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A CERTAIN DESINGULARIZATION THEOREM

BY

MIHAI CIPU and DORIN POPESCU

The Implicit Function Theorem (shortly IFT) states that for a system of polynomials $f=(f_1, f_2, \ldots, f_r)$ in $Y=(Y_1, \ldots, Y_N)$, $r \leq N$, over a henselian local ring $(A, \underline{m})$, every solution $\bar{y}$ of f in $A/\underline{m}$ satisfying

$$(*) \qquad \operatorname{rank}\left(\frac{\partial f}{\partial Y}(\bar{y})\right)=r,$$

can be lifted to a solution y of f in A. In other words, the IFT says that some special systems of polynomials (the number of equations is less or equal to the number of variables) which have some special solutions in $A/\underline{m}$ (satisfying (*)) have also solutions in A. Trying to apply the IFT, the difficult point is to check the condition (*). To this end, a useful tool is the following jacobian criterion of smoothness:

Let A be a ring, $Y=(Y_1, \ldots, Y_N)$ some variables, I a finitely generated ideal of $E:=A[Y]$, $\underline{q}$ a prime ideal from $B:=E/I$ and $\underline{p}:=E \cap \underline{q}$. Then $B_{\underline{q}}$ is a smooth A-algebra iff there exists a system f of m polynomials from I such that

1) $(f)\ E_{\underline{p}}=I_{\underline{p}}$,

2) *the jacobian matrix $(\partial f/\partial Y)$ has an $m \times m$-minor which is invertible in $E_{\underline{p}}$.*

In the above notations, B is a *standard smooth algebra over A* if there exist a system of polynomials $g=(g_1, \ldots, g_r)$ from I and a polynomial h from the ideal Δ_g generated by all $r \times r$-minors from the jacobian matrix $(\partial g/\partial Y)$ such that $B=(E/(g))_h$.

Since the system of polynomials f from $A[Y]$ has a solution in the ring A' iff there exists a morphism of A-algebras from $A[Y]/(f)$ to A', the attempt to reduce the solvability of some polynomial equations to the solvability of another ones for which it is possible to apply the IFT leads to the following:

Question 1. *Given a commutative diagram (D) of A-algebras with B of finite presentation over A,*

$$(D) \qquad \begin{array}{ccc} A & \rightarrow & B \\ & \searrow \ \swarrow \alpha & \\ & A' & \end{array}$$

are there a standard smooth A-algebra B' and two A-morphisms $\beta: B \to B'$, $\gamma: B' \to A'$ such that the diagram (D') commutes?

$$(D') \qquad \begin{array}{ccccc} A & \rightarrow & B & \xrightarrow{\beta} & B' \\ & \searrow & \downarrow \alpha & \swarrow \gamma & \\ & & A' & & \end{array}$$

There are known several circumstances in which the above problem has a positive answer. For example, N é r o n 's p-desingularization [N] says that an unramified extension of discrete valuation rings $R \to R'$ inducing separable extensions on fraction and residue fields is a filtered inductive limit of smooth R-algebras of finite type. This result has been generalized in various ways by many authors (see [K], [B], [P_1], [AD], [Ni]).

A positive answer to Question 1 is the following:

Theorem 2. ([P_2] Th. 2.5). *Let $u: A \to A'$ be a morphism of noetherian rings. Then the following statements are equivalent:*

(i) *u is regular (i.e. for every prime ideal $\underline{q}$ of A' the induced morphism $u_{\underline{q}}: A_{u^{-1}\underline{q}} \to A'_{\underline{q}}$ is formally smooth);*

(ii) *u is a filtered inductive limit of smooth morphisms of finite type;*

(iii) *A' is a filtered inductive limit of standard smooth A-algebras,*

(iv) *for every A-algebra B of finite type and every commutative diagram (D) there exist a standard smooth A-algebra B' and two A-morphisms $\beta: B \to B'$; $\gamma: B' \to A'$ such that the diagram (D') commutes.*

The purpose of this paper is to show that the condition (IV) may be strengthened requiring for β to be as smooth as possible-roughly speaking, except only the singular (nonsmooth) locus of B over A. Let $f=(f_1, \ldots, f_m)$ be a system of polynomials in some variables $Y=(Y_1, \ldots, Y_N)$ over a ring A. Given a system $g=(g_1, \ldots, g_r)$, $r \leq N$ of r-polynomials from the ideal (f), we consider the ideal $H_f := \sqrt{(f) + \sum_g \Delta_g((g):(f))}$, the sum being taken over all systems g of r-polynomials from (f), $1 \leq r \leq N$. The ideal H_f defines the singular (nonsmooth) locus of $B:=A[Y]/(f)$, i.e. if $\underline{q} \in \operatorname{Spec} B$, then $B_{\underline{q}}$ is smooth over A iff $\underline{q} \not\supseteq H_f B$ (simply look at the jacobian criterion of smoothness mentioned above). Then $H_{B/A} := H_f B$ does not depend on the presentation chosen for B over A.

Now the above Theorem may be completed with the following equivalent statement:

(v) *for every A-algebra B of finite type and every A-morphism $\alpha: B \to A'$, there exist a standard smooth A-algebra B' and two A-morphisms $\beta: B \to B'$, $\gamma: B' \to A'$ such that the diagram (D') commutes and $\alpha(H_{B/A}) \subset \sqrt{\gamma(H_{B'/B})}$ (i.e. for every $\underline{q} \in \operatorname{Spec} A'$, the algebra $B'_{\gamma^{-1}\underline{q}}$ is smooth over B iff $B_{\alpha^{-1}\underline{q}}$ is a smooth A-algebra).*

In this form, the Theorem is a positive answer to a conjecture due to M. A r t i n [A]. For some applications of this result see [P_2].

The first step in the proof of the implication (i)⇒(v) is to reduce the problem to the case $A=A'$, $u=1_A$. The next step is to apply:

Proposition 3. *Let A be a ring, B a finitely presented A-algebra, $\alpha: B \to A$ an A-morphism and $I \subset \alpha(H_{B/A})$ a finitely generated ideal of A. Then there exist a standard smooth A-algebra B' and two A-morphisms $\beta: B \to B'$, $\gamma: B' \to A$ such that $I \subset \sqrt{\gamma(H_{B'/B})}$ and $\alpha=\gamma\beta$.*

Remark. When A is a normal domain, this result is proved in ([AD], 2.3).

The proof of the above proposition is given in ([CP], (2.6)+(2.7)).

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Institute of Mathematics
University of Bucharest

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JACOBIAN LOCI AND DETERMINANTAL FORMULA

BY

M. BECHEANU

The jacobian loci of the linear systems of divisors on an algebraic variety X were used by J. A. Todd [7] and M. Eger [2] to define the canonical classes of X. There are well-known formulas connecting the canonical classes of Todd and Eger, the Chern classes of X and the jacobian cycles (see [4] for the formula (III. 42) and the history of the problem).

In this note we give a simple geometric description of the jacobian cycles which gives us the possibility to calculate their equivalence class in terms of Chern classes. This method was used before in [1] and is a generalization of a construction of Gh. Galbura [3].

Let X be a smooth algebraic quasi-projective variety of dimension n over an algebraically closed field k and O_X the sheaf of rational functions on X. We denote by $A(X)$ the Chow ring of X graded by codimension i.e. $A(X)=\bigoplus_{d=0}^{n} A_d(X)$, where $A_d(X)$ is the group of cycles of codimension d on X modulo rational equivalence. For a locally free sheaf of O_X-modules, we denote by $c_i(E)$, $1\leq i\leq n$, the Chern classes of E and by $c_t(E)=\sum_0^n c_i(E)\ t^i$ the Chern polynomial of E. Then the inverse formal power series $c_t(E^*)^{-1}=$ $=s_t(E)=\sum_0^\infty s_i(E)\ t^i$ defines the Segre classes $s_i(E)$ of E.

We will use the following form of the Thom-Porteous formula (see [5]).

Theorem 0. *Let $u: E\to F$ be a homomorphism of locally-free sheaves on X of rank p and q and $p-q\leq i\leq p$. Denote by $Z_i(u)$ the scheme of zeros of the map $\bigwedge^{p-i+1} u$. If $Z_i(u)$ is empty or of codimension $i(q-p+i)$ in X then the cycle associated to $Z_i(u)$ is given by the determinantal formula*

$$\text{Cycle } Z_i(u)=\begin{vmatrix} a_{q-p+i} & a_{q-p+i+1} & \cdots & a_{q-p+2i-1} \\ a_{q-p+i-1} & a_{q-p+i} & \cdots & a_{q-p+2i-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{q-p+1} & a_{q-p+2} & \cdots & a_{q-p+i} \end{vmatrix},$$

where a_j are the coefficients of the product polynomial $c_t(F)\,\sigma_t(E)^{-1}$.

Let L be an invertible sheaf of O_X-modules. If $V\subset\Gamma(X, L)$ is a subvector space of dimension $r+1$, it defines a linear system $|V|$ of effective divisors linearly equivalent with a given divisor D such that $L\simeq L(D)$.