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FROM PRINCIPAL CONNECTIONS TO CONNECTIONS ON PRINCIPAL SHEAVES

BY

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Introduction. The most convenient way to handle a connection on a *vector bundle* is by the corresponding operator of covariant differentiation, acting on the (sheaf) of sections of the bundle. This is also supported by various applications in pure mathematics and theoretical physics.

On the other hand, connections on a *principal bundle* are mainly handled by global and local connection forms, whereas an analogous operator, acting on its sections, is still missing.

The purpose of the present note is to fill in this gap and to establish an *operator-like* description of connections on principal bundles within a *sheaf-theoretic* framework. We note that an early use of sheaf-theoretic methods appears in [1], in which principal connections are considered as sections of the so-called *sheaf of connections*. As it is formally shown in Theorem 3.2 below, the two approaches are equivalent.

Apart from giving a new definition of connections (in spite of M. Spivak's call for ... the summary execution of any geometer proposing such a thing (see [12: vol.V, p.602])), the present setting has another significance: it may serve as the model for the development of a theory of connections on abstract *principal sheaves of structure type a sheaf of groups and with structure sheaf*, the later being defined in the sense of A.Grothendieck [7]. This is the purpose of an ongoing research (cf. [11]), which will be briefly discussed in the end of the present note.

The previous idea is the natural outgrowth and generalization of an analogous geometric structure on *vector sheaves*, originally due to and developed by A.Mallios (cf. [10], [11]). The author is indebted to him for many

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exciting and fruitful discussions on this subject.

1. Preliminaries. In this section we fix the main notations and terminology used throughout. For convenience we are working in the category of smooth real manifolds and bundles. For the theory of sheaves employed hereafter, the reader is referred to [4] and [5].

(A) Let (P, G, X, π) be a principal fibre bundle. We denote by $\mathcal{P}$ the *sheaf of germs of smooth sections* of P, i.e. $\mathcal{P}$ is generated by the presheaf of smooth sections $U \longrightarrow \Gamma(U, P)$ of P, for all open subsets U of X. Thus, for every open $U \subseteq X$,

$$\mathcal{P}(U) \cong \Gamma(U, P). \tag{1.1}$$

We let $\mathcal{A}$ stand for the *sheaf of germs of smooth mappings* on X, customarily written also as $\mathcal{C}^{\infty}_X$. Then, as before,

$$\mathcal{A}(U) := \mathcal{C}^{\infty}_X(U) \cong C^{\infty}(U, \boldsymbol{R}). \tag{1.2}$$

Analogously, $\mathcal{G}$ (resp. $\mathcal{L}$) denotes the *sheaf of germs of smooth G-valued* (resp. $\boldsymbol{G}$-valued) *mapings* on X, if $\boldsymbol{G} \cong T_e G$ is Lie algebra of G. Hence,

$$\mathcal{G}(U) \cong C^{\infty}(U, G), \quad \mathcal{L}(U) \cong C^{\infty}(U, \boldsymbol{G}), \tag{1.3}$$

On the other hand, if $\Lambda^1(U, \boldsymbol{G})$ is the space of $\boldsymbol{G}$-valued differential 1-forms on $U \subseteq X$, we set $\Omega^1_X(\boldsymbol{G})$ for the *sheaf of germs of $\boldsymbol{G}$-valued differential 1-forms* on X. Therefore,

$$\Omega^1_X(\boldsymbol{G})(U) \cong \Lambda^1(U, \boldsymbol{G}) \cong \Omega^1_X(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U), \tag{1.4}$$

where Ω^1_X is the *sheaf of germs of differential 1-forms* on X (see for instance [6; vol.I, p.81]). Since $\mathcal{L} \cong \mathcal{A}^n$, the isomorphisms of (1.4) are now completed also by the following isomorphism of $\mathcal{A}(U)$-modules

$$\Omega^1_X(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U) \cong (\Omega^1_X \otimes_{\mathcal{A}} \mathcal{L})(U). \tag{1.5}$$

As a consequence, we conclude that

$$\Omega^1_X(\boldsymbol{G}) \cong \Omega^1_X \otimes_{\mathcal{A}} \mathcal{L} =: \Omega^1(\mathcal{L}). \tag{1.6}$$

Note that (1.5) does not hold true for arbitrary sheaves (as a result of the very definition of the tensor product of sheaves) and is replaced by an imbedding. However, it is satisfied if $\mathcal{L}$ is a *vector sheaf of finite rank*; that is, if

$\mathcal{L}|U_i \cong \mathcal{A}^n|U_i \cong (\mathcal{A}|U_i)^n$, for some fixed $n \in \boldsymbol{N}$ and U_i running in an open cover of X. Note that $\Omega^1_X(\boldsymbol{G})$ and Ω^1_X are also vector sheaves of respective ranks $n = dim\,\boldsymbol{G} \equiv dim\,G$ and $m = dim\,X$.

(B) Another important entity for our approach is the *total* or *logarithmic differential*. We recall a few basic facts and refer for more details to [2] and [9].

For a smooth mapping $f : X \longrightarrow G$, its total differential is the form $f^{-1} \cdot df \in \Lambda^1(X, \boldsymbol{G})$ given by

$$(f^{-1} \cdot df)_x := T_{f(x)} L_{f(x)^{-1}} \circ T_x f; \quad x \in X,$$

if L_g denotes the left translation of G (by $g \in G$) and $T_x f$ is the ordinary differential of f at x. The previous formula localizes over any open $U \subseteq X$, thus we obtain a mapping

$$\partial_U : C^{\infty}(U, G) \longrightarrow \Lambda^1(U, \boldsymbol{G}) : f \mapsto \partial_U(f) := f^{-1} \cdot df \tag{1.7}$$

satisfying

$$\partial_U(s \cdot g) = Ad(g^{-1}) \cdot \partial_U(s) + \partial_U(g); \quad s, g \in C^{\infty}(U, G), \tag{1.8}$$

if Ad is the adjoint representation of G. Note that, as usual, the form $Ad(g^{-1}) \cdot \partial_U(s)$ is defined by

$$[Ad(g^{-1}) \cdot \partial_U(s)](x) = [Ad(g(x)^{-1}](\partial_U(s)_x); \; x \in U. \tag{1.9}$$

After the identifications (1.3), (1.4) and (1.5), the family (∂_U) determines a morphism of preasheaves

$$\{\partial_U : \mathcal{G}(U) \longrightarrow (\Omega^1_X \otimes_{\mathcal{A}} \mathcal{L})(U) | U \subseteq X \text{ open }\}$$

inducing, in turn, a morphism of sheaves of sets, denoted by

$$\partial : \mathcal{G} \longrightarrow \Omega^1(\mathcal{L}).$$

The sheafification of the adjoint representation yields the morphism of sheaves of groups $\mathcal{A}d : \mathcal{G} \longrightarrow \mathcal{A}ut(\mathcal{L})$ defined by

$$[\mathcal{A}d(g)(f)](x) := [Ad(g(x))]f(x)),$$

for every $g \in \mathcal{G}(U)$, $f \in \mathcal{L}(U)$, $x \in U$, and any open $U \subseteq X$.

Here, as is the custom, we use the same symbol both for a sheaf morphism and the induced morphism between sections, a fact adopted henceforth. We also recall that $\mathcal{A}ut(\mathcal{L})(U) \cong \mathcal{E}nd(\mathcal{L}|U)^{\circ}$, the right–hand side of the previous identification denoting the set of invertible sheaf endomorphisms (automorphisms) of $\mathcal{L}|U$.

Now (1.9) and (1.3)–(1.5) induce a natural action of $\mathcal{G}$ on $\Omega^1(\mathcal{L})$. Namely, for any open $U \subseteq X$, $g \in \mathcal{G}(U)$ and decomposable element $\alpha \otimes u \in \Omega^1_X(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$, we set

$$\mathcal{A}d(g) \cdot (\alpha \otimes u) := \alpha \otimes \mathcal{A}d(g)(u).$$

By linear extension, the expression $\mathcal{A}(g) \cdot \theta$ is defined for every $\theta \in \Omega^1(\mathcal{L})(U)$, thus we get the desired action. It is easily verified that

$$\partial(a \cdot b) = \mathcal{A}d(b^{-1}) \cdot \partial(a) + \partial(b); \quad (a, b) \in \mathcal{G} \times_X \mathcal{G}. \tag{1.10}$$

$$\partial(a^{-1}) = -\mathcal{A}d(a) \cdot \partial(a), \ a \in \mathcal{G}. \tag{1.11}$$

2. The connection operator. We describe the idea motivating the operator–like definition of principal connections.

Let (P, G, X, π) be a principal bundle endowed with an ordinary connection (form) $\omega \in \Lambda^1(P, \mathbf{G})$ and let $\mathcal{C} = \{U_i | i \in I\}$ be a fixed trivializing open cover of X. If s_i are the natural sections of P over $\mathcal{C}$, ω is fully determined by the corresponding *local connection forms* $\omega_i := s_i^* \cdot \omega \in \Lambda^1(U_i, \mathbf{G})$. They satisfy the compatibility condition (recall the notation introduced in (1.7).

$$\omega_j = Ad(g_{ij}^{-1}) \cdot \omega_i + \partial_{U_{ij}}(g_{ij}), \tag{2.1}$$

if $U_{ij} := U_i \cap U_j$ and $g_{ij} \in C^{\infty}(U_{ij}, G)$ are the transition functions of P (cf., for instance, [8], [13]).

Now, let U be any open subset of X. For an arbitrary section $s \in \Gamma(U, P)$, we get the form $s^*\omega \in \Lambda^1(U, \mathbf{G})$. We easily check that

$$(s \cdot g)^* \omega = Ad(g^{-1}) \cdot (s^* \omega) + \partial_U(g), \tag{2.2}$$

for any $g \in C^{\infty}(U, G)$. As a consequence, to every open $U \subseteq X$ there corresponds a mapping

$$\Gamma(U, P) \longrightarrow \Lambda^1(U, \mathbf{G}) : s \mapsto s^* \omega.$$

satisfying (2.2). This produces a morphism of presheaves leading to a morphism between the sheaves $\mathcal{P}$ and $\Omega^1_X(\mathbf{G})$. Therefore, taking into account (1.6), one is led to the following formal definition.

Definition 2.1. *A* **connection operator** *on $\mathcal{P}$ is a morphism of sheaves of sets $\mathbf{D} : \mathcal{P} \longrightarrow \Omega^1(\mathcal{L})$ satisfying condition*

$$\mathbf{D}(w \cdot a) = \mathcal{A}d(a^{-1}) \cdot \mathbf{D}(w) + \partial(a), \ (w, a) \in \mathcal{P} \times_X \mathcal{G}. \tag{2.3}$$

Remarks.

1) Strictly speaking, the term operator is abusive. However, we adopt it here in order to keep in mind that we define a connection as a sheaf morphism acting on the sections of the bundle, in analogy to the operator of covariant differentiation.

2) The use of $\Omega^1(\mathcal{L})$ (instead of its isomorphic, in virtue of (1.6), $\Omega^1(\mathbf{G})$) is motivated by the generalization discussed in the end of the Note.

In the sequel, we denote by $\sigma_i \in \mathcal{P}(U_i)$ and $\gamma_{ij} \in \mathcal{G}(U_{ij})$ the sections corresponding to $s_i \in \Gamma(U_i, P)$ and $g_{ij} \in C^{\infty}(U_{ij}, G)$, respectively, in virtue of the identifications (1.1) and (1.3).

Definition 2.2. *The* **local connection forms** *of $\mathbf{D}$ (with respect to the fixed open cover $\mathcal{C}$) are the 1–forms*

$$\theta_i := \mathbf{D}(\sigma_i) \in \Omega^1(\mathcal{L})(U_i). \tag{2.4}$$

We recall that, after the conventions of Section 1, $\mathbf{D}$ figuring in (2.4) is in fact the induced morphism on sections. Hence, as a direct consequence of (2.4), we obtain

Proposition 2.3. *The forms $(\theta_i)_{i \in I}$ satisfy the compatibility condition*

$$\theta_j = \mathcal{A}d(\gamma_{ij}^{-1}) \cdot \theta_i + \partial(\gamma_{ij}). \tag{2.5}$$

Theorem 2.4. *A family of forms $\{\theta_i \in \Omega^1(\mathcal{L})(U_i) | i \in I\}$ satisfying condition (2.5) determine a unique connection operator $\mathbf{D}$ such that $\mathbf{D}(\sigma_i) = \theta_i$.*

Proof. Let $p : \mathcal{P} \longrightarrow X$ be the projection of $\mathcal{P}$ onto X. For an arbitrary $w \in \mathcal{P}$ with $p(w) = x \in U_i$, we set

$$\mathbf{D}(w) := \mathcal{A}d(a_i^{-1}) \cdot \theta_i(x) + \partial(a_i). \tag{2.6}$$

if $a_i \in \mathcal{G}_x$ (:the fibre of $\mathcal{G}$ over x) is uniquely determined by $w = \sigma_i(x) \cdot a_i$. The previous expression is independent of the open set of $\mathcal{C}$ containing x. Indeed, if $x \in U_{ij}$, then $a_j = \gamma_{ij}(x) \cdot a_i$. Hence, equalities (1.10), (1.11) and (2.5) imply that

$$Ad(a_j^{-1}) \cdot \theta_j(x) + \partial(a_j) =$$

$$Ad(a_i^{-1}) \cdot \theta_i(x) + [Ad(a_i^{-1}) \circ Ad(\gamma_{ij}(x))] \cdot \partial(\gamma_{ij}(x)) +$$

$$Ad(a_i^{-1} \cdot \partial(\gamma_{ij}^{-1}(x)) + \partial(a_i) = Ad(a_i) \cdot \theta_i(x) + \partial(a_i),$$

thus showing that D is well defined. Condition (2.3) is obtained by the same token.

We prove now that D is continous. By the local structure of $\mathcal{P}$, there are open neighbourhoods V and U of the previously chosen w and x, respectively, so that $p|_V$ is a homeomorphism, the inverse of which is denoted by σ. If $W := \sigma(U \cap U_i)$, there is a unique $g \in \mathcal{G}(U \cap U_i)$ such that $\sigma(y) = \sigma_i(y) \cdot g(y)$, for every $y \in U \cap U_i$. Therefore, applying (2.6), we have that

$$D(\sigma(y)) = Ad(g(y)^{-1}) \cdot \theta_i(y) + \partial(g(y) = Ad(g^{-1}) \cdot \theta_i + \partial g)(y);$$

that is,

$$D|_W = (Ad(g^{-1}) \cdot \theta_i + \partial g) \circ p|_W,$$

thus proving the desired continuity at w and, similarly, on the whole of $\mathcal{P}$.

On the other hand, if $1 \in \mathcal{G}(X)$ is the unit section of $\mathcal{G}$ (i.e. $1(x)$ is the unit of $\mathcal{G}_X$), then, for every $x \in X$,

$$(D(\sigma_i)) = D(\sigma_i(x)) = D(\sigma_i(x) \cdot 1(x)) =$$

$$\theta_i(x) + \partial(1(x)) = \theta_i(x),$$

which shows that $D(\sigma_i) = \theta_i$.

Finally, assume there is another connection D' with $D'(\sigma_i) = \theta_i$. Then, for a w as above, $D'(w)$ satisfies (2.6), i.e. $D = D'$ and the proof is now complete. ∎

Theorem 2.5. *An ordinary connection ω on P corresponds in a bijective way to a connection operator D.*

Proof. Let ω be an ordinary connection with local connection forms $(\omega_i)_{i \in I}$ over $\mathcal{C}$. For each i, we denote by $\theta \in \Omega^1(\mathcal{L})(U_i)$ the form corresponding (by (1.4) and (1.5)) to ω_i. Therefore, applying the sheafification of Ad and the induced action on $\Omega^1(\mathcal{L})$, as well as the correspondence between the sections (g_{ij}) and (γ_{ij}) in the discussion before Definition 2.2, we see that (2.4) implies (2.5), which in turn yields an operator D by Theorem 2.4.

The converse is proved by reversing the previous arguments. The bijectivity of the correspondence is immediate by using the local connection forms and the same identifications as before. ∎

3. The sheaf of connections. Following A.Aragnol [1], we define a right action of $\mathcal{G}$ on $\mathcal{P} \times_X \Omega^1(\mathcal{L})$ determined (at the level of sections) by

$$(s, \omega) \cdot g := (s \cdot g, Ad(g^{-1}) \cdot \omega + \partial(g)) \tag{3.1}$$

for every $(s, \omega) \in \mathcal{P}(U) \times \Omega^1(\mathcal{L})(U), g \in \mathcal{G}(U)$ and $U \subseteq X$ open. This induces the following equivalence relation on $\mathcal{P}(U) \times \Omega^1(\mathcal{L})(U)$:

$$(s, \alpha) \sim (t, \beta) \Leftrightarrow \exists!\, g \in \mathcal{G}(U) : (t, \beta) = (s, \alpha) \cdot g \tag{3.2}$$

As a result, we obtain the quotient $Q(U) := (\mathcal{P}(U) \times \Omega^1(\mathcal{L})(U))/\mathcal{G}(U)$ and the (not necessarily complete)) presheaf $Q : U \longrightarrow Q(U)$.

Definition 3.1. *The* **sheaf of connections** *of (P, G, X, π), denoted by $\mathcal{C}(\mathcal{P}) \equiv \mathcal{P} \times_{\mathcal{G}} \Omega^1(\mathcal{L})$, is the (quotient) sheaf generated by Q.*

$\mathcal{C}(\mathcal{P})$ is, in fact, a sheaf of modules associated with $\mathcal{P}$ by the representation Ad and with respect to the action (3.1). It is locally isomorphic to $\Omega^1(\mathcal{L})$.

Theorem 3.2. *Connection operators D correspond bijectively to global sections of $\mathcal{C}(\mathcal{P})$.*

Prof. Let D be a connection operator and let θ_i be the corresponding local connection forms, with respect to the trivializing cover $\mathcal{C}$. We define a section $\tau_i \in \mathcal{C}(\mathcal{P})(U_i)$ by

$$\tau_i(x) := [\sigma_i(x), \theta_i(x)] \tag{3.3}$$

(recall the notations of Section 2). Here $[,]$ denotes equivalence classes on the fibre over x. We immediately check that $\tau_i = \tau_j$ on U_{ij}, as a consequence of (2.3) and (3.1); thus we obtain a global section τ of $\mathcal{C}(\mathcal{P})$. The continuity of τ is ensured by that of each τ_i, which, in turn, is a result of the continuity of the canonical map $\mathcal{P} \times_X \Omega^1(\mathcal{L}) \longrightarrow \mathcal{P} \times_{\mathcal{G}} \Omega^1(\mathcal{L})$.

Conversely, given a $\tau \in \mathcal{C}(\mathcal{P})(X)$, we define the local forms $\theta_i \in \Omega^1(\mathcal{L})(U_i)$ with $\theta_i := \Phi_i^{-1} \circ (\tau|U_i)$, if $\Phi_i : \Omega^1(\mathcal{L})|U_i \xrightarrow{\cong} \mathcal{C}(\mathcal{P})|U_i$ is the local isomorphism $\alpha \mapsto [\sigma_i, \alpha]$ characterizing $\mathcal{C}(\mathcal{P})$ as an associated sheaf of modules (in fact θ_i is the unique form satisfying (3.3)). Since (3.2) implies that condition (2.3) holds true on the overlapping, we get an operator $\boldsymbol{D}$.

Remarks. 1) Considering $\mathcal{C}(\mathcal{P})$ as a sheaf associated with $\mathcal{P}$ and of fibre type $\Omega^1(\mathcal{L})$, Theorem 3.2 can be thought of as the analogue of [3; n° 6.5.6]).

2) Theorems 2.4 and 3.2 provide a complete proof of the equivalence between ordinary connections and global sections of the sheaf of connections discussed, rather sketchily, in [1; p.287].

4. The curvature operator. In this section we outline the notion of curvature of a connection in the present framework.

The ordinary operator of exterior differentiation $d : \Omega^1_X \longrightarrow \Omega^2_X$ extends, in an obvious way, to an exterior differentiation (using the boldface of the same symbol) $\mathbf{d} : \Omega^1(\mathcal{L}) \to \Omega^2(\mathcal{L})$ with the well known properties. On the other hand, extending the notation of [9, p.56], we define the morphism

$$\mathcal{D} : \Omega^1(\mathcal{L}) \longrightarrow \Omega^2(\mathcal{L}) : \alpha \mapsto \mathbf{d}\alpha + \alpha \wedge \alpha = \mathbf{d}\alpha + 1/2 \cdot [\alpha, \alpha], \tag{4.1}$$

for every open $U \subseteq X$ and $\alpha \in \Omega^1(\mathcal{L})(U)$. Here, $\wedge$ and $[,]$ are obtained by the sheafification of the corresponding operations of the ordinary exterior calculus.

Clearly, $\mathcal{D}$ has the following properties (cf. also [9]):

$$\mathcal{D} \circ \partial = 0, \tag{4.2}$$

$$\mathcal{D}(\mathcal{A}d(g) \cdot \alpha) = \mathcal{A}d(g) \cdot \mathcal{D}(\alpha), \quad (\alpha, g) \in \Omega^1(\mathcal{L})(U) \times \mathcal{G}(U). \tag{4.3}$$

Definition 4.1. *The* **curvature** *of a connection operator* $\boldsymbol{D}$ *is the morphism of sheaves of sets* $\boldsymbol{R}^{\boldsymbol{D}} : \mathcal{P} \longrightarrow \Omega^2(\mathcal{L})$ *given by*

$$\boldsymbol{R}^{\boldsymbol{D}} := \mathcal{D} \circ \boldsymbol{D}.$$

$\boldsymbol{R}^{\boldsymbol{D}}$ *is* $\mathcal{G}$*-equivariant, that is*

$$\boldsymbol{R}^{\boldsymbol{D}}(w \cdot g) = \mathcal{A}d(g^{-1}) \cdot \boldsymbol{R}^{\boldsymbol{D}}(w); \ (w, g) \in \mathcal{P} \times_X \mathcal{G}. \tag{4.4}$$

as a consequence of (4.2) and (4.3).

Fixing, as in the previous sections, a trivializing cover $\mathcal{C}$, we give the following

Definition 4.2. *The 2-forms* $\Theta_i := \boldsymbol{R}^{\boldsymbol{D}}(\sigma_i) \in \Omega^2(\mathcal{L})(U_i)$ *are called the* **local curvature forms** *of* $\boldsymbol{D}$ *(with respect to* $\mathcal{C}$*).*

The very definition of $\boldsymbol{R}^{\boldsymbol{D}}$ yields the structural equation

$$\Theta_i = \mathbf{d}\theta_i + 1/2 \cdot [\theta_i, \theta_i] \tag{4.5}$$

and (4.4) implies that

$$\Theta_j = \mathcal{A}d(\gamma_{ij}^{-1}) \cdot \Theta_i \tag{4.6}$$

holds on the overlapping.

Theorem 4.3. *Let* $\boldsymbol{D}$ *be a connection operator on* $\mathcal{P}$ *with corresponding local connection forms* $(\theta_i)_{i \in I}$. *If* $(\theta_i)_{i \in I}$ *is a family of 2-forms defined by (4.5) and satisfying (4.6), then there is a unique operator* $\boldsymbol{R}$ *such that* $\boldsymbol{R} = \boldsymbol{R}^{\boldsymbol{D}}$ *and the* Θ_i*'s coincide with the local curvature forms of* $\boldsymbol{D}$.

Proof. We define $\boldsymbol{R}$ by $\boldsymbol{R}(w) := \mathcal{A}d(g_i^{-1}) \cdot \Theta_i(x)$, if $w \in \mathcal{P}_x$ and $g_i \in \mathcal{G}_x$ is uniquely determined by $w = \sigma_i(x) \cdot g_i$. Then we proceed as in the proof of Theorem 2.4. ∎

Corollary 4.4. *Let* $\boldsymbol{D}$ *be a connection operator corresponding to an ordinary connection* ω. *Then* $\boldsymbol{R}^{\boldsymbol{D}}$ *corresponds to the ordinary curvature (form)* Ω *of* ω.

We shall close this section by some other interpretations of $\boldsymbol{R}^{\boldsymbol{D}}$. To this end we denote by $\mathrm{Hom}_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))$ the set of morphisms of sheaves of sets which are $\mathcal{G}$-equivariant, i.e. they satisfy the analogue of (4.4). Also, $\mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))$ denotes the sheaf generated by the complete presheaf $U \longrightarrow \mathrm{Hom}_{\mathcal{G}|U}(\mathcal{P}|U, \Omega^2(\mathcal{L})|U)$. Finally, we set $\mathcal{A}d(\mathcal{P})$ for the sheaf type $\mathcal{L}$, associated with $\mathcal{P}$ by the representation $\mathcal{A}d$ (cf. also Remark 1, after Theorem 3.2).

Corollary 4.5. *The curvature* $\boldsymbol{R}^{\boldsymbol{D}}$ *has the following properties:*
i) Within an identification, $\boldsymbol{R}^{\boldsymbol{D}} \in \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))(x)$, *i.e.* $\boldsymbol{R}^{\boldsymbol{D}}$ *is a global*

section of $\mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))$.
ii) $\boldsymbol{R}^{\boldsymbol{D}}$ *determines a global 2 form* $\mathcal{R} \in (\Omega^2 \otimes_{\mathcal{A}} Ad(\mathcal{P}))(X)$.

Proof. The first result is obvious by the identification

$$\mathrm{Hom}_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L})) \cong \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))(X).$$

For ii), we denote by $\Psi_i := \mathcal{L}|U_i \xrightarrow{\cong} Ad(\mathcal{P})|U_i$ the local isomorphisms characterizing $Ad(\mathcal{P})$ as an associated sheaf of type $\mathcal{L}$, and we set $\mathcal{R}_i := (id \otimes \Psi_i)(\Theta_i)$ where $id \equiv id|\Omega^2_X$. Now, (4.4) and the equivariance of Ψ_i imply that $\mathcal{R}_i = \mathcal{R}_j$ on U_{ij}, thus defining a global section as in the statement. ∎

5. A sheaf theoretic abstraction. As alluded to in the introduction, we give a short account of an abstraction of the ideas discussed so far, referring for more details to [14]. The notations and terminology applied are deliberately chosen so that they remind of the familiar objects of the classical theory.

We start with an *algebraized space* $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ is a sheaf (over X) of unital, commutative and associative topological $\boldsymbol{K}$-algebras ($\boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C}$). We fix a *differential triad* $(\mathcal{A}, d, \Omega^1)$, where Ω^1 is an $\mathcal{A}-$*module* and $d : \mathcal{A} \longrightarrow \Omega^1$ is an Ω^1-valued *derivation* of $\mathcal{A}$.

Let $\mathcal{P}$ be a *principal sheaf of type* $\mathcal{G}$ and *with structure sheaf* $\mathcal{G}$. By the previous terminology we mean that $\mathcal{G}$ is a sheaf of groups acting on $\mathcal{P}$ (from the right) ad there exists an open cover of X, say $\mathcal{C} = \{U_i | i \in I\}$, such that $\mathcal{P}|U_i \longrightarrow \mathcal{G}|U_i$ is an equivariant isomorphism (for details one may consult [7]).

In order to develop an adequate theory of connections on $\mathcal{P}$, we need two entities playing, respectively, the role of the adjoint representation and that of the total differentiation. For this reason we restrict our attention to structural groups with the following fundamental properties:

(1) there is a *representation* $\rho : \mathcal{G} \longrightarrow Aut(\mathcal{L})$, where $\mathcal{L}$ is a *sheaf of Lie algebras*, which is also an $\mathcal{A}-$*module*.

(2) there is a morphism (of sheaves of sets) $\partial : \mathcal{G} \longrightarrow \Omega^1(\mathcal{L})$ satisfying condition

$$\partial(g \cdot h) = \rho(h^{-1}) \cdot \partial(g) + \partial(h), \tag{5.1}$$

for every $(g, h) \in \mathcal{G} \times_X \mathcal{G}$.

Here, by definition, $\Omega^1(\mathcal{L}) = \Omega^1 \otimes_{\mathcal{A}} \mathcal{L}$ and the first term of the right-hand side of (5.1) denotes the result of the action of $\mathcal{G}$, naturally induced on $\Omega^1(\mathcal{L})$ (see also Section 4).

With the previous assumptions, a *connection* on $\mathcal{P}$ is a morphism (of sheaves of sets) $\boldsymbol{D} : \mathcal{P} \longrightarrow \Omega^1(\mathcal{L})$ satisfying the property

$$\boldsymbol{D}(s \cdot g) = \rho(g^{-1}) \cdot \boldsymbol{D}(s) + \partial(g), \quad (s, g) \in \mathcal{P} \times_X \mathcal{G}.$$

Beside the connection operator studied in Section 2, further examples, fitting in this generalization, are provided by connections on vector sheaves (treated in [10], [11]) and by connections on the projective limit of a projective system of principal bundles, each of which is endowed with ordinary connections ([14]).

In analogy to the results of Sections 2 and 3, we can show that $\boldsymbol{D}$ is equivalent to a family $\theta_i \in \Omega^1(\mathcal{L})$, $i \in I$, as well as to a global section of the *sheaf of connections* $\mathcal{P} \times_{\mathcal{G}} \Omega^1(\mathcal{L})$.

The question of the existence naturally arises here. Since the classical criteria cannot be applied, we consider instead the (generalized) *Atiyah class*

$$a(\mathcal{P}) \in H^1(X, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^1(\mathcal{L}))),$$

where, for any open $U \subseteq X$,

$$\mathrm{Hom}_{\mathcal{G}}(\mathcal{P}, \Omega^1(\mathcal{L}))(U) \cong \mathrm{Hom}_{\mathcal{G}|U}(\mathcal{P}|U, \Omega^1(\mathcal{L})|U)$$

is the set of equivariant morphisms, with respect to the described actions (cf. also the discussion before Corollary 4.5). Therefore, one shows that

$$[\exists \; connections \; on \; \mathcal{P}] \Leftrightarrow [a(\mathcal{P}) = 0].$$

This program carries over to the study of curvature, flat principal bundles, the Chern–Weil homomorphism and so on. Thus we extend a significant part of the theory of connections on ordinary principal bundles to a purely algebro topological setting, without reference to any notion of differentiability; a fact which might be useful to theoretical physics. As one might expect, the generality of the present framework often imposes some cohomological restrictions, depending on how far we wish to extend the classical theory. These matters will be clarified in forthcoming notes.

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INFINITE JETS AND PROJECTIVE LIMITS OF BUNDLES

BY

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Introduction. The purpose of this Note is the presentation of an example of a Fréchet-type vector bundle as the motivation for a general study of a particular class of Fréchet-vector bundles: those which can be obtained as projective limits of Banach-vector bundles.

More precisely, it is well known ([1]) that the set $J^k(\ell)$ of k-jets ($k < +\infty$) of the sections of a Banach-vector bundle ℓ is also a Banach-vector bundle with fibres of type

$$P^k(\boldsymbol{B}, \boldsymbol{E}) := \boldsymbol{E} \times \mathcal{L}(\boldsymbol{B}, \boldsymbol{E}) \times \mathcal{L}_S^2 \times \cdots \times \mathcal{L}_S^k(\boldsymbol{B}, \boldsymbol{E}),$$

where $\boldsymbol{E}$ is the fibre type of ℓ, $\boldsymbol{B}$ the space model of the basis and $\mathcal{L}_S^k(\boldsymbol{B}, \boldsymbol{E})$ the space of symmetric continuous k-linear mappings from $\boldsymbol{B}^k$ to $\boldsymbol{E}$. However, if we move to the case of infinite jets there is not, so far, an analogous study of $J^\infty(\ell) := \varprojlim J^k(\ell)$ (cf. [5]) as a vector bundle since the corresponding fibre type $P^\infty(\boldsymbol{B}, \boldsymbol{E}) = \boldsymbol{E} \times \mathcal{L}(\boldsymbol{B}, \boldsymbol{E}) \times \cdots$ is a Fréchet space and we do not have a reasonable topology for its general linear group. Therefore, $J^\infty(\ell)$ has been studying so far more of a topological point of view. In addition, F.Takens in his paper [5] endows $J^\infty(\ell)$ with a differential structure defining the set of $\boldsymbol{R}$-valued smooth mappings on $J^\infty(\ell)$.

We note that similar problems arise also in the general case of a Fréchet-vector bundle due to the peculiarities of the corresponding structure group.

In this paper we shall equip $J^\infty(\ell)$ with a Fréchet-vector bundle stucture by replacing the pathological general linear group $GL(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ by

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