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INFINITE JETS AND PROJECTIVE LIMITS OF BUNDLES

BY

GEORGE GALANIS

Introduction. The purpose of this Note is the presentation of an example of a Fréchet-type vector bundle as the motivation for a general study of a particular class of Fréchet-vector bundles: those which can be obtained as projective limits of Banach-vector bundles.

More precisely, it is well known ([1]) that the set $J^k(\ell)$ of k-jets ($k < +\infty$) of the sections of a Banach-vector bundle ℓ is also a Banach-vector bundle with fibres of type

$$P^k(\boldsymbol{B}, \boldsymbol{E}) := \boldsymbol{E} \times \mathcal{L}(\boldsymbol{B}, \boldsymbol{E}) \times \mathcal{L}_S^2 \times \cdots \times \mathcal{L}_S^k(\boldsymbol{B}, \boldsymbol{E}),$$

where $\boldsymbol{E}$ is the fibre type of ℓ, $\boldsymbol{B}$ the space model of the basis and $\mathcal{L}_S^k(\boldsymbol{B}, \boldsymbol{E})$ the space of symmetric continuous k-linear mappings from $\boldsymbol{B}^k$ to $\boldsymbol{E}$. However, if we move to the case of infinite jets there is not, so far, an analogous study of $J^\infty(\ell) := \underleftarrow{\lim} J^k(\ell)$ (cf. [5]) as a vector bundle since the corresponding fibre type $P^\infty(\boldsymbol{B}, \boldsymbol{E}) = \boldsymbol{E} \times \mathcal{L}(\boldsymbol{B}, \boldsymbol{E}) \times \cdots$ is a Fréchet space and we do not have a reasonable topology for its general linear group. Therefore, $J^\infty(\ell)$ has been studying so far more of a topological point of view. In addition, F.Takens in his paper [5] endows $J^\infty(\ell)$ with a differential structure defining the set of $\boldsymbol{R}$-valued smooth mappings on $J^\infty(\ell)$.

We note that similar problems arise also in the general case of a Fréchet-vector bundle due to the peculiarities of the corresponding structure group.

In this paper we shall equip $J^\infty(\ell)$ with a Fréchet-vector bundle stucture by replacing the pathological general linear group $GL(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ by

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an appropriate "enlarged" structure group. As a result, $J^\infty(\ell)$ will be characterized by a generalized type of transition functions that we define here.

Using analogous techniques we may extend the previous theory to the general case of projective limits of Banach-vector bundles. We achieve in this way to endow the latter with Fréchet-vector bundles structure.

1. Infinite jets of the sections of a Banach-vector bundle.

Let $\ell = (E, \pi, B)$ be a Banach-vector bundle with fibres of type $\boldsymbol{E}$. Then, as mentioned in the Introduction, the set

$$J^k(\ell) := \{j^k_x s : s \in \Gamma(\ell), x \in B\}$$

of k-jets ($k < +\infty$) of the sections of ℓ is also a Banach-vector bundle over B with fibre type $P(\boldsymbol{B}, \boldsymbol{E}) := \boldsymbol{E} \times \mathcal{L}(\boldsymbol{B}, \boldsymbol{E}) \times \cdots \times \mathcal{L}^k_S(\boldsymbol{B}, \boldsymbol{E})$, where $\boldsymbol{B}$ is the space-model of B (cf. [1]). More precisely, the projection of $J^k(\ell)$ is given by

$$p_k : J^k(\ell) \longrightarrow B : j^k_x s \longmapsto x$$

and its differential structure is constructed by the trivializatons $\{(U, \tau_k)\}$, where

$$\tau_k : p_k^{-1}(U) \to U \times P^k(\boldsymbol{B}, \boldsymbol{E}) : j^k_x \longmapsto (x, s_\alpha(\alpha_0(x)), Ds_\alpha(\alpha_0(x)), \ldots$$
$$, D^k s_\alpha(\alpha_0(x)))$$

if (U, α, α_0) are vector charts of ℓ and $s_\alpha := pr_2 \circ \alpha \circ s \circ \alpha_0^{-1}$ is the principal part of s with respect to α. The bundles $\{J^k(\ell)\}_{k \in \boldsymbol{N}}$ form a projective system with connecting morphisms

$$f_{mk} : J^m(\ell) \longrightarrow J^k(\ell) : j^k_x s \longmapsto j^k_x \quad (m \geq k).$$

The corresponding limit $\varprojlim J^k(\ell)$ is defined to be the set of infinite jets of the sections of ℓ and it is denoted by $J^\infty(\ell)$ (cf. [5]).

On the other hand the Banach space $P^k(\boldsymbol{B}, \boldsymbol{E}), k \in \boldsymbol{N}$, form also a projective system with connecting morphisms the natural projections $r_{mk} : P^m(\boldsymbol{B}, \boldsymbol{E}) \longrightarrow P^k(\boldsymbol{B}, \boldsymbol{E})$, $m \geq k$. The corresponding limit coincides with the Fréchet space $P^\infty(\boldsymbol{B}, \boldsymbol{E})$ via the isomorphism

$$h : P^\infty(\boldsymbol{B}, \boldsymbol{E}) \longrightarrow \varprojlim(P^k(\boldsymbol{B}, \boldsymbol{E})) : (f_0, f_1, \ldots) \longmapsto ((f_0, f_1, \ldots, f_k))_{k \in \boldsymbol{N}}.$$

In order to construct a Fréchet vector bundle structure on $J^\infty(\ell)$ we define the space

$$H(P^\infty(\boldsymbol{B}, \boldsymbol{E})) := \{(g_k)_{k \in \boldsymbol{N}} : g_k \in \mathcal{L}(P^k(\boldsymbol{B}, \boldsymbol{E})) \text{ and } \varprojlim g_k \text{ exists}\}.$$

The previous space is Fréchet since

$$H(P^\infty(\boldsymbol{B}, \boldsymbol{E})) \equiv \varprojlim(H_k(P^\infty(\boldsymbol{B}, \boldsymbol{E}))).$$

where each

$$H_k(P^\infty(\boldsymbol{B}, \boldsymbol{E})) := \{(g_1, \ldots, g_k) :$$
$$g_m \in \mathcal{L}(P^m(\boldsymbol{B}, \boldsymbol{E})) \text{ and } r_{mi} \circ g_m = g_i \circ r_{mi} \quad (m \geq i)\}$$

is a Banach space, as a closed subspace of $\prod_{m=1}^{k} (\mathcal{L}(P^m(\boldsymbol{B}, \boldsymbol{E})))$, and the connecting morphisms are the natural projections

$$p_{mk} : H_m(P^\infty(\boldsymbol{B}, \boldsymbol{E})) \longrightarrow H_k(P^\infty(\boldsymbol{B}, \boldsymbol{E})), \quad m \geq k.$$

Moreover, $H(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ is related to the Hausdorff locally convex (but not Fréchet) topological vector space $\mathcal{L}(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ via the continuous linear map

$$\varepsilon : H(P^\infty(\boldsymbol{B}, \boldsymbol{E})) \longrightarrow \mathcal{L}(P^\infty(\boldsymbol{B}, \boldsymbol{E})) : (g_k)_{k \in \boldsymbol{N}} \longmapsto \varprojlim g_k.$$

Analogously we define

$$H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E})) := \{(g_k)_{k \in \boldsymbol{N}} : g_k \in \mathcal{L}is(P^\infty(\boldsymbol{B}, \boldsymbol{E})) \text{ and } \varprojlim g_k \text{ exists}\},$$

$$H^\circ_k(P^\infty(\boldsymbol{B}, \boldsymbol{E})) := \{(g_1, \ldots, g_k) : g_m \in \mathcal{L}is(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$$
$$\text{and } r_{mi} \circ g_m = g_i \circ r_{mi} \quad (m \geq i)\}$$

Each $H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ is a Lie group modeled on $H_k(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$. Moreover, $\varprojlim(H^\circ_k(P^\infty(\boldsymbol{B}, \boldsymbol{E})))$ can be defined, with connecting morphisms $p_{mk}|_{H^\circ_k(P^\infty(\boldsymbol{B}, \boldsymbol{E}))} (m \geq k)$, and coincides with $H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$. Therefore, the latter is a topological group.

Theorem 1.1. *$J^\infty(\ell) := \varprojlim J_k(\ell)$ is a Fréchet-vector bundle over B with fibres of type $P^\infty(\boldsymbol{B}, \boldsymbol{E})$. The corresponding transition functions $\{T_{UV}\}$, with respect to an open covering $\{U\}$ of B, can be factored in the form $\varepsilon \circ T^*_{UV} = T_{UV}$, where $T^*_{UV} : U \cap V \longrightarrow H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ are smooth mappings if we consider them as taking values in $H(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$.*

Proof. First we observe that the projections p_k of $J^k(\ell)$ $(k \in \boldsymbol{N})$ satisfy $p_k \circ f_{mk} = p_m$ $(m \geq k)$. As a result, the continuous map $p :=$

$\varprojlim p_k : J^\infty(\ell) \longrightarrow B$ can be defined. If $\{(U, \tau_k)\}_{k\in \boldsymbol{N}}$ are the local trivializatons of $J_k(\ell)$ constructed by the same vector chart (U, α, α_0) of ℓ, then we check that they form a projective system with corresponding limit the homeomorphism

$$\tau := \varprojlim \tau_k : p^{-1}(U) \longrightarrow U \times P^\infty(\boldsymbol{B}, \boldsymbol{E}).$$

The couples $\{(U, \tau)\}$ define a Fréchet manifold structure on $J^\infty(\ell)$ (concerning the differentiability of J.Leslie ([4])). With respect to this structure p is smooth since $p|_{p^{-1}(U)} = pr_1 \circ \tau$. On the other hand, $p^{-1}(x) = \varprojlim(p_k^{-1}(x))$, $x \in B$, hence the fibres of $J^\infty(\ell)$ are Fréchet spaces. The diffeomorphisms $\{\tau := \varprojlim \tau_k\}$, as above, are the local trivializatons of $J^\infty(\ell)$ since each $\tau_x := \tau|_{p^{-1}(x)}$ equals to the linear isomorphism $\varprojlim(\tau_k|_{p_k^{-1}(x)})$. Therefore, to complete the proof it suffices to check the differentiability of the transition functions. To this end, let $(U, \tau = \varprojlim \tau_k)$, $(V, \sigma = \varprojlim \sigma_k)$ be local trivialization function of $J^\infty(\ell)$ and $T_{UV}(x) = \tau_x \circ \sigma_x^{-1}$ the corresponding transition function. Setting, for any $k \in \boldsymbol{N}$,

$$(T^*_{UV})_k : U \cap V \longrightarrow H^\circ_k(P^\infty(\boldsymbol{B}, \boldsymbol{E})) : x \longmapsto (\tau_{1x} \circ \sigma_{1x}^{-1}, \ldots, \tau_{kx} \circ \sigma_{kx}^{-1})$$

we check that $\rho_{mk} \circ (T^*_{UV})_m = (T^*_{UV})_k$, $m \geq k$. As a result the mapping

$$T^*_{UV} := \varprojlim (T^*_{UV})_k : U \cap V \longrightarrow H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$$

exists and satisfies

$$(\varepsilon \circ T^*_{UV})(x) = \varepsilon(((T^*_{UV})_k(x))_{k\in \boldsymbol{N}}) \overset{h}{\equiv} \varepsilon((\tau_{kx} \circ \sigma_{kx}^{-1})_{k\in \boldsymbol{N}}) =$$

$$= \varprojlim \tau_{kx} \circ \varprojlim \sigma_{kx}^{-1} = \tau_x \circ \sigma_x^{-1} = T_{UV}(x).$$

Furthermore, T^*_{UV} is a smooth mapping, as a projective limit of smooth mappings, if we consider it as taking values in $H(P^\infty(\boldsymbol{B}, \boldsymbol{E})) \supseteq$ $\supseteq H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$. Therefore, T_{UV} is smooth. ■

Remarks 1.2. (i) The previous mapping $\{T^*_{UV}\}$, which characterize $\{T_{UV}\}$ and $J^\infty(\ell)$, will be called *generalized transition functions*. It is obvious that the structure group of $J^\infty(\ell)$ is, in fact, the topological group $H^\circ(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$ and not any more the pathological general linear $GL(P^\infty(\boldsymbol{B}, \boldsymbol{E}))$.

(ii) The differential structure that we define on $J^\infty(\ell)$ is larger than of Taken's in [5]. Indeed, the latter is determined by the condition $g \in C^\infty(J^\infty(\ell), \boldsymbol{R}) \Leftrightarrow \exists k \in \boldsymbol{N}$, $U_k \subseteq J^k(\ell)$ open and $g_k \in C^\infty(U_k, \boldsymbol{R})$ such that $g|_{f_k^{-1}(U_k)} = g_k \circ f_k$, where $f_k : J^\infty(\ell) \longrightarrow J^k(\ell)$ are the canonical projections. This conditions is satisfied if and only if g is the limit of the projective system of smooth mappings $\{g_k \circ f_{ik}\}_{i\geq k}$. Therefore, Taken's $\boldsymbol{R}$-valued smooth maps on $J^\infty(\ell)$ are projective limits of smooth maps hence smooth in the sense of J.Leslie too(cf. [4], [2]). Taking into account that is not necessary for a smooth map (in Leslie's sense) to be projective limit, we get the assertion. **2. Generalization to the case of projective limits of Banach-vector bundles.** Motivated by the previous construction we study in [2] the general case of a projective limit of Banach-vector bundles. Namely, let $\{E_i; f_{ji}\}_{i,j\in \boldsymbol{N}}$ be a projective system of Banach-vector bundles, over the same basis B, which satisfy the following conditions :

(I) The corresponding fibre types $\{\boldsymbol{E}_i\}_{i\in \boldsymbol{N}}$ form a projective system with connecting morphisms $r_{ji} : \boldsymbol{E}_j \longrightarrow \boldsymbol{E}_i$ $(j \geq i)$.

(II) For any $x \in B$ there exist (U, τ_i) local trivializatons of E_i through x $(i \in \boldsymbol{N})$ such that $(id_U \times r_{ji}) \circ \tau_j = \tau_i \circ f_{ji}$ $(j \geq i)$.

We assume, in other words, that the analogous to the special case of infinite jets, conditions hold.

Then, $E := \varprojlim E_i$ is a Fréchet-vector bundle over B with fibres of type $E := \varprojlim \boldsymbol{E}_i$. The corresponding transition functions $\{T_{UV}\}$ can be factored in the form $T_{UV} = \varepsilon \circ T^*_{UV}$

where $H^\circ(\boldsymbol{E}) := \{(g_i)_{i\in \boldsymbol{N}} : g_i \in \mathcal{L}is(\boldsymbol{E}_i)$ and $\varprojlim g_i$ exists$\}$ is a topological group, $\varepsilon((g_i)_{i\in \boldsymbol{N}} := \varprojlim g_i$ and $T^*_{UV} : U \cap V \longrightarrow H^\circ(E)$ are smooth mappings if we consider them as taking values in the Fréchet space

$$H(\boldsymbol{E}) := \{(g_i)_{i\in \boldsymbol{N}} : g_i \in \mathcal{L}(\boldsymbol{E}_i) \text{ and } \varprojlim g_i \text{ exists}\} \supseteq H^\circ(\boldsymbol{E}).$$

Moreover $\{T^*_{UV}\}$, which will be called *generalized* transition functions, characterize E since they fully recover it. Indeed, next Proposition is valid.

Proposition 2.1. ([2]). *Let B be a Banach manifold, $\{U_a\}_{a\in\mathcal{A}}$ an open covering of B, $\boldsymbol{E} = \varprojlim \boldsymbol{E}_i$ a Fréchet space and*

$$\{T^*_{ab} : U_a \cap U_b \longrightarrow H^\circ(\boldsymbol{E})\}_{a,b\in\mathcal{A}}$$

a family of smooth mappings such that $T^*_{ab}(x) \circ T^*_{bc}(x) = T^*_{ac}(x), T^*_{aa}(x) = (id_{\boldsymbol{E}_i})_{i\in\boldsymbol{N}}$. Then, there exists a (uniquely determined) Fréchet-vector bundle E, which is projective limit of Banach vector bundles, with generalized transition functions $\{T^*_{ab}\}_{a,b\in\mathcal{A}}$ and classical (in Lang's sense) transition functions $\{\varepsilon \circ T^*_{ab}\}_{a,b\in\mathcal{A}}$.

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SOME REMARKS ON METRIC STRUCTURE IN THE GENERALIZED FINSLER SPACE

BY

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1. Introduction. Let $\xi = (E, \pi, M)$ be a C^∞ vector bundle with an n-dimensional base manifold M, type fibre $F = R^m$ and a total space of dimension $(n+m)$. In some local chart the point $u \in E$ has the coordinates

$$(x^1, \dots, x^n, y^1, \dots, y^m) = (x^i, y^a) = (x, y)$$

$$i, j, h, k, l = 1, \dots, n \qquad a, b, c, d, e, f = 1, \dots, m.$$

The allowable coordinate transformations $(x, y) \to (x', y')$ are given by

$$x^{i'} = x^{i'}(x^1, \dots, x^n) \qquad y^{a'} = M^{a'}_a(x) y^a. \tag{1.1}$$

where

$$\operatorname{rank}[\partial x^{i'}/\partial x^i] = n \qquad \operatorname{rank}[M^{a'}_a] = m.$$

The inverse transformation $(x', y') \to (x, y)$ exists and is determined by

$$x^i = x^i(x^{1'}, \dots, x^{n'}) \qquad y^a = M^a_{a'}(x') y^{a'}. \tag{1.2}$$

The usual relations

$$\begin{matrix} M^{a'}_a M^a_{b'} = \delta^{a'}_{b'} & B^{i'}_j B^j_{j'} = \delta^{i'}_{j'} & B^{i'}_j = \partial x^{i'}/\partial x^j \\ M^{a'}_b M^a_{a'} = \delta^a_b & B^j_{k'} B^{k'}_j = \delta^j_j & B^j_{j'} = \partial x^j/\partial x^{j'} \end{matrix} \tag{1.3}$$

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