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TYPES OF NON-ISOMORPHIC TWO-DIMENSIONAL REAL DIVISION ALGEBRAS

BY

ILIE BURDUJAN

In their paper [1], [2], G. M. Benkart and J. M. Osborn give a classification for finite dimensional real division algebras having as a classification criterion the type of Lie derivation algebra associated with a real division algebra. The authors find out all types of real division algebras corresponding to the admissible types of derivation algebras. In [3] it is showed that in general the real division algebras having the same derivation algebra are not isotopic. Thus, the determination of all types of non-isomorphic finite dimensional real algebras remains very important. Moreover, the basic tool used in these papers was the representation theory of Lie (derivation) algebras. But, for a two-dimensional real division algebra A, the derivation algebra $\mathrm{Der}\, A$ is $\{0\}$. Hence it is not possible to use this tool in the study of these algebras.

In this paper we deal with the determination of all non-isomorphic types of two-dimensional real division algebras.

Let $A(\cdot)$ be a two-dimensional real division algebra. If $A(\cdot)$ has a unity element, then the algebra is associative and will be isomorphic to the algebra of complex numbers (over the reals). On the other hand, any nonzero element in a real division algebra has a unique left-identity and a unique right-identity and it will be important to use the notion of idempotent element, because it is weaker than the notion of unity element, but stronger than the one of left-or right-identity. Everywhere in what follows, we deal only with the two-dimensional algebras, so we shall say "real division algebra" instead of "two-dimensional real division algebra".

We now deal with the existence of the idempotent elements in a real division algebra.

Let $A(.)$ be a real division algebra, having — in the basis $\{a, b\}$ — the following multiplication table

I	a	b
a	$\alpha a+\beta b$	$\gamma a+\delta b$
b	$\varepsilon a+\theta b$	$\varkappa a+\varphi b$

It is well known that $A(\cdot)$ is a division algebra if and only if

(1) $$(\alpha\varphi+\delta\varepsilon-\varkappa\theta-\beta\varkappa)^2-4(\alpha\delta-\beta\gamma)(\varepsilon\varphi-\theta\varkappa)<0.$$

Proposition 1. *Any two-dimensional real division algebra have at least an idempotent element and not more than three idempotent elements.*

By proof of this proposition it is shown that the number of idempotent elements is the same with the number of real roots of the algebraic equation:

(2) $$\beta t^3+(\delta+\theta-\alpha)t^2+(\varphi-\gamma-\varepsilon)t-\varkappa=0.$$

The determination of non-isomorphic types of two-dimensional real division algebras is based on the use of idempotents and of eigenvalues for left and right translations associated with the (at most) three idempotents.

Let us introduce the notations:
e, e_1 and e_2 — the idempotents (if there are three of them), $(1, s)$, $(1, s_1)$ and $(1, s_2)$ — the pairs of (real) eigenvalues for L_e, L_{e_1} and L_{e_2} respectively, $(1, \sigma)$, $(1, \sigma_1)$ and $(1, \sigma_2)$ — the pairs of (real) eigenvalues for R_e, R_{e_1} and R_{e_2} respectively, $V=\{v\in A/L_e(v)=sv\}$, $V'=\{v\in A/R_e(v)=\sigma v\}$ — the eingenspaces for L_e and R_e respectively, $d=\dim V$, $d'=\dim V'$.

As regards the idempotent e, there exist the following two possibilities:

$$\mathrm{I}: s=1 \qquad \mathrm{II}: s\neq 1.$$

There are also, in relation with I and II, the folowing situations: (Table).

Corresponding to each i_0 of the 22 situations, it is possible to determine a suitable basis for which we have the corresponding i^0-multiplication table and the corresponding restrictions, as below:

1°	e	a
e	e	a
a	a	$-e$

2°	e	a
e	e	a
a	$e+a$	$\varkappa e$

$1+4\varkappa<0$

3°	e	a
e	e	a
a	σa	$-\mathrm{sign}\sigma.e+\varphi a$

$\varphi^2-4\sigma.\mathrm{sign}\sigma<0$
$\varphi\geq 0$

4°	e	a
e	e	$e+a$
a	a	$\varkappa e$

$1+4\varkappa<0$

5°	e	a
e	e	$e+a$
a	$\alpha e+a$	$\varkappa e$

$(1-\alpha)^2+4\varkappa<0$
$(1+\alpha)^2+4\varkappa<0$

6°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$(1-\sigma_1)e+e_1$	e_1

$\sigma_1\in(0, 1)$
$s_1=1-\sigma_1$

7°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$e+e_1$	e_1

$\sigma_1\in(0, 4)$

8°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$s_1 e+e_1$	e_1

$(1+s_1-\sigma_1)^2-4s_1<0$
$1-2(s_1+\sigma_1)\neq 0, s_1\neq 1-2\sigma_1$
$\sigma_1\neq 1-2s_1$

9°	e	a
e	e	$e+a$
a	σa	$\varkappa e+\varphi a$

$(\varphi-1)^2+4\sigma\varkappa<0$
$(\varphi-\sigma)^2+4\sigma\varkappa<0$

10°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$e+\sigma e_1$	e_1

$\sigma_1<\sigma;\ \sigma+\sigma_1+\sigma\sigma_1=0$
$-4<\sigma+\sigma_1<0$

11°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$s_1 e+\sigma e_1$	e_1

$(1+s_1-\sigma\sigma_1)^2-4s_1<0$
$1-(1+\sigma)(s_1+\sigma_1)=0$

12°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$e+\sigma e_1$	e_1

$(2-\sigma\sigma_1)^2-4<0$
$\sigma+\sigma_1+\sigma\sigma_1\neq 0$

13°	e	e_1
e	e	$\sigma_1 e+e_1$
e_1	$s_1 e+e_1$	e_1

$(1+s_1-\sigma\sigma_1)^2-4s_1<0$
$1-(1+\sigma)(s_1+\sigma_1)\neq 0$
$s_1<(1-\sigma_1-s_1-\sigma\sigma_1)/(1-\sigma_1-s_1-\sigma\sigma_1-\sigma s_1)$
$s_1\neq 1-2_1$

14°	e	a
e	e	sa
a	a	$-\mathrm{sign}s.e+\varphi a$

$\varphi^2-4s.\mathrm{sign}s<0$
$\varphi\geq 0$

15°	e	a
e	e	sa
a	$\alpha e+a$	$-\mathrm{sign}s.e+\varphi a$

$(\varphi-\alpha s)^2-4s.\mathrm{sign}s<0$
$(\varphi-\alpha)^2-4s.\mathrm{sign}s<0$
$\sigma\neq 0,\ \varphi\geq 0$

16°	e	e_1
e	e	$e+es_1$
e_1	$s_1 e+e_1$	e_1

$s+s_1+ss_1=0$
$s^2 s_1-4ss_1<0$

17°	e	e_1
e	e	$\sigma_1 e + e_1$
e_1	$s_1 e + e_1$	e_1

$(1-\sigma_1+ss_1)^2-4ss_1<0$
$(s_1+\sigma_1)(1+s)=1$
$s_1\neq 1$

18°	e	e_1
e	e	$e+se_1$
e_1	$s_1 e + e_1$	e_1

$s^2s_1^2-4ss_1<0$
$(1+s_1)(1+s)\neq 1,$
$s_1\neq 1$

19°	e	e_1
e	e	$\sigma_1 e + se_1$
e_1	$s_1 e + e_1$	e_1

$(1-\sigma_1+ss_1)^2-4ss_1<0$
$(\sigma_1+s_1)(1+\sigma\neq)1$
$s_1<(-ss_1)/(1-(s_1+\sigma_1)(1+s)),$
$s_1\neq 1$

20°	e	a
e	e	sa
a	$\alpha e+sa-\mathrm{sign}\,s\,(e+\gamma a)$	

$(\gamma-\alpha s)^2-4s\cdot\mathrm{sign}\,s<0$
$(\gamma-\alpha)^2-4(s+\sigma-1)\,\mathrm{sign}\,s<0$
$\gamma\geqslant 0$

21°	e	e_1
e	e	$\sigma_1 e + se_1$
e_1	$s_1 e+\sigma e_1$	e_1

$(1+ss_1-\sigma\sigma_1)^2-4ss_1<0$
$1=(s+\sigma)(s_1+\sigma_1)$
$s<s_1$ or $\sigma\leqslant\sigma_1$
$s=s_1$

22°	e	e_1
e	e	$\sigma_1 e + se_1$
e_1	$s_1 e+e_1$	e_1

$(1+ss_1-\sigma\sigma_1)^2-4ss_1<0$
$(\sigma+s)(\sigma_1+s_1)\neq 1$
$s<s_1<s_2$ or $s=s_1<s_2$
$\sigma\leqslant\sigma_1\leqslant\sigma_2$

Remarks: 1. The algebra 1° is isomorphic to the algebra of complex numbers (over the reals).

2. The algebras of type 2° corresponding to admissible values $\varkappa_1\neq\varkappa_2$ are non-isomorphic.

Similar statements are true for the other 20 types of algebras.

3. The algebras of distinct types are non-isomorphic.

4. In the case $I_{211(2)}$, for $\alpha=1$ and $\varkappa<-1$, we shall obtain a commutative algebra.

5. In the case I_{2122} we have $s_1,\sigma_1\in(0,1)$, and $s_1+\sigma_1=1$. If $\sigma_1=s_1=1/2$ we shall obtain a commutative algebra.

6. In the case I_{21231} we have $s_1=1$, $\sigma_1\in(0,4)$, $s_2\in(1/5,1)$, $\sigma_2\in(0,4/5)$. If $\sigma_1=1$ the corresponding algebra is commutative.

7. In the case I_{21232} we have $s_1<s_2=s_1/[2(s_1+\sigma_1)-1]$, $\sigma_2=\sigma_1/[2(s_1+\sigma_1)-1]$, $s_2\neq 1$, $\sigma_2\neq 1$.
If $s_1=\sigma_1$ the algebra $A(\cdot)$ will be also commutative.

8. In the case I_{2231} we have $s_2=(\sigma+\sigma_1-\sigma\sigma_1)/(\sigma+\sigma_1+\sigma\sigma_1)\neq 1$.

9. In the case I_{2232} we have $1\neq s_2=(1-\sigma_1-s_1-\sigma+\sigma\sigma_1)/(1-\sigma_1-s_1-\sigma\sigma_1-s_1\sigma)>s_1$.
If $s_1=\sigma_1$ it will be obtained a commutative algebra.

Finally, as a conclusion, our main result is:

Theorem 1. *Any two-dimensional real division algebra is isomorphic to an algebra of anyone of the 22 types of algebras specified by their corresponding multiplication tables above listed. Only the algebras of types* I_{2121} *with* $\alpha=1$, $n<-1$, I_{2122} *with* $\sigma_1=1/2$, I_{21231} *with* $\sigma_1=1$, I_{21232} *with* $s_1=\sigma_1$ *and* II_{22} *&* II_{23} *with* $s_1=\sigma_1$, $s=\sigma$ *are commutative.*

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Polytechnic Institute Iași
Str. 23 August Nr 11
6600 — Iași