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ABOUT R_n-MORPHISMS

BY

CRISTODOR IONESCU

All rings are assumed to be commutative, with unity and noetherian. The definitions are those of [2].

Let **P** be a property of noetherian local rings (like for instance regular, normal, Gorenstein, Cohen-Macaulay, (R_n), (S_n) etc.). If k is a field, a noetherian k-algebra A is called a *geometrically* **P** k-algebra, if for any field k' finite extension of k, the ring $A\otimes_k k'$ has the property **P**. A ring morphism $u\colon A\to B$ is called a **P**-*morphism* if it is flat and for any prime ideal Q of A, $B\otimes_A k(Q)$ is a geometrically **P** k-algebra. A noetherian ring A is called a **P**-*ring* if for any prime ideal Q of A, the canonical morphism $A_Q\to(A_Q)^\wedge$ is a **P**-morphism. A. Grothendieck was the first who developed the theory of **P**-rings and formal fibers and he thought that, if a noetherian ring A has good formal fibers, then any ideal-adic completion of A has good formal fibers. More generally, in [2], he left us the following questions:

Question 1 ([2], 7.5.4). *Let $u\colon A\to B$ be a flat and local morphism of local rings. Suppose that:*

a) A is a **P**-*ring;*

b) $u\otimes k(\underline{m})\colon A/\underline{m}\to B/\underline{m}B$ is a **P**-*morphism, where $\underline{m}$ is the maximal ideal of A.*

Does it follow that u is a **P**-*morphism?*

Question 2 ([2], 7.4.8). *Let A be a noetherian ring, I an ideal of A, such that A is I-adically complete. If A/I is a* **P**-*ring, does it follow that A is a* **P**-*ring?*

We are now interested in the property (R_n) of Serre. For other properties, one can see [1], [3].

The following proposition, which shows that question 1 has a positive answer, was obtained by the author, together with A. Brezuleanu.

Proposition 1. *Let $u\colon A\to B$ be a local and flat morphism of noetherian local rings. Suppose that: a) A is an R_n-ring; b) $u\otimes k(m)$ is an (R_n)-morphism. Then u is an (R_n)-morphism.*

In [1] we were able to prove proposition 1 only when B is a catenarian domain.

Proposition 1 enables us to give the following answer to question 2, when A is a semilocal ring.

Proposition 2. *Let A be a semilocal ring, I an ideal of A. Suppose that: a) A is I-adically complete; b) A/I is a universally japanese (R_n)-ring. Then A is an (R_n)-ring.*

In the general non-semilocal case, question 2 has a negative answer, but we are able to prove that, in a special case, the answer is affirmative. The result is given by the following proposition.

Proposition 3. *Let A be a noetherian ring, I an ideal of A. Suppose that*

a) A is I-adically complete;

b) For every maximal ideal $\underline{m}$ of A and for any finite $A_{\underline{m}}$-algebra B, Reg B is open in Spec B.

c) For any maximal ideal $\underline{m}$ of A, the completion of $A_{\underline{m}}$ in the radical topology is equidimensional;

d) A/I is an (R_n)-ring.

Then A is an (R_n)-ring.

The condition c) is fulfilled, for example, when A is a universally catenarian domain. The condition c) probably can be replaced by: A is a universally catenarian nor-ring of finite dimension, but the case seems to become too special.

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Institute of Mathematics
University of Bucharest