[5] and the above proved theorem, it follows that $(\mathbf{A}_h)_{h\in H}$ is a bistable family if there is an asymptotical and inversely asymptotical equicontinuous family on convergent sequences.

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DUALITY BETWEEN CATEGORIES OF MODULES OVER A UNIVERSAL ALGEBRA

BY

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If $\mathcal{A}$ and $\mathcal{B}$ are categories, then a *duality* between $\mathcal{A}$ and $\mathcal{B}$ is a pair of contravariant functors $(\mathbf{D}_1 ; \mathbf{D}_2)$ with $\mathbf{D}_1 : \mathcal{A}\to\mathcal{B}$ and $\mathbf{D}_2 : \mathcal{B}\to\mathcal{A}$ such that the covariant functors $\mathbf{D}_1\circ\mathbf{D}_2$ and $\mathbf{D}_2\circ\mathbf{D}_1$ are naturally equivalent to the identity functors $1_{\mathcal{B}}$, respectively $1_{\mathcal{A}}$.

Two dualities $(\mathbf{D}_1 ; \mathbf{D}_2)$ and $(\mathbf{E}_1 ; \mathbf{E}_2)$ between the categories $\mathcal{A}$ and $\mathcal{B}$ are named *equivalent* if the functors $\mathbf{D}_i$ and $\mathbf{E}_i$ are naturally equivalent for $i=\overline{1,2}$.

If **A** and **B** are rings with unity element, we denote by ${}_A\mathcal{C}$ the category of unitary left **A**-modules and by $\mathcal{C}_B$ the category of unitary right **B**-modules.

In [1] the following theorem is proved :

Theorem 1. *If $\mathcal{A}$ and $\mathcal{B}$ are two full subcategories of the category ${}_A\mathcal{C}$, respectively $\mathcal{C}_B$ with properties :*
(i) ${}_AA\in|\mathcal{A}|$, ${}_BB\in|\mathcal{B}|$; (ii) *if* $\mathbf{X}\in|\mathcal{A}|$, $\mathbf{X}'\in|{}_A\mathcal{C}|$ *and* $\mathbf{X}$ *is isomorphic with* $\mathbf{X}'$ *then* $\mathbf{X}'\in|\mathcal{A}|$; (iii) *if* $\mathbf{Y}\in|\mathcal{B}|$, $\mathbf{Y}'\in|\mathcal{C}_B|$ *and* $\mathbf{Y}$ *is isomorphic with* $\mathbf{Y}'$ *then* $\mathbf{Y}'\in|\mathcal{B}|$, *and if* $(\mathbf{D}_1 ; \mathbf{D}_2)$ *is a duality between the categories* $\mathcal{A}$ *and* $\mathcal{B}$, $\mathbf{D}_1, \mathbf{D}_2$ *being additive functors, then there is a bimodule* ${}_A\mathbf{U}_B$ *such that this duality is equivalent to the duality realized by the functors* $\mathrm{Hom}_A(., \mathbf{U}) : \mathcal{A}\to\mathcal{B}$ *and* $\mathrm{Hom}_B(., \mathbf{U}) : \mathcal{B}\to\mathcal{A}$.

Let **X** be a universal algebra having the set of operations $\Omega=\{\omega_1^{n_1},\ldots,\omega_k^{n_k}\}$, with $n_1,\ldots,n_k\in\mathbf{N}^*(\omega_i^{n_i} : \mathbf{X}^{n_i}\to\mathbf{X})$.

Further on, we assume that **X** and all the other universal algebras of the same type verify the relation :

$$\omega_i^{n_i}[\omega_j^{n_j}(x_{11},\ldots,x_{1n_j}),\ldots,\omega_j^{n_j}(x_{n_i1},\ldots,x_{n_in_j})]= \tag{1}$$
$$=\omega_j^{n_j}[\omega_i^{n_i}(x_{11},\ldots,x_{n_i1}),\ldots,\omega_i^{n_i}(x_{1n_j},\ldots,x_{n_in_j})]$$

for every $i, j\in\{1,\ldots,k\}$ and every $x_{lm}\in\mathbf{X}$, with $l=\overline{1, n_i}$, $m=\overline{1, n_j}$. Let Γ be a σ-algebra with $\sigma=\sigma_1\cup\sigma_2$ where $\sigma_1=\{\sigma_1^{n_1},\ldots,\sigma_k^{n_k}\}$ and $\sigma_2=\{m^2\}$, $m^2 : \Gamma\times\Gamma\to\Gamma$. We assume that Γ verifies (1) as σ_1-algebra} and also verifies the conditions :

$$m^2[\gamma, \sigma_i^{n_i}(\gamma_1,\ldots,\gamma_{n_i})]=\sigma_i^{n_i}[m^2(\gamma, \gamma_1),\ldots,m^2(\gamma, \gamma_{n_i})], \tag{2}$$

$$m^2[\sigma_i^{n_i}(\gamma_1,\ldots,\gamma_{n_i}), \gamma]=\sigma_i^{n_i}[m^2(\gamma_1, \gamma),\ldots, m^2(\gamma_{n_i}, \gamma)], \tag{3}$$

$$(\exists)1\in\Gamma, \text{ such that } m^2(1,\gamma)=m^2(\gamma,1)=\gamma, \tag{4}$$

$$m^2[m^2(\gamma_1,\gamma_2),\gamma_3]=m^2[\gamma_1, m^2(\gamma_2,\gamma_3)], \tag{5}$$

for every $i=\overline{1,k}$ and every $\gamma,\gamma_1,\gamma_2,\gamma_3,\gamma_j\in\Gamma$, with $j=\overline{1,n_i}$.

Let $\varphi:\Gamma\times\mathbf{X}\to\mathbf{X}$ be a function. We consider the properties :

$$\varphi[\gamma,\omega_i^{n_i}(x_1,\ldots,x_{n_i})]=\omega_i^{n_i}[\varphi(\gamma,x_1),\ldots,\varphi(\gamma,x_{n_i})], \tag{6}$$

$$\varphi[\sigma_i^{n_i}(\gamma_1,\ldots,\gamma_{n_i}),x]=\omega_i^{n_i}[\varphi(\gamma_1,x),\ldots,\varphi(\gamma_{n_i},x)], \tag{7}$$

$$\varphi(1,x)=x, \tag{8}$$

$$\varphi[m^2(\gamma_1,\gamma_2),x]=\varphi[\gamma_1,\varphi(\gamma_2,x)], \tag{9 l}$$

$$\varphi[m^2(\gamma_1,\gamma_2),x]=\varphi[\gamma_2,\varphi(\gamma_1,x)], \tag{9 r}$$

for every $i=\overline{1,k}$, $x, x_j\in\mathbf{X}$, $\gamma,\gamma_1,\gamma_2,\gamma_3,\gamma_j\in\Gamma$, with $j=\overline{1,n_i}$.

Definition 2. *A left (right) module over the universal algebra* Γ *is a universal algebra* $\mathbf{X}$ *on which an external operation* φ *is defined having the properties* 6, 7, 8, 9l (9r). *To simplify,* $\mathbf{X}$ *will be called left (right)* Γ*-module.*

Definition 3. *Let* $\mathbf{X}$ *and* $\mathbf{X}'$ *be two modules over the same universal algebra* Γ. *We say that* $f:\mathbf{X}\to\mathbf{X}'$ *is a left (right)* Γ*-module homomorphism if* f *is a homomorphism of universal algebras and the relation*

$$f[\varphi(\gamma,x)]=\varphi[\gamma,f(x)], \tag{10}$$

for every $x\in\mathbf{X}$ *and every* $\gamma\in\Gamma$, *is true.*

Definition 4. *We say that* $\mathbf{X}$ *is a* Γ-Γ'*-module if* $\mathbf{X}$ *is a left* Γ*-module, right* Γ'*-module and the relation* (11) *is fulfilled:*

$$\varphi[\gamma,\varphi'(\gamma',x)]=\varphi'[\gamma',\varphi(\gamma,x)], \tag{11}$$

for every $\gamma\in\Gamma$, $\gamma'\in\Gamma'$, $x\in\mathbf{X}$, *where by* φ *and* φ' *we denote the external laws corresponding to* Γ, *respectively* Γ'. Let ${}_{\Gamma}\mathcal{C}(\mathcal{C}_{\Gamma'})$ be the category which has as objects the left Γ-modu'es (the right Γ-modules) and as morphisms the left Γ-modules (right Γ'-moou'es) homomorphisms. Let $\mathbf{U}$ be a Γ-Γ'- module. If $\mathbf{X}$ is a left Γ-module and $\mathbf{Y}$ is a right Γ'-module, we denote $\mathrm{Char}_{\,U}\mathbf{X}=\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(\mathbf{X},\mathbf{U})$ and $\overline{\mathrm{Char}}_{\,U}\mathbf{Y}=\mathrm{Hom}_{\mathcal{C}_{\Gamma'}}(\mathbf{Y},\mathbf{U})$. For $f\in\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(\mathbf{X},\mathbf{X}')$ and $g\in\mathrm{Hom}_{\mathcal{C}_{\Gamma'}}(\mathbf{Y},\mathbf{Y}')$ we define $\mathrm{Char}_U\, f:\mathrm{Char}_U\,\mathbf{X}'\to\mathrm{Char}_U\,\mathbf{X}$ and $\overline{\mathrm{Char}}_U\, g:\overline{\mathrm{Char}}_{\,U}\mathbf{Y}'\to\overline{\mathrm{Char}}_{\,U}\mathbf{Y}$ by the relations $\mathrm{Char}_U f(\alpha)=\alpha\circ f$ and $\overline{\mathrm{Char}}_U g(\beta)=\beta\circ g$, for every $\alpha\in\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(\mathbf{X}',\mathbf{U})$, respectively $\beta\in\mathrm{Hom}_{\mathcal{C}_{\Gamma'}}(\mathbf{Y}',\mathbf{U})$.

Proposition 5. $\mathrm{Char}_U:{}_{\Gamma}\mathcal{C}\to\mathcal{C}_{\Gamma'}$ and $\overline{\mathrm{Char}}_U:\mathcal{C}_{\Gamma'}\to{}_{\Gamma}\mathcal{C}$ *are contravariant functors.*

In the following, we assume that the functor $\mathbf{D}_1:{}_{\Gamma}\mathcal{C}\to\mathcal{C}_{\Gamma'}$ and $\mathbf{D}_2:\mathcal{C}_{\Gamma'}\to\mathcal{C}_{\Gamma}$ verify the relations .

$$\mathbf{D}_1[\Pi_i^{n_i}(f_1,\ldots,f_{n_i})]=\Pi_i^{n_i}[\mathbf{D}_1(f_1),\ldots,\mathbf{D}_1(f_{n_i})], \tag{12}$$

for every $i=\overline{1,k}$ and every $f_j\in\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(\mathbf{X},\mathbf{X}')$, $j=\overline{1,x_{ni}}$,

$$D_2[\Pi_i^{n_i}(g_1,\ldots,g_{n_i})]=\Pi_i^{n_i}[D_2(g_1),\ldots,D_2(g_{n_i})] \tag{13}$$

for every $i=\overline{1,k}$ and every $g_j\in\mathrm{Hom}_{\mathcal{C}_{\Gamma'}}(\mathbf{Y},\mathbf{Y}')$, $j=\overline{1,n_i}$.

The functors $\mathrm{Cha}:{}_{\Gamma}\mathcal{C}\to\mathcal{C}_{\Gamma'}$ and $\overline{\mathrm{Char}}_I:\mathcal{C}_{\Gamma'}\to{}_{\Gamma}\mathcal{C}$ verify also the relations (12), respectively (13).

The relations :

$$f\circ\Pi_i^{n_i}(f_1,\ldots,f_{n_i})=\Pi_i^{n_i}(f\circ f_1,\ldots,f\circ f_{n_i}), \tag{14}$$

$$\Pi_i^{n_i}(f_1,\ldots,f_{n_i})\circ f'=\Pi_i^{n_i}(f_1\circ f',\ldots,f_{n_i}\circ f'), \tag{15}$$

for every $i=\overline{1,k}$, $f_j\in\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(X_2,X_3)$, $j=\overline{1,n_i}$, $f\in\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(X_3,X_4)$, $f'\in\mathrm{Hom}_{\,{}_{\Gamma}\mathcal{C}}(X_1,X_2)$, $X_1,X_2,X_3,X_4\in|{}_{\Gamma}\mathcal{C}|$, hold in $\mathcal{C}$.

Let Γ and Γ' be two universal algebras which verify the conditions (1) considered as σ_1-algebras and (2), (3), (4, (5). Then $\Gamma(\Gamma')$ can be organized with a structure of left Γ-module (right Γ'-module) the scalar multiplication $\varphi:\Gamma\times\Gamma\to\Gamma(\varphi':\Gamma'\times\Gamma'\to\Gamma')$ being given by : $\varphi(\gamma_1,\gamma_2)=m^2(\gamma_1,\gamma_2)$ $(\varphi'(\gamma_1',\gamma_2')=m^2(\gamma_2',\gamma_1'))$.

Theorem 6. *If* $\mathcal{A}$ *is a full subcategory of* ${}_{\Gamma}\mathcal{C}$, $\mathcal{B}$ *is a full subcategory of* $\mathcal{C}_{\Gamma'}$, *with properties :*

(i) $\Gamma\in|\mathcal{A}|$, $\Gamma'\in|\mathcal{B}|$; (ii) *if* $\mathbf{X}\in|\mathcal{A}|$, $\mathbf{X}'\in|{}_{\Gamma}\mathcal{C}|$ *and* $\mathbf{X}$ *is isomorphic with* $\mathbf{X}'$ *then* $\mathbf{X}'\in|\mathcal{A}|$; (iii) *if* $\mathbf{Y}\in|\mathcal{B}|$, $\mathbf{Y}'\in|\mathcal{C}_{\Gamma'}|$ *and* $\mathbf{Y}$ *is isomorphic with* $\mathbf{Y}'$ *then* $\mathbf{Y}'\in|\mathcal{B}|$; *then every duality* $(\mathbf{D}_1,\mathbf{D}_2)$ *between* $\mathcal{A}$ *and* $\mathcal{B}$, $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *verifying the relations* (12) *respectively* (13), *is equivalent to the duality realized by the functors* Char_U *and* $\overline{\mathrm{Char}}_U$ *corresponding to a certain* Γ-Γ'*-module.*

The proof is a matter of skilled application of the definitions and the relations given above and we omit it here.

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