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THE RANK OF THE COMPOSITION OF TWO MORPHISMS IN EXACT CATEGORIES WITH SERRE CLASS

BY

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1. Introduction. If $\mathcal{C}$ is an abelian category with Serre class $\mathcal{S}$ and Euler-Poincaré mapping ε, then the Euler-Poincaré rank of a morphism f with $\mathrm{Im}\, f \in \mathcal{S}$ is defined as $\varepsilon(\mathrm{Im}\, f)$. (see [2]). More generally, an exact category $\mathcal{E}$ may be considered instead of the abelian category $\mathcal{C}$. For example, the category Bij of partial bijections between sets, which is an exact and inverse category, not being abelian, may be provided with Serre class and Euler-Poincaré mapping. The Euler-Poincaré rank of a morphism f can be defined just as in the abelian case. The purpose of this paper is to give a direct proof of the rank inequality for the composition of two morphisms, in exact categories with Serre class and Euler-Poincaré mapping.

2. Exact categories with Serre class. An exact category is a normal and conormal category, with kernels and cokernels, where any morphism f can be written as a composition $f = uh$, h being an epimorphism and u a monomorphism. (Basic properties for exact categories can be found in [1] pp. 18–24).

Let $\mathcal{E}$ be an exact category. A class $\mathcal{S}$ of objects from $\mathcal{E}$ is a *Serre class* of $\mathcal{E}$ if : $0 \in \mathcal{S}$, and for any exact sequence of $\mathcal{E}$ of the form: $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, $Y \in \mathcal{S}$ iff $X \in \mathcal{S}$ and $Z \in \mathcal{S}$.

Let $(G, +)$ be an abelian group and $\mathcal{S}$ a Serre class of the exact category $\mathcal{E}$. By definition $\varepsilon : \mathcal{S} \to G$ is an **Euler Poincaré mapping** if for any exact sequence of $\mathcal{E}$ of the form : $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ where $Y \in \mathcal{S}$, $\varepsilon(Y) = \varepsilon(X) + \varepsilon(Z)$.

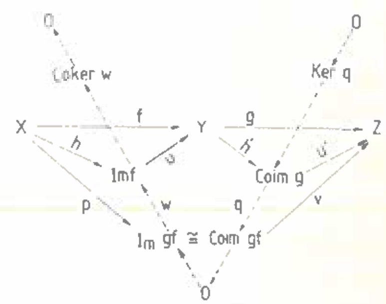
Proposition. *Let $\mathcal{E}$ be an exact category with Serre class $\mathcal{S}$ and Euler-Poincaré mapping $\varepsilon : \mathcal{S} \to (G, +)$. Then 1). $\varepsilon(0) = 0$; 2). If $f : A \to B$ is an isomorphism and one of the objects A or B belongs to $\mathcal{S}$, then the other one also belongs to $\mathcal{S}$ and $\varepsilon(A) = \varepsilon(B)$; 3). If $A \in \mathcal{S}$ then any subobject of A and any quotient object of A also belongs to $\mathcal{S}$; 4). If in the exact category $\mathcal{E}$, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence and $X, Z \in \mathcal{E}$, then $Z \in \mathcal{S}$.*

3. The rank inequality (for the composition of two morphisms). In this part of our paper, we consider by $\mathcal{E}$ an exact category with Serre class $\mathcal{S}$ and **Euler-Poincaré mapping** $\varepsilon : \mathcal{S} \to (\mathbf{Z}, +)$, where $(\mathbf{Z}, +)$ is the additive

group of integers and $\varepsilon(X) \geq 0$ for any $X \in \mathcal{S}$. The Euler-Poincaré rank of a morphism f with $\mathrm{Im}\, f \in \mathcal{S}$ is defined as $\mathrm{rank}_\varepsilon f = \varepsilon(\mathrm{Im}\, f)$.

The category of R-modules over a ring R, where the class of all R-modules of finite length is a Serre class $\mathcal{S}$ and the mapping $\mathrm{long}_R : \mathcal{S} \to (\mathbf{Z}, +)$ which identifies a module X of finite length with its length $\mathrm{long}_R(X)$, is an abelian category with Serre class and Euler-Poincaré mapping. Now we give an example of an exact category with Serre class and Euler-Poincaré mapping which is not abelian. This is the category of partial bijections between sets. In this category the Serre class is the class of all finite sets, and $|\,.\,| : \mathcal{S} \to (\mathbf{Z}, +)$ (which identifies a finite set X with its number of elements $|X|$) is the Euler-Poincaré mapping.

Proposition. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of the category $\mathcal{E}$, with $\mathrm{Im}\, f \in \mathcal{S}$ and $\mathrm{Im}\, g \in \mathcal{S}$. Then the Euler-Poincaré rank of the composition gf is less or equal with the rank of each morphism* : $\mathrm{rank}_\varepsilon(gf) \leq \mathrm{rank}_\varepsilon f$ *and* $\mathrm{rank}_\varepsilon(gf) \leq \mathrm{rank}_\varepsilon g$.



Proof. Let us consider the diagram : where $f = uh$ and $gf = vp$ are mono-epi factorizations of f and respectively of gf. Since $gf = g(uh) = (gu)h$, it follows that there exists a morphism (necessarilly a monomorphism) $w : \mathrm{Im}\, gf \to \mathrm{Im}\, f$ such that $(gu)w = v$. Then the sequence $0 \to \mathrm{Im}\, gf \to \mathrm{Im}\, f \to \mathrm{Coker}\, w \to 0$ is exact and since $\mathrm{Im}\, f \in \mathcal{S}$, it follows that $\mathrm{Im}\, gf \in \mathcal{S}$, $\mathrm{Coker}\, w \in \mathcal{S}$ and $\varepsilon(\mathrm{Im}\, f) = \varepsilon(\mathrm{Im}\, gf) + \varepsilon(\mathrm{Coker}\, w)$ which means $\mathrm{rank}_\varepsilon(gf) \leq \mathrm{rank}_\varepsilon f$.

Let us now consider $g = u'h'$ a mono-epi factorization of g. Since $gf = (u'h')f = u'(h'f)$, it follows that there exists a morphism (necessarily an epimorphism) q : $\mathrm{Coim}\, g \to \mathrm{Coker}\, gf$ so that $q(h'f) = p$. Then the sequence $0 \to \mathrm{Ker}\, q \to \mathrm{Coim}\, s \xrightarrow{q} \mathrm{Coim}\, gf \to 0$ is exact. Since $\mathrm{Coim}\, gf \cong \mathrm{Im}\, g$, $\mathrm{Coim}\, gf \cong \mathrm{Im}\, gf$ and $\mathrm{Im}\, g \in \mathcal{S}$, it follows that $\mathrm{Im}\, gf \in \mathcal{S}$, $\mathrm{Ker}\, q \in \mathcal{S}$ and $\varepsilon(\mathrm{Im}\, g) = \varepsilon(\mathrm{Ker}\, q) + \varepsilon(I\mathrm{Im}\, gf)$ which means $\mathrm{rank}_\varepsilon(gf) \leq \mathrm{rank}_\varepsilon$.

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