1) K is $\mathcal{P}$-adically closed in K o c h e n-R o q u e t t e sense [7].

2) K is a Becker's generalized real closed field ; thus we get an alternative to the Nullstellensatz over such fields given in [5].

3) k is real closed and G is a Z-group ; in this case we get an alternative to the Nullstellensätze from [6] and [3].

4) K is equipped with an algebraically maximal t-ordering [2].

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VARIETIES OF SMALL DEGREE

BY

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This is a short report on our recent work [7], to which we refer for complete proofs.

Let $X^r_d \subset \mathbf{P}^{r+s}_{\mathbf{C}}$ be a smooth, connected, nondegenerate complex projective algebraic variety of dimension r and degree d. H a r t s h o r n e and B a r t h-V a n d e V e n proved that if d is „small" with respect to r, X is a complete intersection. It is conjectured that the same conclusion holds if s is „small" with respect to r. The result of Hartshorne and Barth-Van de Ven would be a consequence of this conjecture, since one has the elementary inequality $d \geqslant s+1$. Our main result is theorem 1 below, which gives a (sharp) bound such that if d is „small" with respect to s, than both the abstract structure and the embedding of X could be determined. Before stating the theorem we give two definitions. Let $X \subset \mathbf{P}^n$ be a smooth, connected, projective variety of dimension $r \geqslant 2$.

Definitions. a) *X is called a scroll if there is a surjective morphism $f . X \to Y$ onto some smooth variety Y of dimension $s > 0$, such that the fibres of f are linear varieties of dimension $r - s > 0$.*

b) *X is called a hyperquadric fibration if there is a surjective morphism $f : X \to C$ onto a smooth curve C such that the closed fibres of f are hyperquadrics. In this case one can prove that the singular fibres of f are ordinary cones.*

Theorem 1. *Let $X^r_d \subset \mathbf{P}^{r+s}_{\mathbf{C}}$ be as above, $r \geqslant 2$; let H be its general hyperplane section.*

If $d \leqslant 2s+1$, X is one of the following :

a) *A (birationally) ruled surface;*

b) *A scroll over a curve or a (birationally) ruled surface ;*

c) *A hyperquadric fibration ;*

d) *A variety with $\omega_X^{-1} \simeq \mathcal{O}_X(H)^{r-1}$; these are completely classified* (see [8], [4], [5]) ;

e) $\mathbf{P}^r$, $H \in |\mathcal{O}(1)|$, *or Q^r (the hyperquadric), $H \in |\mathcal{O}(1)|$;*

f) *There is a morphism $\sigma : X \to X'$ expressing X as the blowing up of X' with center finitely many points $P_1, \ldots, P_k$ and a divisor H' on X' with $H = \sigma^*(H') - \sigma^{-1}(P_1) - \cdots - \sigma^{-1}(P_k)$ such that :*

$$X' \simeq \mathbf{P}^3,\ H' \in |\mathcal{O}(3)|,\ or\ X' \simeq Q^3,\ H' \in |\mathcal{O}(2)|,\ or\ X' \simeq \mathbf{P}^4,$$

$H' \in |\mathcal{O}(2)|$, or X' is a $\mathbf{P}^2$-bundle over a smooth curve such that $\mathcal{O}_X(H')$ induces $\mathcal{O}(2)$ on each fibre :

If $d=2s+2$ and X is not as above, then either:

g) *X is a $K3$ surface, or*

h) *$r\geq 3$, X satisfies $\omega_X^{-1}\simeq \mathcal{O}_X(H)^{r-2}$.*

Since surfaces of degree 5 in $\mathbf{P}^3$ are of general type, the bound $d\leq 2s+2$ is sharp. The above result „explains" the possibility of classifying varieties of small degree. Note that if $s\geq 3$ and $d\leq 8$ we are in the range $d\leq 2s+2$, while for $s=2$ some specific techniques (such as „linkage") are available. This makes $d=8$ a natural bound. In [6] we have treated the cases when $d\leq 7$. The case $d=8$ will be published separately.

An important step in proving theorem 1 is theorem 2 below which describes all (smooth) threefolds whose hyperplane section is a (birationally) ruled surface.

Theorem 2. *Let X be a smooth, connected, complex projective threefold and H a smooth very ample divisor on it. Then, H is a (birationally) ruled surface if and only if X is of one of the following types:*

a) *A scroll over a curve or over a (birationally) ruled surface;*

b) *A hyperquadric fibration;*

c) *A threefold with $\omega_X^{-1}\simeq \mathcal{O}_X(H)^2$; these are classified in* [8];

d) *P^3, $H\in|\mathcal{O}(1)|$ or Q^3, $H\in|\mathcal{O}(1)|$;*

e) *There is a morphism $\sigma: X\to X'$ expressing X as the blowing up of X' with center finitely many points $P_1, \ldots, P_k$ and a divisor H' on X' with $H=\sigma^*(H')-\sigma^{-1}(P_1)-\ldots-\sigma^{-1}(P_k)$ such that:*

$X'\simeq P^3$, $H'\in|\mathcal{O}(3)|$ or $X'\simeq Q^3$, $H'\in|\mathcal{O}(2)|$, or X' is a $\mathbf{P}^2$-bundle over a smooth curve such that $\mathcal{O}_{X'}(H')$ induces $\mathcal{O}(2)$ on each fibre.

Theorem 2 may be viewed as a refinement, when H is a very ample divisor, of a result proved differently by S o m m e s e (see [11], Th. (2.4)) in case of ample divisors.

The proof of both theorems depends on the study of the adjunction mapping (see [10], [12], [6]). Moreover, to prove theorem 2 one needs the powerful results due to S. M o r i [9] on threefolds whose canonical bundles are not numerically effective and a theorem of L. B ă d e s c u (see [1], [2], [3]) which describes all (smooth) threefolds supporting a geometrically ruled surface as an ample divisor.

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