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ULTRAPRODUCTS AND HOCHSTER'S MODIFICATIONS

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Let $(R, \underline{m})$ be a noetherian local ring and $x=(x_1, \ldots, x_n)$ a system of parameters on R. Let M be an R-module, a an element from M, $k \in \{0,., n-1\}$ and $r=(m_1, \ldots, m_{k+1}) \in M^{k+1}$ a system of elements such that $\sum_{i=1}^{k+1} x_i m_i = 0$.

Consider the R-module $M' := M \oplus R^k / R(m_{k+1} + (x, \ldots, x)))$ and let a' be the image of a in M' under the composed map $M \to M \oplus R^k \to M'$. Then (M', a') is a *modification of type k of* (M, a) *corresponding to* x. If

$$(M, a) = (M_0, a_0) \to (M_1, a_1) \to \ldots \to (M_t, a_t) = (M', a')$$

is a sequence, where for all i (M_i, a_i) is a modification of type k_i of (M_{i-1}, a_{i-1}) corresponding to x, then (M', a') *is a modification of type* $(k_1, \ldots, k_t)$ *of* (M, a) *corresponding to* x (shortly a *modification of* (M, a)). A modification (M', a') of $(R, 1)$ is *good* if $a' \notin (x)M'$, otherwise it is *bad*. An R-module is a *big CM-module corresponding to* x (*shortly a big CM-module*) if x is an M-sequence.

The main results on modifications are given by the following theorems due to Hochster [H₁], [H₂].

Theorem A. *Let R be a noetherian local ring and x a system of parameters on R. Then the following statements are equivalent:*

i) *every modification of $(R, 1)$ corresponding to x is good,*

ii) *there exists a big CM-module corresponding to x.*

Theorem B. *Let R be a noetherian ring containing a field and x a system of parameters on R. Then every modification of $(R, 1)$ corresponding to x is good. Hence there exists a big CM-module corresponding to x (see Theorem A).*

Remark. There are many types for which it is easy to check that the modifications are always good, for example $\underbrace{(1, \ldots, 1)}_{t\text{-times}}$.

Let $(R, \underline{m})$ be a noetherian local ring with $\dim R = n$, x a system of parameters on R and $k = (k_1, \ldots, k_t)$, $0 \leq k_i \leq n-1$ for all i, a certain type of modification. Define $\mu(x) := \min \{r \in \mathbb{N} \mid \underline{m}^r \subseteq (x) R\}$,

$$\mu(R) := \min \{\mu(y) \mid y \text{ is a system of parameters on } R\}.$$

Using ultraproducts (see [AK], [BDLV]) we obtain the following result (see [MP] for its proof).

Theorem 1. *Let $n, r, s, t, k_1, \ldots, k_t$ be some nonnegative integers such that $n \leq r$, $n, s, t \geq 1$ and for all i, $0 \leq k_i \leq n-1$. Then there exists a prime number p such that for every noetherian local ring $(R, \underline{m})$, and for every system of parameters x on R satisfying*

i) $\dim R = n$ *and* $\operatorname{rank}_{R/\underline{m}} \underline{m}/\underline{m}^2 \leq r$,

ii) $\mu(x) \leq s$,

iii) $\operatorname{char} R/\underline{m} \geq p$,

every modification of type $(k_1, \ldots, k_t)$ of $(R, 1)$ corresponding to x is good.

REFERENCES

[AK] Ax, J., Kochen, S. — *Diophantine problems over local fields* I, II Amer. J. Math. 87 (1965), 605—648.

[BDLV] Becker, J., Denef, J., Lipshitz, L., van den Dries, L. — *Ultraproducts and approximation in local rings I*, Inventiones Math. 51 (1979), 189—203.

[H₁] Hochster, M. — *Topics in the homological theory of modules over commutative rings.* C.B.M.S. Reg. Conf. Series in Math., No. 24, Amer. Math. Soc. Providence R.I., 1975.

[H₂] Hochster, M. — *Some applications of the Frobenius in characteristic zero*, Bul. A. M.S. 84 (1978), 886—912.

[MP] Mateescu, C., Popescu, D. — *Ultraproducts and big Cohen-Macaulay modules*, Stud. Cerc. Mat. 36 (1984), 424—428.

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