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FUNCTORIAL BISTABILITY

BY

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In [2] we constructed an abstract theory of approximation schemes like in [1] (Aubin), [6] (Marinescu) and [7] (Stummel) using categorical concepts only. In [3], [4] and [5] the ideas concerning the stability and inverse stability of some functors among the approximating categories of some aproximation schemes have been developed. In this paper we define the bistability of a functor in the same manner as in [6], but in a categorial way and we study the connection of this concept with stability and inverse stability.

We use notations, definitions, theorems and examples from [2], [3], [4] and [5].

Let $(\mathbf{X}, \mathbf{AX}, \mathbf{PSX})$ and $(\mathbf{Y}, \mathbf{AY}, \mathbf{PSY})$ be two approximation schemes and $\mathbf{T} : \mathbf{SX} \to \mathbf{SY}$ a functor. Assume that the aproximation schemes verifies the feature of limit unicity (T. 5.6., [2]):

$$\text{(u)} \qquad \begin{aligned} &\mathbf{AX}(a) \cap \mathbf{AX}(b) \neq 0 \Rightarrow a = b, \\ &\mathbf{AX}(f)\,(s) = \mathbf{AX}(g)\,(s) \Rightarrow f = g, \end{aligned}$$

where $a, b, \in \mathrm{Ob}\,\mathbf{X}$, $f, g \in \mathbf{X}(a, b)$ and s is an element of $\mathbf{AX}(a)$.

Definition. *The functor* $\mathbf{T}$ *is bistable iff there exists a subcategory* $\mathbf{X}_0 \subset \mathbf{X}$ *and a functor* $\mathbf{L} : \mathbf{X} \to \mathbf{Y}$ *having the following features:*

(i) *for any* $a \in \mathrm{Ob}\,\mathbf{X}_0$ *and* $\mathbf{U} \subset \mathbf{AX}(a)$. $\mathbf{T}(\mathbf{U}) \subset \mathbf{AY}(\mathbf{L}(a))$ *iff* $\mathbf{T}^{-1}\mathbf{AY}(\mathbf{L}(a)) \subset \mathbf{AX}(\mathrm{a})$.

(ii) *for any* $f \in \mathbf{X}_0(a, b)$ *and* $\mathbf{F} \in \mathbf{PSX}(\mathbf{U}, \mathbf{V})$ *with* $\mathbf{U} \subset \mathbf{AX}(a)$ $\mathbf{V} \subset \mathbf{AX}(b)$ *the diagram* (1) *is commutative iff the diagram* (2) *is commutative:*

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ {\scriptstyle IU}\downarrow & (1) & \downarrow{\scriptstyle IV} \\ AX(a) & \xrightarrow[AX(f)]{} & AX(b) \end{array} \qquad\qquad \begin{array}{ccc} U & \xrightarrow{F} & V \\ {\scriptstyle TX}\downarrow & (2) & \downarrow{\scriptstyle TV} \\ AY(L(a)) & \xrightarrow[AY(L(f))]{} & AY(L(b)) \end{array}$$

We denote by Ob **Bistab** the class of the pairs (a, x), where $a \in \mathrm{Ob}\,\mathbf{X}_0$ and $x = \mathbf{L}(a)$ and by Hom **Bistab** the class of all pairs (f, l), where $f \in \mathbf{X}_0(a, b)$ and $l = L(f)$.

Proposition. If $\mathbf{T}$ *is a bistable monofunctor, then* **Bistab** *is a subcategory of the category* $\mathbf{X} \times \mathbf{Y}$.

Proof. Let $(a, x) \in \mathrm{Ob}$ Bistab and $F \in \mathbf{PSX}(\mathbf{U}, \mathbf{V})$. Assume the commutativity of the diagram (1), where $f = 1_a$; then $\mathbf{F}$ is the inclusion functor $\mathbf{U} \subset \mathbf{V}$, hence $\mathbf{T}(\mathbf{U}) \subset \mathbf{T}(\mathbf{V})$ and $\mathbf{TV} \circ \mathbf{F} = \mathbf{AY}(1_x) \circ \mathbf{TU}$, that is, (2) is commutative for $\mathbf{L}(f) = 1_x$. Conversely, if $\mathbf{TV} \circ \mathbf{F} = \mathbf{AY}(1_x)\ \mathbf{TU}$, then $\mathbf{T}(\mathbf{U}) \subset \mathbf{T}(\mathbf{V})$; but $\mathbf{T}$ is a monofunctor, hence $\mathbf{U} \subset \mathbf{V}$ and the diagram (1) is commutative. It follows that $(1_a, 1_x) \in$ Bistab $((a, x), (a, x))$.

Let now $(g, p) \in$ Bistab $((a, x), (b, y))$, $(h, q) \in$ Bistab $((b, y), (c, z))$ and $\mathbf{F} \in \mathbf{PSX}(\mathbf{U}, \mathbf{W})$. According to the Proposition 3.3. [2] and Proposition 1, [4], it follows (ii), for $\mathbf{V} = \mathbf{W}$, $f = h \circ g$ and $\mathbf{L}(f) = q \circ p$, hence $(h \circ g, q \circ p) \in$ Bistab $((a, x), (c, z))$.

Theorem. *The functor* $\mathbf{T}$ *is bistable if and only if it is stable and inversely stable.*

Proof. Let $\mathbf{T}$ be a bistable functor and $\mathbf{F} \in \mathbf{PSX}(\mathbf{U}, \mathbf{V})$ a functor T-convergent to $f \in \mathbf{X}(a, b)$ and $1 \in \mathbf{Y}(x, y)$. Hence, $\mathbf{U} \subset \mathbf{AX}(a)$, $\mathbf{V} \subset \mathbf{AX}(b)$ and $\mathbf{T}(\mathbf{U}) \subset \mathbf{AY}(x)$, $\mathbf{T}(\mathbf{V}) \subset \mathbf{AY}(y)$; $\mathbf{T}$ being bistable: $\mathbf{T}(\mathbf{U}) \subset \mathbf{AY}(\mathbf{L}(a))$ and $\mathbf{T}(\mathbf{V}) \subset \mathbf{AY}(\mathbf{L}(b))$, whence, according to (u): $\mathbf{L}(a) = x$ and $\mathbf{L}(b) = y$. Because of the T-convergency of $\mathbf{F}$ and the bistability of $\mathbf{T}$ it follows that $\mathbf{IV} \circ \mathbf{F} = \mathbf{AX}(f) \circ \mathbf{IU}$ and $\mathbf{TV} \circ \mathbf{F} = \mathbf{AY}(l) \circ \mathbf{TU} = \mathbf{AY}(\mathbf{L}(f)) \circ \mathbf{TU}$, consequently, according to (u): $\mathbf{L}(f) = l$ it follows that $f \in \mathbf{X}_0(a, b)$.

Let now $\mathbf{G} \in \mathbf{PSX}(\mathbf{P}, \mathbf{Q})$. If $\mathbf{IQ} \circ \mathbf{G} = \mathbf{AX}(f)\mathbf{IP}$ then this is the commutativity of a type (1) diagram for $\mathbf{U} = \mathbf{P}$, $\mathbf{V} = \mathbf{Q}$ and $\mathbf{F} = \mathbf{G}$; then from (ii): $\mathbf{TQ} \circ \mathbf{G} = \mathbf{AY}(l) \circ \mathbf{TP}$ and accordingly $\mathbf{T}$ is stable around $\mathbf{F}$. If $\mathbf{IQ} \circ \mathbf{TG} = \mathbf{AY}(l) \circ \mathbf{IP}$ it results that $\mathbf{TQ} \circ \mathbf{G} = \mathbf{AY}(l) \circ \mathbf{TP}$ and this implies the commutativity of the type (2) diagram and from (ii) it follows thar $\mathbf{IQ} \circ \mathbf{G} = \mathbf{AX}(f) \circ \mathbf{IU}$; according to Definition 2, [3], the functor $\mathbf{T}$ is inversely stable around $\mathbf{F}$.

Conversely, let $\mathbf{T}$ be a stable and inversely stable functor. We mark by $\mathrm{Ob}\,\mathbf{X}_0$ and $\mathrm{Ob}\,\mathbf{Y}_0$ the classes of objects $a \in \mathrm{Ob}\,\mathbf{X}$, respectively, $x \in \mathrm{Ob}\,\mathbf{Y}$ for which there exists a subcategory $\mathbf{U} \in \mathrm{Ob}\,\mathbf{PSX}$ which is T-convergent to a and x; we define $\mathbf{L}(a) = x$; by $\mathbf{X}_0(a, b)$ and $\mathbf{Y}_0(x, y)$ we denote the sets of morphisms $f \in \mathbf{X}(a, b)$, respectively, $l \in \mathbf{Y}(x, y)$ for which there exists a functor $\mathbf{F} \in \mathbf{PSX}(\mathbf{U}, \mathbf{V})$ which is T-convergent to f and l; we define $\mathbf{L}(f) = l$. If for $a \in \mathrm{Ob}\mathbf{X}_0$ there are two objects $x, x' \in \mathrm{Ob}\mathbf{Y}_0$ and two subcategories $\mathbf{U}, \mathbf{U}' \in \mathrm{Ob}\,\mathbf{PSX}$ which are T-convergent to a and x, respectively, a and x', then $\mathbf{T}(\mathbf{U}) \subset \mathbf{T}(\mathbf{AX}(a)) \subset \mathbf{AY}(x)$ and $\mathbf{T}(\mathbf{U}') \subset \mathbf{T}(\mathbf{AX}(a)) \subset AY(x')$ and from (u): $x = x'$; if for $f \in \mathbf{X}_0(a, b)$ there exists $l, l' \in \mathbf{Y}_0(x, y)$ and $\mathbf{F} \in \mathbf{PSX}(\mathbf{U}, \mathbf{V})$ which is T-convergent to f and l and $\mathbf{F}' \in \mathbf{PSX}(\mathbf{U}', \mathbf{V}')$ which is T-convegent to f and l' from the stability of $\mathbf{T}$ it follows that: $\mathbf{TAX}(b) \circ \mathbf{AX}(f) = \mathbf{AY}(l) \circ \mathbf{TAX}(a) = \mathbf{AY}(l') \circ \mathbf{TAX}(a)$, hence $\mathbf{AY}(l)\,(s) = \mathbf{AY}(l')\,(s)$ if s is an element from $\mathbf{T}(\mathbf{AX}(a))$ then according to (u) it follows that $l = l'$. Now, according to the proposition 3.3 from [2], 1 and 2 from [4] and 2 from [5] the proof of the conditions (i) and (ii) is a matter of evidence.

Corollary. *If* $\mathbf{T}$ *is a monofunctor* Bistab = Stab Stabinv.

Example. Let $(\mathbf{E}, \mathbf{E}_h, p_h, r_h)$, $(\mathbf{F}, \mathbf{F}_h, q_h, s_h)$, $h \in \mathbf{H}$, two internal approximations, according to Aubin, [1] and $\mathbf{A}_h : \mathbf{E}_h \to \mathbf{F}_h$, $h \in \mathbf{H}$, an operator family. We obtain sufficient conditions for the bistability of the family $(\mathbf{A}_h)_{h \in H}$, as in [6] or [7] if we choose the approximation schemes and the functor $\mathbf{T}$ as in [2] and [5]. According to the examples 1.5, 2.3, 3.4 from [2],

[5] and the above proved theorem, it follows that $(A_h)_{h\in H}$ is a bistable family if there is an asymptotical and inversely asymptotical equicontinuous family on convergent sequences.

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DUALITY BETWEEN CATEGORIES OF MODULES OVER A UNIVERSAL ALGEBRA

BY

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If $\mathcal{A}$ and $\mathcal{B}$ are categories, then a *duality* between $\mathcal{A}$ and $\mathcal{B}$ is a pair of contravariant functors $(\mathbf{D}_1 ; \mathbf{D}_2)$ with $\mathbf{D}_1 : \mathcal{A}\to\mathcal{B}$ and $\mathbf{D}_2 : \mathcal{B}\to\mathcal{A}$ such that the covariant functors $\mathbf{D}_1\circ\mathbf{D}_2$ and $\mathbf{D}_2\circ\mathbf{D}_1$ are naturally equivalent to the identity functors $1_{\mathcal{B}}$, respectively $1_{\mathcal{A}}$.

Two dualities $(\mathbf{D}_1 ; \mathbf{D}_2)$ and $(\mathbf{E}_1 ; \mathbf{E}_2)$ between the categories $\mathcal{A}$ and $\mathcal{B}$ are named *equivalent* if the functors $\mathbf{D}_i$ and $\mathbf{E}_i$ are naturally equivalent for $i=\overline{1,2}$.

If **A** and **B** are rings with unity element, we denote by ${}_A\mathcal{C}$ the category of unitary left **A**-modules and by $\mathcal{C}_B$ the category of unitary right **B**-modules.

In [1] the following theorem is proved :

Theorem 1. *If $\mathcal{A}$ and $\mathcal{B}$ are two full subcategories of the category ${}_A\mathcal{C}$, respectively $\mathcal{C}_B$ with properties :*
(i) ${}_AA\in|\mathcal{A}|$, ${}_BB\in|\mathcal{B}|$; (ii) *if* $\mathbf{X}\in|\mathcal{A}|$, $\mathbf{X}'\in|{}_A\mathcal{C}|$ *and* $\mathbf{X}$ *is isomorphic with* $\mathbf{X}'$ *then* $\mathbf{X}'\in|\mathcal{A}|$; (iii) *if* $\mathbf{Y}\in|\mathcal{B}|$, $\mathbf{Y}'\in|\mathcal{C}_B|$ *and* $\mathbf{Y}$ *is isomorphic with* $\mathbf{Y}'$ *then* $\mathbf{Y}'\in|\mathcal{B}|$, *and if* $(\mathbf{D}_1 ; \mathbf{D}_2)$ *is a duality between the categories* $\mathcal{A}$ *and* $\mathcal{B}$, $\mathbf{D}_1, \mathbf{D}_2$ *being additive functors, then there is a bimodule* ${}_A\mathbf{U}_B$ *such that this duality is equivalent to the duality realized by the functors* $\mathrm{Hom}_A(.,\mathbf{U}) : \mathcal{A}\to\mathcal{B}$ *and* $\mathrm{Hom}_B(.,\mathbf{U}) : \mathcal{B}\to\mathcal{A}$.

Let **X** be a universal algebra having the set of operations $\Omega=\{\omega_1^{n_1},\ldots,\omega_k^{n_k}\}$, with $n_1,\ldots,n_k\in\mathbf{N}^*(\omega_i^{n_i} : \mathbf{X}^{n_i}\to\mathbf{X})$.

Further on, we assume that **X** and all the other universal algebras of the same type verify the relation :

$$\begin{aligned}(1)\qquad &\omega_i^{n_i}[\omega_j^{n_j}(x_{11},\ldots,x_{1n_j}),\ldots,\omega_j^{n_j}(x_{n_i1},\ldots,x_{n_in_j})]=\\ &=\omega_j^{n_j}[\omega_i^{n_i}(x_{11},\ldots,x_{n_i1}),\ldots,\omega_i^{n_i}(x_{1n_j},\ldots,x_{n_in_j})]\end{aligned}$$

for every $i, j\in\{1,\ldots,k\}$ and every $x_{lm}\in\mathbf{X}$, with $l=\overline{1,n_i}$, $m=\overline{1,n_j}$. Let Γ be a σ-algebra with $\sigma=\sigma_1\cup\sigma_2$ where $\sigma_1=\{\sigma_1^{n_1},\ldots,\sigma_k^{n_k}\}$ and $\sigma_2=\{m^2\}$, $m^2 : \Gamma\times\Gamma\to\Gamma$. We assume that Γ verifies (1) as σ_1-algebra} and also verifies the conditions :

$$(2)\qquad m^2[\gamma, \sigma_i^{n_i}(\gamma_1,\ldots,\gamma_{n_i})]=\sigma_i^{n_i}[m^2(\gamma,\gamma_1),\ldots,m^2(\gamma,\gamma_{n_i})],$$

$$(3)\qquad m^2[\sigma_i^{n_i}(\gamma_1,\ldots,\gamma_{n_i}),\gamma]=\sigma_i^{n_i}[m^2(\gamma_1,\gamma),\ldots,m^2(\gamma_{n_i},\gamma)],$$