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ENERGY IN RIEMANNIAN GEOMETRY

BY

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Introduction. The use of energy to identify geometric objects with "nice" properties is nothing new; familiar examples include geodesics, harmonic differential forms, harmonic maps, and more recently Yang–Mills connections. It is the purpose of this paper to describe some new ways of bringing energy into Riemannian geometry, within a unified framework which encompasses many different types of geometric structure. Just about everything we have to say takes place in the smooth (C^∞) category, and will be devoid of all but the simplest details, which may be found elsewhere. We begin with a simple example which illustrates the basic idea.

1. The energy of a unit vector field. A vector field X on a manifold M is a section of the tangent bundle TM; that is, a mapping $X: M \to TM$ which is a right inverse for the natural projection $\pi: TM \to M$. If M has a Riemannian metric g, then TM acquires a compatible metric h, called the Sasaki metric (by "compatible" we mean that $\pi: (TM, h) \to (M, g)$ is a Riemannian submersion). It is therefore possible to form the **energy** of X (assuming for convenience that M is compact, and without boundary):

$$E(X) = \frac{1}{2}\int_M |\, dX\,|^2\, dx$$

where dX is the derivative of the mapping X and dx is the Riemannian volume element of (M, g). We say that X is a **harmonic section of** TM if X is a critical point for E with respect to variations through nearby vector fields, a natural generalization of X being a harmonic map. Now this is a linear variational problem, and may be solved as follows. If

$$dX = d^v X + d^h X$$

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denotes the orthogonal splitting into vertical and horizontal components, then since π is a Riemannian submersion:

$$|d^h X|^2 = |d\pi \circ d^h X|^2 = |d\pi \circ dX|^2 = |1_{TM}|^2 = m, \text{ the dimension of } M.$$

On the other hand, if $K: TTM \to TM$ is the connection map for the Levi–Civita connection then

$$|d^v X|^2 = |K \circ d^v X|^2 = |K \circ dX|^2 = |\nabla X|^2.$$

Therefore

$$E(X) = \frac{1}{2}\int_M |\nabla X|^2\, dx + \frac{m}{2}\mathrm{Vol}(M,g) = Q(X) + \frac{m}{2}\mathrm{Vol}(M).$$

where Q is a quadratic form, which may be extended to the Hilbert space of L_1^2 vector fields. The variational problem is therefore that of criticalizing Q, and this occurs precisely when $Q(X) = 0$, because

$$dQ(X)(X) = 2Q(X).$$

In other words, X is a harmonic section of TM if and only if X is parallel. (This result, and various generalizations, should be attributed to O.Nouhaud [N], T.Ishihara [I] and J.Konderak [K]. The condition $\nabla X = 0$ is of course nothing new, and rather restrictive, especially on compact manifolds with zero first Betti number. A more interesting criterion can be obtained by imposing further constraints. First, note that the Euler-Lagrange equations may be expressed in divergence form after an integration by parts:

$$dE(X)(Y) = \int_M \langle \nabla X, \nabla Y\rangle dx = -\int_M \langle Tr\nabla^2 X, Y\rangle dx$$

so that in the non-compact case X is a harmonic section of TM if and only if

$$0 = \nabla^*\nabla X = -Tr\nabla^2 X, \qquad \text{the so-called } \textbf{rough Laplacian}.$$

Now suppose that the Euler characteristic of M is zero and X is a unit vector field. Say that X is a **harmonic section of T_1M**, the unit tangent bundle, if X is a critical point of E with respect to nearby unit vector fields. Introducing a Lagrange multiplier λ, from the preceding linear analysis, this is the case precisely when

$$(*) \qquad \nabla^*\nabla X = \lambda X = |\nabla X|^2 X.$$

This equation is analogous to the equation for a harmonic map $f: M \to S^n$:

$$(**) \qquad \Delta F = |dF|^2 F,$$

where $F: M \to \boldsymbol{R}^{n+1}$ is the natural composition, and Δ is the Laplace-Beltrami operator. Indeed, the fibres of T_1M are spheres S^{m-1}, and if M was flat we could find local trivializations such that (*) becomes (**). This indicates a second, and more profound way in which harmonic sections may be considered as generalizations of harmonic maps.

Example. If $M = S^{2n+1}$ and X is a Hopf vector field, then it turns out that X is a harmonic section of T_1M (although of course X is not a harmonic section of TM). Details of the calculation may be found in [W7], but in summary, if Ω denotes the pullback to S^{2m+1} of the Kähler 2-form on $\boldsymbol{C}^{n+1}$, then (*) reads

$$\delta\Omega(Y) = 2n\langle X, Y\rangle = 2n\omega(Y), \quad \forall Y \in TS^{2n+1}.$$

Now ω is the Lie algebra–valued 1-form for a connection in the Hopf fibration $S^{2n+1} \to \boldsymbol{CP}^n$, a principal S^1-bundle, and Ω is the curvature form. Moreover $\delta\Omega = 2n\omega$ are the Yang–Mills equations. But ω is known to be a Yang–Mills connection [B]. This link with Yang–Mills theory will reappear later. An interesting open question is whether Hopf vector fields are the only harmonic sections of T_1M.

Having found some critical points, it is natural to ask whether or not they are stable, relative to the variational constraints. It is therefore desirable to know the formula for the second variation. Those familiar with the second variational formula for harmonic maps [S] will expect the appearance of the Riemann curvature tensor R of (M,g), and doubtless make the comparison for themselves:

$$\frac{\partial^2 E(X_{s,t})}{\partial t\partial s}\Big|_{(s,t)=(0,0)} = \int_M \langle \nabla^*\nabla Y - |\nabla X|^2 Y - R(Y,\nabla X)\nabla X, Z\rangle dx$$

where $X_{s,t}$ it a 2-parameter variation through unit vector fields, with

$$Y = \frac{\partial X_{s,t}}{\partial s}\Big|_{(s,t)=(0,0)} \text{ and } Z = \frac{\partial X_{s,t}}{\partial t}\Big|_{(s,t)=(0,0)}$$

and

$$R(Y,\nabla X)\nabla X = \sum_j R(Y,\nabla_{E_j}X)\nabla_{E_j}X$$

for any local orthonormal tangent frame field $\{E_i\}$ on M. See [W7] for further details.

Example (continued). We know that a harmonic map $f: S^m \to N$ is unstable if $m \geq 3$ [X], with E-decreasing variations f_t obtained by "reparametrizing" f using the flow φ_t of a gradient vector field on S^m; thus $f_t = f \circ \varphi_t$. It is clear that this won't work for a Hopf vector field X, because $X \circ \varphi_t$ is no longer a vector field. There is however an obvious modification:

$$X_t(x) = (\pi_t)^{-1} \circ X \circ \varphi_t(x)$$

where π_t is parallel translation in T_1M down the integral curve from x to $\varphi_t(x)$. (Note that the variation field for X_t is ∇X). However, it is now not clear that for small t the energy of X_t is less than that of X. Scrutiny of the second variation reveals that when $n > 1$ this is indeed the case, yielding a $(2n+2)$-dimensional family of E-decreasing variations through unit vector felds. This raises the questions of whether there exist any harmonic sections of T_1M whose energy is less than that of the Hopf vector fields, and whether or not the infimum of E over all sections is the theoretical absolute minimum, $\frac{m}{2}\,\mathrm{Vol}(M)$. When $n = 1$ the second variation of $E(X_t)$ vanishes, allowing the possibility (but of course not proving) that the Hopf vector fields on S^3 are stable harmonic sections of T_1S^3. This is another interesting question. In fact, it was shown by H.Gluck & W.Ziller [GZ] that if we use the volume functional V instead of E, then the Hopf vector fields on S^3 are indeed stable. Interestingly, their technique (based on calibrations) cannot work in higher dimensions, and it has subsequently been shown that Hopf vector fields on S^5 are V-unstable [J]. However, these results for volume cannot be automatically transferred to energy, because as immersions $M \hookrightarrow T_1M$ the Hopf vector fields are not isometric.

2. The Energy of Foliations and Almost-Product Structures. The discussion of unit vector fields may be generalized by observing that T_1M double covers the Grassmann bundle $G_1(TM)$, whose fibres are the 1-dimensional linear subspaces of each tangent space. Then, given a k-dimensional foliation $\mathcal{F}$ of M there is a corresponding section σ of $G_k(TM) \to M$. The Riemannian metric on M induces a compatible metric on $G_k(TM)$, in the same way as the Sasaki metric on TM, and we are therefore again able to measure the energy of σ and look for harmonic sections. It should be pointed out that if σ is a harmonic section, then $\mathcal{F}$ is not necessarily a harmonic foliation in the sense of F.Kamber & P.Tondeur [KT]. (A harmonic foliation in the sense of [KT] is one for which the projection

$TM \to TM/\mathcal{F}$ is a harmonic bundle valued 1-form, with respect to the adapted connection in $TM/\mathcal{F}$. It is easily shown that harmonic foliations are characterized by all leaves being minimal submanifolds). The following two special cases give some idea of the kind of foliated geometry encoded by the harmonic section equations.

Theorem [W2]. *Suppose $\mathcal{F}$ is Riemannian foliation. (By this we mean that the geometric normal bundle is totally geodesic: any geodesic which cuts a leaf orthogonally will do so for all other leaves).*

(i) If the leaves of $\mathcal{F}$ are totally geodesic, then σ is a harmonic section if and only if

$$\mathrm{Ric}(X, Y) = 0, \quad \forall X \in \mathcal{F},\ \forall Y \in \mathcal{F}^\perp,$$

where Ric is the Ricci curvature of (M, g).

(ii) If the normal bundle $\mathcal{F}^\perp$ is integrable, then σ is a harmonic section if and only if

$$\mathrm{Ric}(X, Y) = k\langle \nabla^\perp_X H, Y\rangle, \quad \forall X \in \mathcal{F},\ \forall Y \in \mathcal{F}^\perp,$$

where H is the mean curvature vector of the leaves.

For foliations which are not Riemannian, geometric characterizations of the harmonic section equations get more complicated. We note that (i) gives a direct proof that Hopf vector fields are harmonic sections of T_1S^{2n+1}, since S^{2n+1} is an Einstein manifold. Furthermore, in the following special case the vanishing of the "off-diagonal" components of the Ricci curvature is known to be equivalent to the Yang-Mills equations.

Corollary. *Suppose M is the total space of a fibre bundle with connection, and g is a Kaluza-Klein metric. If $\mathcal{F}$ is the foliation by fibres, then σ is a harmonic section if and only if the connection is Yang-Mills.*

The appearance of $\nabla^\perp H$ in (ii) reminds us of the clasical Ruh-Vilms theorem [RV], which states that the Gauss map $\gamma: M \to G_{m,n}$ of an isometric immersion $f: M \to \boldsymbol{R}^n$ is a harmonic map if and only if f has parallel mean curvature. If the target of f is an arbitrary Riemannian manifold N, the Gauss map may be replaced by the Gauss lift $\gamma: M \to G_m(TN)$, and it is possible to generalize [RV] by considering the condition that γ is a harmonic section of the pullback bundle $f^{-1}G_m(TN)$ [W1].

Given an arbitrary section σ of $G_k(TM)$, there is no reason to expect that σ parametrizes a foliation of M: indeed, all one gets in general is a pair of ortho-complementary distributions on M, otherwise known as an **almost-product structure** (AP-structure). A convenient way to define an AP– structure is by an isometric involution in TM; that is, a section P of $T^*M \otimes TM$ satisfying $P^2 = 1, P \neq \pm 1$ and $\langle PX, PY\rangle = \langle X, Y\rangle$. The distributions are then taken to be the eigenbundles of P. There is a neat characterization of the harmonic section equations in terms of P, which is similar to that for harmonic sections of T_1M (to which it reduces when an eigenbundle is 1– dimensional), and which clearly demonstrates the non–linearity of the system.

Theorem [W5]. *σ is a harmonic section if and only if the rough Laplacian $\nabla^*\nabla P$ commutes with P i.e. $[\nabla^*\nabla P, P] = 0$.*

A more geometric characterization is available if we introduce the **total second fundamental form** α of the AP–structure:

$$\alpha(X, Y) = \frac{1}{4}(\nabla_X P(PY) + \nabla_{PX} P(Y)), \quad \forall X, Y \in TM.$$

The skew–symmetric component of α is the integrability tensor for the distributions, and the symmetric component tels us whether or not the distributions are totally geodesic. The energy of σ may be expressed in terms of α, as follows:

$$E(\sigma) = \int_M |\alpha|^2 \, dx + \frac{m}{2} \operatorname{Vol}(M).$$

Theorem [W8]. *σ is a harmonic section if and only if the endomorphism field $\delta\alpha$ is self-adjoint.*

Example [W5]. An AP–structure is said to be **totally geodesic** if α is skew–symetric. This will be the case for any left–invariant AP-structure on a Lie group with bi–invariant metric, since in this case

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \text{for all left-invariant vector fields } X, Y$$

which is skew–simetric. (Similarly for right–invariant AP–structures). Neither distribution need be integrable. Examples arise in clasical mechanics, from systems with non–holonomic constraints. For example, suppose that two spheres in $\boldsymbol{R}^3$ of radii r_1 and r_2 have a point of contact, and are allowed to rotate about their centres subject to the constraint that the point of contact is "absolutely rough". The configuration space is $M = SO(3) \times SO(3)$, equipped with a metric which reflects the relative diameters of the spheres, and the constraint defines a 3–dimensional distribution, which is integrable if and only if $r_1 = r_2$. It turns out that σ is also a harmonic section of

$G_3(TM)$ if and only if $r_1 = r_2$. One wonders whether the harmonic section equations inevitably force integrability. However, it is possible to construct another 3– dimensional AP–structure on $SO(3) \times SO(3)$ which is totally non integrable, and yet parametrized by a harmonic section.

3. The Energy of an Almost–Complex Structure. Let J be an almost structure on M, and suppose g is chosen so that J is orthogonal; otherwise said, J is an **almost–Hermitian structure** on (M, g). Then J parametrized by a section σ of the homogeneous fibre bundle associated to the orthonormal frame bundle of (M, g), with fibre $O(m)/U(k)$ where $2k = m$. We refer to this bundle as the **twistor bundle** for (M, g). If we define $P = -iJ$, then P is an AP–structure in the complexified tangent bundle $T^{\boldsymbol{C}}M$, and it can be shown that σ is a harmonic section of the twistor bundle if and only if the section $\tilde{\sigma}$ parametrizing P is a harmonic section of the complex Grassmann bundle $G_k(T^{\boldsymbol{C}}M)$. From a previous Theorem it follows that:

Theorem. *σ is a harmonic section and only if $[\nabla^*\nabla J, J] = 0$.*

Clearly, every Kähler structure ($\nabla J = 0$) is parametrized by harmonic section. In general, the energy functional measures the deviation from Kählerness:

$$E(\sigma) = \frac{1}{8}\int_M |\nabla J|^2 \, dx + \frac{m}{2} \operatorname{Vol}(M, g).$$

It should be pointed out that a variational problem for almost–Kähler structures has been propounded by D.Blair & S.Ianus [BI]. The energy is essentially the same as above, but variations involve both J and g, couple by the constraint that the symplectic 2–form is left invariant. Critical points for this variational problem are the almost–Kähler manifolds (M, g, J) with invariant Ricci tensor:

$$\operatorname{Ric}(JX, JY) = \operatorname{Ric}(X, Y), \quad \forall X, Y \in TM.$$

It turns out that these do not correspond to harmonic sections, unless of course (M, g, J) is Kähler.

Recall that an almost–Hermitian structure is said to be **integrable** or **(1,2)–symplectic** according as

$$\nabla_{JX} J(JY) = \nabla_X J(Y) \text{ or } \nabla_{JX} J(JY) = -\nabla_X J(Y).$$

Otherwise said, the vertical differential $d^v\sigma$ of the section parametrizing J is holomorphic or anti– holomorfic respectively, with respect to J on M and the natural Hermitian symmetric space structure on the fibre $O(m)/U(k)$. (Another way of saying this is that σ is J_1-holomorphic or J_2-holomorphic, respectively). It is known that holomorphic or anti-holomorphic *mappings* between almost-Hermitian manifolds are E-minimizing, under certain conditions [L] (namely, when the domain is cosymplectic, and the codomain is (1,2)-symplectic). The techniques of [L] can be adapted to give necessary and sufficient conditions for σ to be a harmonic section, and in some cases a local minimum. In contrast to mappings, the only case when σ is automatically a harmonic section is when J is a nearly-Kähler struture, and in this case σ need not be a local minimum (see below). We denote by ϕ the 2-form $\mathcal{R}(\Omega)$ on M, where Ω it the Kähler 2-form, and $\mathcal{R}$ is the curvature operator mapping 2-forms to 2-forms. Call ϕ the **Ricci form** of (M, g, J). Let Φ denote the corresponding endomorphism field: thus

$$\Phi(X) = \frac{1}{2}\sum_j R(E_j, JE_j)X.$$

Say that ϕ is **reducible** if it has type (1,1): equivalently, if Φ commutes with J. Another way to say this is that the * **Ricci curvature** is invariant:

$$\mathrm{Ric}^*(JX, JY) = \mathrm{Ric}^*(X, Y).$$

Theorem [W6]. *(i) If J is integrable then σ is a harmonic section if and only if $[J, \Phi] = \nabla_{\delta J}J$, where δJ is the vector field dual to the 1-form $\delta\Phi$ (the coderivative).*

(ii) If J is (1,2)-symplectic then σ is a harmonic section if and only if ϕ is reducible.

(iii) If J is integrable and cosymplectic, or (1,2)-symplectic, then σ minimizes energy in its vertical homotopy class if $\phi = 0$.

Since every nearly-Kähler structure is known to have reducible Ricci form [G], it follows from (ii) that the corresponding σ is a harmonic section. However, the well-known nearly-Kähler structure on S^6 is E-unstable, by an argument similar to that mentioned in §1 for Hopf vector fields [W4]. It is not surprising to find that the same situation arises with the Calabi-Eckmann complex structures on products of odd-dimensional spheres [W6], whose parametrizing sections are all harmonic by (i), but unstable except in the low-dimensional cases $S^1 \times S^3$ and $S^3 \times S^1$ (and of course $S^1 \times S^1$) where the gradient flow technique for generating E-decreasing variations is inconclusive. (It should be mentioned that the open questions raised for Hopf vector fields are also relevant here). From (ii) it follows that another

example of a harmonic section σ is that parametrizing the almost-Kähler structure J of E.Abbena [A] on the symplectic 4-manifold of W.Thurston [T]. Moreover, although not a consequence of (iii), it turns out that σ is a local minimum of E. It is interesting to note that the Ricci curvature is not J-invariant, so J is not a critical point of the Blair-Ianus variational principal.

The hypothesis $\phi = 0$ required in (iii) for σ to be an E-minimizer is rather strong. The following result describes some weaker conditions, under which σ is a local E-minimizer.

Theorem [W6]. *Suppose that the almost-Hermitian structure J parametrizing by σ has either of the fillowing properties:*

(i) J is 4-dimensional almost-Kähler, and ϕ is anti-self-dual;

(ii) $\phi = f \cdot \Omega$ for some smooth function $f\colon M \to \mathbf{R}$, where $f \geq 0$ if J is integrable and cosymplectic, or $f \leq 0$ if J is (1,2)-symplectic.
Then σ is a local minimum of E.

Regarding (i), we note that in 4-dimensions every (1,2)-symplectic structure is almost-Kähler. The relation $\phi = f \cdot \Omega$ in (ii) is equivalent to the proportionality of Ric^* and g, and is therefore sometimes called the bf * Einstein condititon: in contrast to an Einstein manifold, the function f need not be constant. The nearly-kähler structure on S^6 is Einstein whit $f > 0$, and therefore provides a partial converse to part (ii) of the Theorem. It is interesting to note that the Abbena-Thurston almost-Kähler structure satisfies neither (i) nor (ii). In fact, the existence of almost-Hermitian structures with either of these properties is another open problem.

Conclusion. We have described how the notion of energy may be introduced into some familiar areas of Riemannian geometry, and shown that the geometric structures corresponding to critical points satisfy conditions which are in many cases novel, and interesting enough to warrant further study. There is the potential for new applications of the calculus of variations to Riemannian geometry, and conversely the possibility of using geometric techniques to construct harmonic sections, which are generalization of harmonic maps. (The existence problem for harmonic maps, and hence also for harmonic sections, is a major question. The familiar heat flow method may be applied successfully to sections of fibre bundles with compact non-positively curved fibres [W3], but when the fibres have some positive curvature there is little than can be said in general). The scenario extends to any Riemannian G-structure, which may be parametrized by a section σ of the homogeneous fibre bundle $O(M, g)/G \to M$, where $O(M, g)$ is the principal bundle of orthonormal tangent frames of (M, g). It is possible to give a general characterization of the harmonic section equations

[W8]; indeed, one which is valid for harmonic sections of any reductive homogeneous bundle over M, generalizing a characterization of harmonic maps into a reductive Riemannian homogeneous space given by J.Rawnsley [R].

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