Z)$ such that $t_i \in A$. Substituting Y by h we get $t_i(h) \equiv Z_i \bmod bB$, $i = 1, \ldots, r$ which says that f is also surjective.

(2.3) **Remark**. A nice geometric application of Theorem (1.0) is given in Theorem (3.8) from [A_2]. There it is shown that isolated singularities are in fact algebraic.

REFERENCES

[A_1] Artin, M. — *On the solution of analytic equations*, Inv. math., 5 (1968), 277—291.

[A_2] Artin, M. — *Algebraic approximation of structures over complete local rings*, Publ. Math. IHES, 36 (1969), 23—58.

[A_3] Artin, M. — *Construction techniques for algebraic spaces*, Actes Congrès Intern. Math., tome I (1970), 419—423.

[B] Becker, J. — *A counterexample to Artin Approximation with respect to subrings*, Math. Ann. 230, (1977), 195—196.

[Br] Brown, M. L. — *Artin's approximation property*, Thesis, Cambridge, Peterhouse, August 1979.

[CP] Cipu, M. and Popescu, D. — *Some extensions of Néron's p-desingularization and approximation*, Rev. Roum. Math. Pures et Appl., t. XXIV, 10(1981), 1299—1304.

[DL] Denef, J. and Lipshitz, L. — *Ultraproducts and approximation in local rings II*, Math. Ann. 253 (1980), 1—28.

[G] Gabrielov, A. M. — *The formal relations between analytic functions* (Russian), Funktional, Anal. i Priložen. 5 (1971), no 4, 64—65.

[K] Kurke, H., Mostowski, T., Pfister, G., Popescu, D. and Roczen, M. — *Die Approximationseigenschaft lokaler Ringe*, Lect. Notes in Math. 634, Springer-Verlag, 1978.

[L] Lipman, J. — *Rational singularities with applications to algebraic surfaces and unique factorization*, Publ. Math. IHES 36 (1969), 195—279.

[PP] Pfister, G. and Popescu, D. — *Die strenge Approximationseigenschaft lokaler Ringe*. Invent. Math. 30 (1975), 145—174.

[P_1] Popescu D. — *Algebraically pure morphisms*, Rev. Roum. Math. Pures et Appl., t. 24, 6(1979), 947—977.

[P_2] Popescu, D. — *General Néron desingularization and approximation II*, INCREST Prepr. 49/1983.

[R] Rotthaus, C — *Potenzreihenerweiterung und formale Fasern in lokalen Ringen mit Approximationseigenschaft*, Manuscripta Math. 42 (1983), 53—65.

Institute of Mathematics
University of Bucharest

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, 1985 (supliment)

A NULLSTELLENSATZ OVER SOME HENSELIAN VALUED FIELDS

BY

ŞERBAN A. BASARAB

A basic tool of algebraic geometry is Hilbert's Nullstellensatz. It has been observed by A. Robinson [9] that this classical statement is equivalent in model theoretic terms with the fact that an algebraically closed field is existentially complete in every field extension. In the last time the methods of model theoretic algebra were successfully applied in order to get suitable Nullstellensätze over various fields.

The aim of the present work is to apply the general model theoretic approach from [1] to the case of a base field K of characteristic zero equipped with a proper Henselian valuation such that the residue field $k = k_v$ and the value group $G = G_v$ satisfy some supplementary conditions. We consider two types of restrictions on k. First we assume that k is a pseudo-real closed (prc) field [8] of characteristic zero such that $-1 \notin k^2$; in particular we consider the case when k is a formally real prc field. Next we assume that k is a finite field. As for the value group G we assume that the following finiteness conditions are satisfied.

Given a prime p and a natural number r, let us denote by $\Delta_{p,r} = \Delta_{p,r}(G)$ the maximal convex subgroup U of G such that dim $U/pU \leq r$. Consider the next restrictions on G:

i) For every convex subgroup U of G such that G/U is not divisible, U/pU are finite for all primes p.

ii) The chain of the convex subgroups $\Delta_{p,r}$ for all p, r is finite.

The regular ordered groups [10], [11] satisfy the conditions i) and ii) above.

Given such a base field K and an ideal $\mathfrak{a}$ of the polynomial ring $K[z]$, $z = (z_1, \ldots, z_n)$, we consider the set $V_K(\mathfrak{a}) = \{a \in K^n \mid f(a) = 0$ for all $f \in \mathfrak{a}\}$ of K-rational zeros of $\mathfrak{a}$ and the corresponding ideal of $K[z]$

$$I\,V_K(\mathfrak{a}) = \{f \in K[z] \mid f = 0 \text{ on } V_K(\mathfrak{a})\}.$$

We explicitely construct an overring H of $K[z]$ contained in $F = K(z)$ in such a way that for every ideal $\mathfrak{a}$ of $K[z]$, $I\,V_K(\mathfrak{a}) = K[z] \cap \sqrt{\mathfrak{a}}$.

The ring H turns out to be the intersection of the valuation rings corresponding to the K — rational places of F/K lying over the polynomial ring $K[z]$. In other words, H is the holomorphy ring of certain subspace of the Riemann space of all places of F/K which are rational over K.

The details and the proofs are given in [4].

As remarkable special cases of the present general approach we mention the following ones.