Remark. When A is a normal domain, this result is proved in ([AD], 2.3).

The proof of the above proposition is given in ([CP], (2.6)+(2.7)).

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JACOBIAN LOCI AND DETERMINANTAL FORMULA

BY

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The jacobian loci of the linear systems of divisors on an algebraic variety X were used by J. A. Todd [7] and M. Eger [2] to define the canonical classes of X. There are well-known formulas connecting the canonical classes of Todd and Eger, the Chern classes of X and the jacobian cycles (see [4] for the formula (III. 42) and the history of the problem).

In this note we give a simple geometric description of the jacobian cycles which gives us the possibility to calculate their equivalence class in terms of Chern classes. This method was used before in [1] and is a generalization of a construction of Gh. Galbura [3].

Let X be a smooth algebraic quasi-projective variety of dimension n over an algebraically closed field k and O_X the sheaf of rational functions on X. We denote by $A(X)$ the Chow ring of X graded by codimension i.e. $A(X)=\bigoplus_{d=0}^{n}A_d(X)$, where $A_d(X)$ is the group of cycles of codimension d on X modulo rational equivalence. For a locally free sheaf of O_X-modules, we denote by $c_i(E)$, $1\leqslant i\leqslant n$, the Chern classes of E and by $c_t(E)=\sum_0^n c_i(E)\ t^i$ the Chern polynomial of E. Then the inverse formal power series $c_t(E^*)^{-1}=$ $=s_t(E)=\sum_0^\infty s_i(E)\ t^i$ defines the Segre classes $s_i(E)$ of E.

We will use the following form of the Thom-Porteous formula (see [5]).

Theorem 0. *Let $u: E\to F$ be a homomorphism of locally-free sheaves on X of rank p and q and $p-q\leqslant i\leqslant p$. Denote by $Z_i(u)$ the scheme of zeros of the map $\bigwedge^{p-i+1} u$. If $Z_i(u)$ is empty or of codimension $i(q-p+i)$ in X then the cycle associated to $Z_i(u)$ is given by the determinantal formula*

$$\text{Cycle } Z_i(u)=\begin{vmatrix} a_{q-p+i} & a_{q-p+i+1} & \cdots & a_{q-p+2i-1} \\ a_{q-p+i-1} & a_{q-p+i} & \cdots & a_{q-p+2i-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{q-p+1} & a_{q-p+2} & \cdots & a_{q-p+i} \end{vmatrix},$$

where a_j are the coefficients of the product polynomial $c_t(F)\, s_t(E)^{-1}$.

Let L be an invertible sheaf of O_X-modules. If $V\subset\Gamma(X, L)$ is a subvector space of dimension $r+1$, it defines a linear system $|V|$ of effective divisors linearly equivalent with a given divisor D such that $L\cong L(D)$.

We define $\dim |V| = r$. Given a global section $s \in \Gamma(X, L)$ we denote by (s) the effective divisor of zeros of s.

Definition. *A point $x \in X$ is a jacobian point of the linear system $|V|$ if there is a section $s \in V$ such that $(s) = D$ and x is a singular point of the scheme (D, O_D).*

Geometrically, this means that there is a divisor class $D \in |V|$ such that a simple component of D has a singularity in x or x is a point of a multiple component of D or there is at least two components of D through x. We denote by $J(V)$ the closed set of jacobian points of V. The first problem is to give for $J(V)$ a natural structure of scheme. For this we prove the following theorem :

Theorem 1. *Under the above assumptions, let $|V|$ be a linear system of dimension r on X. Then there exist a locally free sheaf F on X of rank $n+1$ and a homomorphism $u: F \to O_X^{r+1}$ such that $J(V) = Z_{n-r+1}(u)$ as a set.*

Proof. One considers an open covering of X by affine sets $\{U_\alpha\}$ such that the sheaves L and Ω_X^1 are trivial on every U_α. Let $s_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, O_X^*)$ the transition functions of L and $\{x_1^\alpha, \ldots, x_n^\alpha\}$ be a local system of parameters of X on U_α. We define the modules $E_\alpha = L | U_\alpha \oplus \Omega_X^1 | U_\alpha$, α, and we patch them together on $U_\alpha \cap U_\beta$ using the automorphisms $f_{\beta\alpha}: E_\alpha | U_\alpha \cap U_\beta \to$ $\to E_\beta | U_\alpha \cap U_\beta$ defined by $f_{\beta\alpha}(s_\alpha) = s_{\beta\alpha} s_\alpha \oplus ds_{\beta\alpha}$, $f_{\beta\alpha}(dx_i^\alpha) = s_{\beta\alpha} \sum_1^n (\partial x_i^\alpha / \partial x_j^\beta) dx_j^\beta$. One obtains a locally free sheaf E on X of rank $n+1$. If $s \in \Gamma(X, L)$ is a global section and $s | U_\alpha = s_\alpha$, one has $s_\alpha \oplus ds_\alpha \in \Gamma(U_\alpha, E)$. The local sections $s_\alpha \oplus ds_\alpha$ patch together to a global section $\bar{s} \in \Gamma(X, E)$. Moreover, we have the following characterisation of the jacobian points of V: $x \in J(V)$ iff there is a section $s \in V$ such that $\bar{s}(x) = 0$. Let $s_0, s_1, \ldots, s_r$ be a k-linear basis of V. Then $x \in J(V)$ iff the global sections $\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_r$ are linearly dependent in $E(x)$. That means that the matrix :

$$\left\| \begin{matrix} s_0 & \partial s_0/\partial x_1 & \cdots & \partial s_0/\partial x_n \\ s_1 & \partial s_1/\partial x_1 & \cdots & \partial s_1/\partial x_n \\ \cdots & \cdots & \cdots & \cdots \\ s_r & \partial s_r/\partial x_1 & \cdots & \partial s_r/\partial x_n \end{matrix} \right\|$$

drops rank in the point x. The sections $\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_r$ define a homomorphism $v: O_X^{r+1} \to E$ where O_X^{r+1} is the trivial sheaf. Considering the dual sheaves and the dual homomorphism $v^* = u$, one obtains $u: E^* \to O_X^{r+1}$ such that $J(V) =$ $= \{x \in X \mid u(x)$ is not surjective$\} = \{x \in X \mid \bigwedge^{r+1} u(x) = 0\} = Z_{n-r+1}(u)$. Thus the required sheaf is $F = E^*$ and u is locally given by the above matrix.

Corollary. *There is an exact sequence of O_X-modules $0 \to \Omega_X^1 \otimes L \to E \to L \to 0$.*
This results from the construction of E.

It is natural to consider $J(V)$ with the structure scheme of $Z_{n-r+1}(u)$ which is independent of the choice of a basis of V.

Theorem 2. *Let X, L and V as above and suppose $r \leqslant n$. If the codimension of $J(V)$ in X is $n-r+1$ then the class of* Cycle $J(V)$ *is given by the formula :*

$$\text{Cycle } J(V) = c_{n-r+1}(E) = \sum_{i=0}^{n-r+1} \binom{r+i}{r} c_{n-r+1-i}(\Omega_X^1)\, c_1(L)^i.$$

Proof. By theorems 0 and 1, one has

$$\text{Cycle } J(V) = \begin{vmatrix} s_1 & s_2 & \cdots & s_{n-r+1} \\ 1 & s_1 & \cdots & s_{n-r} \\ 0 & 1 & \cdots & s_{n-r-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots 1 & s_1 \end{vmatrix}$$

where $s_i = s_i(E)$ are the Segre classes of E because $c_t(O_X^{r+1})\, c_t(F)^{-1} = c_t(E^*)^{-1} =$ $= s_t(E)$. Denote $c_i = c_i(E)$.
In the above determinant we multiply for all $2 \leqslant i \leqslant n-r+1$ the i-row by $(-1)^{i-1} c_{i-1}$ and add all to the first row. Using the above identities we obtain the formula

$$\text{Cycle } J(V) = \begin{vmatrix} 0 & 0 & \cdots & (-1)^{n-r} c_{n-r+1} \\ 1 & s_1 & \cdots & s_{n-r} \\ 0 & 1 & \cdots & s_{n-r-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots 1 & s_1 \end{vmatrix} = c_{n-r+1}(E).$$

From the exact sequence in the corollary we have $c_t(E) = c_t(\Omega_X^1 \otimes L)\, c_t(L)$ and then $c_{n-r+1}(E) = c_{n-r+1}(\Omega_X^1 \otimes L) + c_1(L)\, c_{n-r}(\Omega_X^1 \otimes L)$. Using the Chern class formula for a tensor product by an invertible sheaf: $c_p(\Omega_X^1 \otimes L) =$ $= \sum_{i=0}^{p} \binom{n-p+i}{i} c_{p-i}(\Omega_X^1)\, c_1(L)^i$, one deduces the required formula

$$c_{n-r+1}(E) = \sum_{i=0}^{n-r+1} \binom{r+i}{r} c_{n-r+1-i}(\Omega_X^1)\, c_1(L)^i.$$

Remark. The exact sequence given in the corollary gives the possibility to obtain the inverse formula connecting the canonical classes $c_p(\Omega_X^1)$ and jacobian cycles. Assume $0 \leqslant r \leqslant n$ and let $|V_i|$ be a family of linear systems, where $0 \leqslant i \leqslant n-r$. We assume that $\dim |V_i| = r+i+1$ and that codimension of $J(V_i)$ in X is $n-r-i$. Then $c_{n-r}(\Omega_X^1) = \sum_{i=0}^{n-r} \binom{r+i+1}{r+1}$ Cycle $J(V_i)\, c_1(L^{-1})^i$.

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