Theorem 4.9. *A fts X is f. almost if and only if $\overline{A} = [A]_\delta$ for any f. semi-open set A in X.*

Proof. Clearly from Theorem 4.7. and Lemma 4.8.

Theorem 4.10. *A fts X is f. semi-regular if and only if $\overline{A} = [A]_\delta$.*

Proof. Let P be a fuzzy point with $P \in \overline{A}$, there exists a fuzzy open $q-nbd\ V$ of P such that VqA. Since X is f-semi-regular, there exists a fuzzy open set U such that $PqU \leq \overline{U}^{\circ} \leq V$, $V \not{q} A \Rightarrow (\overline{U})^{\circ} \leq V \leq A' \Rightarrow (\overline{U})^{\circ} \not{q} A \Rightarrow P \in [A]_\delta \Rightarrow [A]_\delta \leq \overline{A}$, but $\overline{A} \leq [A]_\delta$. Then $\overline{A} = [A]_\delta$.

Conversely, Let P be a fuzzy point and V be a fuzzy open set with PqV. Since $P \in \overline{V'} = [V']_\delta$, then there exists a fuzzy open $q-nbd\ U$ of P such that $(\overline{V})^{\circ} \not{q} V'$. Then $(\overline{U})^{\circ} \leq V$. Therefore X is f. semi-regular.

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Department of Mathematics
Faculty of Sciences
University of Assiut
Assiut, Egypt

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
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THE INTEGRABILITY OF ALMOST 2 - π STRUCTURES IN THE BUNDLE OF ACCELERATIONS

BY

MONICA PURCARU AND VICTOR BLĂNUŢĂ

The generalization of almost complex and almost product structures have been made by C.J.Hsu which introduce the notion of $r-\pi$ structure, [5].

In the present paper we consider the group G of transformation of N-linear connections compatible with an almost $2-\pi$ structure given on the bundle of accelerations and prove that it has forty-three invariants, which are d-tensor fields. By means of these invariants, we get the characterizations of the integrability of type I, II, II, IV, or V for the almost $2-\pi$ d-structures. All this integrability theory is based only on geometry of the bundle of accelerations (cf.with Gh.Atanasiu [1] and R.Miron and Gh.Atanasiu [3], [6–9]).

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1. Preliminaries. [1]: Let M be a real C^∞-manifold with n-dimensions $(n = 2n')$, let OSC^2M be the osculator bundle of second order (or the bundle of accelerations) and let $N(\underset{(1)}{N}, \underset{(2)}{N})$ be a nonlinear connection on OSC^2M. We denote by $(x^i, y^{(1)i}, y^{(2)i})$ the canonical coordinates on OSC^2M. Then, an adapted basis to the direct sum of the vector spaces

(1.1)
$$T_u(OSC^2M) = N_u(OSC^2M) \oplus V_u(OSC^2M), \forall u = (x, y^{(1)}, y^{(2)}) \in OSC^2M$$

is given by $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\}$, where

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$$(1.2)\qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}},$$

$$(1.3)\qquad \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}}.$$

The dual basis is $\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}$, where

$$(1.4)\qquad \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{N}{}^i{}_m dx^m$$

$$(1.5)\qquad \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{N}{}^i{}_m dy^{(1)m} + (\underset{(1)}{N}{}^i{}_m + \underset{(1)}{N}{}^i{}_s \underset{(1)}{N}{}^s{}_m) dx^m.$$

We have

$$(1.6)\qquad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \overset{(1)}{\underset{(0)}{R}}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{\underset{(0)}{R}}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}},$$

$$(1.7)\qquad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)j}}\right] = (\overset{(1)}{\underset{(1)}{P}}{}^m{}_{ij} + L^m_{ji}) \frac{\delta}{\delta y^{(1)m}} + \overset{(2)}{\underset{(1)}{P}}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}},$$

$$(1.8)\qquad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^{(2)j}}\right] = \overset{(1)}{\underset{(2)}{P}}{}^m{}_{ij} \frac{\delta}{\delta y^{(1)m}} + (\overset{(2)}{\underset{(2)}{P}}{}^m{}_{ij} + L^m_{ji}) \frac{\partial}{\partial y^{(2)m}},$$

$$(1.9)\qquad \left[\frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(1)j}}\right] = \overset{(2)}{\underset{(1)}{S}}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}} = \overset{(1)}{\underset{(1)}{R}}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}},$$

$$(1.10)\qquad \left[\frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)j}}\right] = \overset{(1)}{\underset{(2)}{P}}{}^m{}_{ij} \frac{\partial}{\partial y^{(2)m}}, \quad \left[\frac{\partial}{\partial y^{(2)i}}, \frac{\partial}{\partial y^{(2)j}}\right] = 0.$$

We consider an almost $2-\pi$ structure F on OSC^2M :

$$(1.11)\qquad F \circ F = \lambda^2 I$$

where λ is a non vanish complex number. Its integrability tensor field $\tilde{N}$ is given by

$$(1.12)\qquad \tilde{N}(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + \frac{1}{\lambda^2}[X,Y],$$
$$\forall X,Y \in \mathcal{X}(OSC^2M).$$

In the basis $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\}$ *and* $\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}$, F can be represented by

$$(1.13)\qquad F = \overset{00}{F}{}^i{}_j \frac{\delta}{\delta x^i} \otimes dx^j + \overset{01}{F}{}^i{}_j \frac{\delta}{\delta x^i} \otimes \delta y^{(1)j} + \overset{02}{F}{}^i{}_j \frac{\delta}{\delta x^i} \otimes \delta y^{(2)j}$$
$$+ \overset{10}{F}{}^i{}_j \frac{\delta}{\delta y^{(1)i}} \otimes dx^j + \overset{11}{F}{}^i{}_j \frac{\delta}{\delta y^{(1)i}} \otimes \delta y^{(1)j} + \overset{12}{F}{}^i{}_j \frac{\delta}{\delta y^{(1)i}} \otimes \delta y^{(2)j} +$$
$$+ \overset{20}{F}{}^i{}_j \frac{\partial}{\partial y^{(2)i}} \otimes dx^j + \overset{21}{F}{}^i{}_j \frac{\partial}{\partial y^{(2)i}} \otimes \delta y^{(1)j} + \overset{22}{F}{}^i{}_j \frac{\partial}{\partial y^{(2)i}} \otimes \delta y^{(2)j}$$

In this expression $\overset{\alpha\beta}{F}$ $(\alpha, \beta = 0,1,2.)$ are d-tensor field on OSC^2M. We have

$$(1.14)\qquad \begin{cases} F(\frac{\delta}{\delta x^j}) = \overset{00}{F}{}^i{}_j \frac{\delta}{\delta x^i} + \overset{10}{F}{}^i{}_j \frac{\delta}{\delta y^{(1)i}} + \overset{20}{F}{}^i{}_j \frac{\partial}{\partial y^{(2)i}} \\ F(\frac{\delta}{\delta y^{(1)j}}) = \overset{01}{F}{}^i{}_j \frac{\delta}{\delta x^i} + \overset{11}{F}{}^i{}_j \frac{\delta}{\delta y^{(1)i}} + \overset{21}{F}{}^i{}_j \frac{\partial}{\partial y^{(2)i}} \\ F(\frac{\delta}{\delta y^{(2)j}}) = \overset{02}{F}{}^i{}_j \frac{\delta}{\delta x^i} + \overset{12}{F}{}^i{}_j \frac{\delta}{\delta y^{(1)i}} + \overset{22}{F}{}^i{}_j \frac{\partial}{\partial y^{(2)i}} \end{cases}$$

The components of $\tilde{N}$ with respect to the adapted basis can be easily obtained.

F is a $2-\pi$ structure (a integrable almost $2-\pi$ structure) on OSC^2M if and only if $\tilde{N} = 0$.

2. Almost $2-\pi$ d-structure. The Atanasiu operators.

Definition 2.1. *A d-tensor field $\varphi^i_j(x, y^{(1)}, y^{(2)})$ of type (1, 1) on the differentiable manifold OSC^2M, which satisfies the property*

$$(2.1)\qquad \varphi^m_j \varphi^i_m = \lambda^2 \delta^i_j,\ \lambda \in C \setminus \{0\}$$

is called an almost $2-\pi$ *d-structure on* OSC^2M.

Evidently rank

$$\| \varphi^i_j(x, y^{(1)}, y^{(2)}) \| = 6n'$$

and the map $\varphi : X \in \mathcal{X}_d(OSC^2M) \dashrightarrow \varphi X \in \mathcal{X}_d(OSC^2M)$ is an isomorphism, where $\mathcal{X}_d(OSC^2M)$ is the module of d-vector fields on the manifold OSC^2M.

Let $\underset{1}{\Phi}, \underset{2}{\Phi}$ be the Obata operators of the almost $2-\pi$ d-structure φ, [4] :

$$(2.2) \qquad \underset{1}{\Phi}{}^{pq}_{ij} = \frac{1}{2}(\delta^p_i \delta^q_j + \frac{1}{\lambda^2}\varphi^p_i \varphi^q_j), \quad \underset{2}{\Phi}{}^{pq}_{ij} = \frac{1}{2}(\delta^p_i \delta^q_j - \frac{1}{\lambda^2}\varphi^p_i \varphi^q_j).$$

Definition 2.2. *The linear maps* $\overset{\alpha}{A} : \tau^1_2(OSC^2M) \dashrightarrow \tau^1_2(OSC^2M)$ $\alpha \in \overline{(1,4)}$, *given by*

$$(2.3) \qquad \overset{1}{A}(t^m_{pq}) = \lambda^2 t^m_{pq} + \varphi^r_p \varphi^s_q t^m_{rs} - \varphi^m_r \varphi^s_q t^r_{ps} - \varphi^r_p \varphi^m_s t^s_{rq}.$$

$$(2.4) \qquad \overset{2}{A}(t^m_{pq}) = \lambda^2 t^m_{pq} + \varphi^r_p \varphi^s_q t^m_{rs} + \varphi^m_r \varphi^s_q t^r_{ps} + \varphi^r_p \varphi^m_s t^s_{rq},$$

$$(2.5) \qquad \overset{3}{A}(t^m_{pq}) = \lambda^2 t^m_{pq} - \varphi^r_p \varphi^s_q t^m_{rs} + \varphi^m_r \varphi^s_q t^r_{ps} - \varphi^r_p \varphi^m_s t^s_{rq}.$$

$$(2.6) \qquad \overset{4}{A}(t^m_{pq}) = \lambda^2 t^m_{pq} - \varphi^r_p \varphi^s_q t^m_{rs} - \varphi^m_r \varphi^s_q t^r_{ps} + \varphi^r_p \varphi^m_s t^s_{rq}.$$

are called **the Atanasiu operators** *of the* $2-\pi$ *d-structure* φ.

Proposition 2.1. *The Atanasiu operators of the almost product d-structure* $P^i_j(x, y^{(1)}, y^{(2)})$ *are given by :*

$$(2.7) \qquad \overset{1}{A}(t) = \underset{2}{O}\underset{1}{O}t, \quad \overset{2}{A}(t) = \underset{1}{O}\underset{1}{O}t, \quad \overset{3}{A}(t) = \underset{1}{O}\underset{2}{O}t, \quad \overset{4}{A}(t) = \underset{2}{O}\underset{2}{O}t.$$

where $\underset{1}{O}, \underset{2}{O}$ *are the Obata operators of* P *and* $(\underset{\alpha}{O}\underset{\beta}{O}t)^m_{pq} = \underset{\alpha}{O}{}^{rm}_{ps} \underset{\beta}{O}{}^{su}_{rq} t^s_{ru}$ $(\alpha, \beta = 1, 2)$.

The statement results immediately from (2.1), (2.2) and (2.3)-(2.6) for $\lambda^2 = 1$ and $\varphi = P$.

Also, for $\lambda^2 = -1$ and $\varphi = f$, by direct calculations we have

Proposition 2.2. *The Atanasiu operators of the almost complex d-structure* $f^i_j(x, y^{(1)}, y^{(2)})$ *are given by :*

$$(2.8) \qquad \overset{1}{A}(t) = -\underset{2}{Q}\underset{1}{Q}t, \quad \overset{2}{A}(t) = -\underset{1}{Q}\underset{1}{Q}t, \quad \overset{3}{A}(t) = -\underset{1}{Q}\underset{2}{Q}t, \quad \overset{4}{A}(t) = -\underset{2}{Q}\underset{2}{Q}t,$$

where $\underset{1}{Q}, \underset{2}{Q}$ *are the Obata operators of* f *and* $(\underset{\alpha}{Q}\underset{\beta}{Q})^m_{pq} = \underset{\alpha}{Q}{}^{rm}_{ps} \underset{\beta}{Q}{}^{su}_{rq} t^s_{ru}$ $(\alpha, \beta = 1, 2)$.

Let us consider a nonlinear connection $N(\underset{(1)}{N}, \underset{(2)}{N})$ arbitrary fixed on OSC^2M.

Definition 2.3. *A N-linear connection* $D\Gamma(N) = (L^m_{ij}, \underset{(1)}{C}{}^m_{ij}, \underset{(2)}{C}{}^m_{ij})$ *on* OSC^2M *is called an almost* $2-\pi$ *N-linear connection with respect to the almost* $2-\pi$ *d-structure* φ, *(2.1), if the* h- *and* v_1-, v_2- *covariant derivatives vanish :*

$$(2.9) \qquad \varphi^i_{j|m} = 0, \quad \varphi^i_j \overset{(1)}{|}_m = 0 \quad \varphi^i_j \overset{(2)}{|}_m = 0$$

Theorem 2.1. *([4]) The set of all almost* $2-\pi$ *N-linear connections is given by*

$$L^m_{ij} = \overset{0}{L}{}^m_{ij} - \frac{1}{2\lambda^2}\varphi^r_i \varphi^m_{r\overset{0}{|}j} + \underset{1}{\Phi}{}^{sm}_{ir} Y^r_{sj},$$

$$\underset{(\alpha)}{C}{}^m_{ij} = \underset{(\alpha)}{\overset{0}{C}}{}^m_{ij} - \frac{1}{2\lambda^2}\varphi^r_i \varphi^i_r \underset{(\alpha)}{\overset{0}{|}}_j + \underset{1}{\Phi}{}^{sm}_{ir} \underset{(\alpha)}{Z}{}^n_{sj}, \quad \alpha = 1, 2,$$

where $D\overset{0}{\Gamma}(N) = (\overset{0}{L}{}^m_{ij}, \underset{(1)}{\overset{0}{C}}{}^m_{ij}, \underset{(2)}{\overset{0}{C}}{}^m_{ij})$ is a fixed N-linear connection, $\overset{0}{|}$ and $\underset{(\alpha)}{\overset{0}{|}}$ $(\alpha = 1, 2)$ denote h- and $v_\alpha-$covariant derivatives with respect to $D\overset{0}{\Gamma}(N)$ and $Y^m_{ij}, \underset{(\alpha)}{Z}{}^m_{ij}$ are arbitrary d-tensor fields.

3. The group of transformations of almost 2–π N–linear connections. Let us consider the transformation $D\Gamma(N) \to D\overline{\Gamma}(N)$ of almost 2–π N–linear connections, which preserve the nonlinear connection $N(\underset{(1)}{N}, \underset{(2)}{N})$. Owing to the Theorem 2.1 they are given by

$$(3.1)\qquad \overline{L}^{m}_{ij} = L^{m}_{ij} + (\underset{i}{\Phi} Y)^{m}_{ij},\quad \underset{(\alpha)}{\overline{C}}{}^{m}_{ij} = \underset{(\alpha)}{C}{}^{m}_{ij} + (\Phi \underset{i\,(\alpha)}{Z})^{m}_{ij},\quad \alpha = 1, 2.$$

Theorem 3.1. *([4]) The set of all transformations (3.1) with the mapping product as law of composition form an Abelian group G, which is isomorphic with the additive group of the set of the d-tensor fields having the form $(\underset{i}{\Phi} Y, \underset{1}{\Phi}\underset{(1)}{Z}, \underset{1}{\Phi}\underset{(2)}{Z})$.*

We shall pay attention to the invariants of the group G. By straightforward calculus we have

Theorem 3.2. *There exist forty three d-tensor fields which are left invariants by the group G. The first twenty six invariants are given by the following actions of the Atanasiu operators on the d-tensor fields of torsion:*

$$(3.2)\qquad \begin{cases} \overset{1}{A}(T),\quad \overset{1}{A}(\overset{(1)}{\underset{(0)}{R}}),\quad \overset{1}{A}(\overset{(2)}{\underset{(0)}{R}}) \\ \overset{1}{A}(\underset{(1)}{C}),\quad \overset{1}{A}(\overset{(1)}{\underset{(1)}{P}}),\quad \overset{1}{A}(\overset{(2)}{\underset{(1)}{P}}) \\ \overset{1}{A}(\underset{(2)}{C}),\quad \overset{1}{A}(\overset{(1)}{\underset{(2)}{P}}),\quad \overset{1}{A}(\overset{(2)}{\underset{(2)}{P}}) \\ \overset{1}{A}(\underset{(1)}{S}),\quad \overset{1}{A}(\overset{(2)}{\underset{(1)}{S}}),\quad \overset{1}{A}(\underset{(2)}{S}) \end{cases}$$

$$(3.3)\qquad \begin{cases} \overset{2}{A}(\overset{(1)}{\underset{(0)}{R}}),\quad \overset{2}{A}(\overset{(2)}{\underset{(0)}{R}}) \\ \overset{3}{A}(\underset{(1)}{C}),\quad \overset{4}{A}(\overset{(1)}{\underset{(1)}{P}}),\quad \overset{3}{A}(\overset{(2)}{\underset{(1)}{P}}) \\ \overset{3}{A}(\underset{(2)}{C}),\quad \overset{2}{A}(\overset{(1)}{\underset{(2)}{P}}),\quad \overset{2}{A}(\overset{(2)}{\underset{(1)}{S}}),\quad \overset{4}{A}(\overset{(1)}{\underset{(2)}{P}} + \underset{(1)}{S} - \underset{(1)}{C}) \end{cases}$$

$$(3.4)\qquad \overset{2}{A}(\overset{(2)}{\underset{(1)}{P}}),\quad \overset{3}{A}(\overset{(1)}{\underset{(2)}{P}}),\quad \overset{4}{A}(\overset{(2)}{\underset{(2)}{P}})$$

$$(3.5)\qquad \overset{4}{A}(\overset{(2)}{\underset{(1)}{P}}),\quad \overset{4}{A}(\overset{(1)}{\underset{(2)}{P}})$$

and the others invariants are given by

$$(3.6)_1\qquad \begin{cases} \overset{51}{T}{}^{m}_{pq} = \lambda^2 T^{m}{}_{pq} + \varphi^{m}_{s}(\varphi^{r}_{p}\overset{(2)}{\underset{(2)}{P}}{}^{s}{}_{qr} - \varphi^{r}_{q}\overset{(2)}{\underset{(2)}{P}}{}^{s}{}_{pr}), \\ \overset{51}{R}{}^{m}_{pq} = \lambda^2 \overset{(1)}{\underset{(0)}{R}}{}^{m}{}_{pq} + \varphi^{m}_{s}(\varphi^{r}_{p}\overset{(1)}{\underset{(2)}{P}}{}^{s}{}_{qr} - \varphi^{r}_{q}\overset{(1)}{\underset{(2)}{P}}{}^{s}{}_{pr}) \\ \underset{52}{R}{}^{m}_{pq} = \lambda^2 \overset{(2)}{\underset{(0)}{R}}{}^{m}{}_{pq} + \varphi^{m}_{s}(\varphi^{r}_{p}\underset{(2)}{C}{}^{s}{}_{qr} - \varphi^{r}_{q}\underset{(2)}{C}{}^{s}{}_{pr}), \end{cases}$$

$$(3.6)_2\qquad \begin{cases} \overset{51}{\underset{1}{C}}{}^{m}_{pq} = \lambda^2 \underset{(1)}{C}{}^{m}_{pq} + \varphi^{m}_{s}\{\varphi^{r}_{p}(\overset{(1)}{\underset{(2)}{P}}{}^{s}{}_{qr} - \underset{(1)}{C}{}^{s}_{rq}) - \varphi^{r}_{q}\overset{(2)}{\underset{(1)}{P}}{}^{s}{}_{pr}\}, \\ \overset{51}{\underset{1}{P}}{}^{m}_{pq} = \lambda^2 \overset{(1)}{\underset{(1)}{P}}{}^{m}{}_{pq} - \varphi^{r}_{p}\varphi^{s}_{q}\underset{(2)}{C}{}^{m}_{sr} + \varphi^{m}_{s}(\varphi^{r}_{p}\underset{(2)}{C}{}^{s}_{qr} - \varphi^{r}_{q}\overset{(1)}{\underset{(1)}{P}}{}^{s}{}_{pr}), \\ \overset{52}{\underset{1}{P}}{}^{m}_{pq} = \lambda^2 \overset{(2)}{\underset{(1)}{P}}{}^{m}{}_{pq} - \varphi^{r}_{p}\varphi^{s}_{q}(\overset{(1)}{\underset{(2)}{P}}{}^{m}{}_{sr} - \underset{(1)}{C}{}^{m}_{rs}) - \varphi^{r}_{q}\varphi^{m}_{s}\underset{(1)}{C}{}^{s}_{pr}, \end{cases}$$

$$(3.6)_3\qquad \begin{cases} \overset{52}{\underset{1}{C}}{}^{m}_{pq} = \lambda^2 \underset{(2)}{C}{}^{m}_{pq} - \varphi^{r}_{p}\varphi^{s}_{q}\underset{(2)}{C}{}^{m}_{sr} + \varphi^{m}_{s}(\varphi^{r}_{p}\underset{(2)}{S}{}^{s}{}_{qr} - \varphi^{r}_{q}\overset{(2)}{\underset{(0)}{R}}{}^{s}{}_{pr}), \\ \overset{51}{\underset{2}{P}}{}^{m}_{pq} = \lambda^2 \overset{(1)}{\underset{(2)}{P}}{}^{m}{}_{pq} - \varphi^{r}_{p}\varphi^{s}_{q}\overset{(1)}{\underset{(2)}{P}}{}^{m}{}_{rs} - \varphi^{r}_{q}\varphi^{m}_{s}\overset{(1)}{\underset{(0)}{R}}{}^{s}{}_{pr}, \\ \overset{52}{\underset{2}{P}}{}^{m}_{pq} = \lambda^2 \overset{(2)}{\underset{(2)}{P}}{}^{m}{}_{pq} - \varphi^{r}_{p}\varphi^{s}_{q}(\overset{(2)}{\underset{(2)}{P}}{}^{m}{}_{rs} - T^{m}{}_{rs}) - \varphi^{r}_{q}\varphi^{m}_{s}T^{s}{}_{pr}, \end{cases}$$

$$(3.6)_4\qquad \begin{cases} \overset{52}{T}{}^{m}_{pq} = \varphi^{m}_{s}(\varphi^{r}_{p}\overset{(2)}{\underset{(1)}{S}}{}^{s}{}_{qr} - \varphi^{r}_{q}\overset{(2)}{\underset{(1)}{S}}{}^{s}{}_{pr}), \\ \overset{52}{\underset{1}{S}}{}^{m}_{pq} = \lambda^2 \overset{(2)}{\underset{(1)}{S}}{}^{m}{}_{pq} + \varphi^{r}_{p}\varphi^{s}_{q}\overset{(2)}{\underset{(1)}{S}}{}^{m}{}_{rs}, \end{cases}$$

$$(3.6)_5\qquad \begin{cases} \overset{53}{T}{}^{m}_{pq} = \varphi^{r}_{p}\varphi^{s}_{q}\underset{(1)}{C}{}^{m}_{sr} + \varphi^{m}_{s}\{\varphi^{r}_{p}(\overset{(1)}{\underset{(2)}{P}}{}^{s}{}_{rq} - \underset{(1)}{C}{}^{s}_{qr}) - \varphi^{r}_{q}\overset{(2)}{\underset{(1)}{P}}{}^{s}{}_{rp}\}, \\ \overset{51}{\underset{2}{C}}{}^{m}_{pq} = \lambda^2(\underset{(1)}{C}{}^{m}_{qp} - \overset{(1)}{\underset{(2)}{P}}{}^{m}{}_{pq}) + \varphi^{r}_{p}\varphi^{s}_{q}\overset{(2)}{\underset{(1)}{P}}{}^{m}{}_{sr} - \varphi^{r}_{q}\varphi^{m}_{s}\underset{(1)}{C}{}^{s}_{rp}, \\ \overset{52}{\underset{2}{C}}{}^{m}_{pq} = \lambda^2 \underset{(2)}{C}{}^{m}_{pq} - \varphi^{r}_{p}\varphi^{s}_{q}\overset{(1)}{\underset{(1)}{P}}{}^{m}{}_{sr} - \varphi^{m}_{s}(\varphi^{r}_{p}\underset{(2)}{C}{}^{s}_{rq} - \varphi^{r}_{q}\overset{(1)}{\underset{(1)}{P}}{}^{s}{}_{rp}) \end{cases}$$

$$(3.6)_6\qquad \begin{cases} \overset{54}{T}{}^{m}_{pq} = \varphi^{r}_{s}\varphi^{s}_{q}T^{m}{}_{rs} - \varphi^{m}_{s}(\varphi^{r}_{p}\overset{(2)}{\underset{(2)}{P}}{}^{s}{}_{rq} - \varphi^{r}_{q}\overset{(2)}{\underset{(2)}{P}}{}^{s}{}_{rp}), \\ \overset{55}{T}{}^{m}_{pq} = \varphi^{r}_{p}\varphi^{s}_{q}\overset{(1)}{\underset{(0)}{R}}{}^{m}{}_{rs} - \varphi^{m}_{p}(\varphi^{r}_{p}\overset{(1)}{\underset{(2)}{P}}{}^{s}{}_{rq} - \varphi^{r}_{q}\overset{(1)}{\underset{(2)}{P}}{}^{s}{}_{rp}), \\ \overset{52}{\underset{2}{S}}{}^{m}_{pq} = \lambda^2 \underset{(2)}{S}{}^{m}{}_{pq} + \varphi^{r}_{p}\varphi^{s}_{q}\overset{(2)}{\underset{(0)}{R}}{}^{m}{}_{rs} - \varphi^{m}_{s}(\varphi^{r}_{p}\underset{(2)}{C}{}^{s}_{rq} - \varphi^{r}_{q}\underset{(2)}{C}{}^{s}_{rp}) \end{cases}$$

The integrability of an almost 2–π d–structure. Assume that a nonlinear connection $N(\underset{(1)}{N}, \underset{(2)}{N})$ is given in the bundle of accelerations OSC^2M. Then, an almost 2-π d-structure φ on OSC^2M is lifted to an almost 2-π structure F on $T(OSC^2M)$ in five manner, pointed out by Atanasiu–Păun–Stoica [2], Stoica–Atanasiu, [10]:

$$\begin{aligned}
F^{I} &= \varphi^i_j\frac{\delta}{\delta x^i}\otimes dx^j + \varphi^i_j\frac{\delta}{\delta y^{(1)i}}\otimes\delta y^{(1)j} + \varphi^i_j\frac{\partial}{\partial y^{(2)i}}\otimes\delta y^{(2)j},\\
F^{II} &= \varphi^i_j\frac{\delta}{\delta x^i}\otimes dx^j - \varphi^i_j\frac{\delta}{\delta y^{(1)i}}\otimes\delta y^{(1)j} + \varphi^i_j\frac{\partial}{\partial y^{(2)i}}\otimes\delta y^{(2)j},\\
F^{III} &= \varphi^i_j\frac{\delta}{\delta x^i}\otimes dx^j + \varphi^i_j\frac{\delta}{\delta y^{(1)i}}\otimes\delta y^{(1)j} - \varphi^i_j\frac{\partial}{\partial y^{(2)i}}\otimes\delta y^{(2)j},\\
F^{IV} &= \varphi^i_j\frac{\delta}{\delta x^i}\otimes dx^j - \varphi^i_j\frac{\delta}{\delta y^{(1)i}}\otimes\delta y^{(1)j} - \varphi^i_j\frac{\partial}{\partial y^{(2)i}}\otimes\delta y^{(2)j},\\
F^{V} &= \varphi^i_j\frac{\delta}{\delta x^i}\otimes\delta y^{(2)j} + \varphi^i_j\frac{\delta}{\delta y^{(1)i}}\otimes\delta y^{(1)j} + \varphi^i_j\frac{\partial}{\partial y^{(2)i}}\otimes dx^j
\end{aligned}\tag{4.1}$$

Then the values of the d-components of F, from (1.13) are given in the table

F	$\overset{00}{F}$	$\overset{01}{F}$	$\overset{02}{F}$	$\overset{10}{F}$	$\overset{11}{F}$	$\overset{12}{F}$	$\overset{20}{F}$	$\overset{21}{F}$	$\overset{22}{F}$
F^{I}	φ^i_j	0	0	0	φ^i_j	0	0	0	φ^i_j
F^{II}	φ^i_j	0	0	0	$-\varphi^i_j$	0	0	0	φ^i_j
F^{III}	φ^i_j	0	0	0	φ^i_j	0	0	0	$-\varphi^i_j$
F^{IV}	φ^i_j	0	0	0	$-\varphi^i_j$	0	0	0	$-\varphi^i_j$
F^{V}	0	0	φ^i_j	0	φ^i_j	0	φ^i_j	0	0

Definition 4.1. *An almost 2-π d-structure φ on OSC^2M is called integrable of the type I, II, III, IV or V with respect to the nonlinear connection $N(\underset{(1)}{N}, \underset{(2)}{N})$, if the corresponding lifted almost 2-π structures $F^{I}, F^{II}, F^{III}, F^{IV}$ or F^{V} are integrable.*

We can characterize these cases of integrability using only the invariants of the group G. Indeed, we have

Theorem 4.2. *The almost 2-π d-structure φ is integrable of the type I, II, III, IV or V if and only if the invariants of the group G have the values given in the following table*

type	characterization
I	$\overset{1}{A}(T)=0, \overset{1}{A}(\overset{(1)}{\underset{(0)}{R}})=0, \overset{1}{A}(\overset{(2)}{\underset{(0)}{R}})=0, \overset{1}{A}(\underset{(1)}{C})=0, \overset{1}{A}(\overset{(1)}{\underset{(1)}{P}})=0, \overset{1}{A}(\overset{(2)}{\underset{(1)}{P}})=0$ $\overset{1}{A}(\underset{(2)}{C})=0, \overset{1}{A}(\overset{(1)}{\underset{(2)}{P}})=0, \overset{1}{A}(\overset{(2)}{\underset{(2)}{P}})=0, \overset{1}{A}(\underset{(1)}{S})=0, \overset{1}{A}(\overset{(2)}{\underset{(1)}{S}})=0, \overset{1}{A}(\underset{(2)}{S})=0$
II	$\overset{1}{A}(T)=0, \overset{2}{A}(\overset{(1)}{\underset{(0)}{R}})=0, \overset{2}{A}(\overset{(2)}{\underset{(0)}{R}})=0, \overset{3}{A}(\underset{(1)}{C})=0, \overset{4}{A}(\overset{(1)}{\underset{(1)}{P}})=0, \overset{3}{A}(\overset{(2)}{\underset{(1)}{P}})=0$ $\overset{1}{A}(\underset{(2)}{C})=0, \overset{2}{A}(\overset{(1)}{\underset{(2)}{P}})=0, \overset{1}{A}(\overset{(2)}{\underset{(2)}{P}})=0, \overset{1}{A}(\underset{(1)}{S})=0, \overset{2}{A}(\overset{(2)}{\underset{(1)}{S}})=0, \overset{1}{A}(\underset{(2)}{S})=0$ $\overset{3}{A}(\underset{(2)}{C})=0, \overset{4}{A}(\overset{(1)}{\underset{(2)}{P}}+\underset{(1)}{S}-\underset{(1)}{C})=0$
III	$\overset{1}{A}(T)=0, \overset{1}{A}(\overset{(1)}{\underset{(0)}{R}})=0, \overset{2}{A}(\overset{(2)}{\underset{(0)}{R}})=0, \overset{1}{A}(\underset{(1)}{C})=0, \overset{1}{A}(\overset{(1)}{\underset{(1)}{P}})=0, \overset{2}{A}(\overset{(2)}{\underset{(1)}{P}})=0$ $\overset{3}{A}(\underset{(2)}{C})=0, \overset{3}{A}(\overset{(1)}{\underset{(2)}{P}})=0, \overset{4}{A}(\overset{(2)}{\underset{(2)}{P}})=0, \overset{1}{A}(\underset{(1)}{S})=0, \overset{2}{A}(\overset{(2)}{\underset{(1)}{S}})=0, \overset{1}{A}(\underset{(2)}{S})=0$ $\overset{4}{A}(\overset{(1)}{\underset{(2)}{P}}+\underset{(1)}{S}-\underset{(1)}{C})=0$
IV	$\overset{1}{A}(T)=0, \overset{2}{A}(\overset{(1)}{\underset{(0)}{R}})=0, \overset{2}{A}(\overset{(2)}{\underset{(0)}{R}})=0, \overset{3}{A}(\underset{(1)}{C})=0, \overset{4}{A}(\overset{(1)}{\underset{(1)}{P}})=0, \overset{4}{A}(\overset{(2)}{\underset{(1)}{P}})=0$ $\overset{3}{A}(\underset{(2)}{C})=0, \overset{4}{A}(\overset{(1)}{\underset{(2)}{P}})=0, \overset{4}{A}(\overset{(2)}{\underset{(2)}{P}})=0, \overset{1}{A}(\underset{(1)}{S})=0, \overset{2}{A}(\overset{(2)}{\underset{(1)}{S}})=0, \overset{1}{A}(\underset{(2)}{S})=0$ $\overset{4}{A}(\underset{(1)}{S}-\underset{(1)}{C})=0$

type	characterization
V	$\overset{51}{T}=0, \overset{51}{R}=0, \overset{52}{R}=0, \underset{1}{\overset{51}{C}}=0, \underset{1}{\overset{51}{P}}=0, \underset{1}{\overset{52}{P}}=0$ $\overset{52}{T}=0, \overset{53}{T}=0, \overset{54}{T}=0, \underset{1}{\overset{52}{C}}=0, \underset{2}{\overset{51}{P}}=0, \underset{2}{\overset{52}{P}}=0$ $\underset{1}{\overset{52}{S}}=0, \underset{2}{\overset{52}{S}}=0, \overset{55}{T}=0, \underset{2}{\overset{51}{C}}=0, \underset{2}{\overset{52}{C}}=0$

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Transilvania University
Department of Geometry
2200 Braşov, Romania

University of Bacău
Department of Mathematics
5500 - Bacău, Romania

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SOME PROPERTIES OF β–CONTINUOUS MULTIFUNCTIONS

BY

VALERIU POPA AND TAKASHI NOIRI

1. Introduction. As a generalization of semi-continuity [8] and precontinuity [9], Abd El-Monsef et al. [11] defined β-continuous functions. Quite recently, Przemski [20] has introduced the notion of pre-semi-continuous multifunctions. The purpose of the present paper is to introduce and investigate the notion of β-continuous multifunctions. We obtain several characterizations of such multifunctions and show that pre-semi-continuity of multifunctions due to Przemski [20] is equivalent to β-continuity of multifunctions.

2. Preliminaries. Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be β-open [11] or presemiopen [1] (resp. preopen [9], semi-open [8], α-open [13]) if $A \subset \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$ (resp. $A \subset \mathrm{Int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{cl}(\mathrm{Int}(A)))$). The family of β-open (resp. preopen, semi-open, α-open) sets of X is denoted by $\beta O(X)$ (resp. $PO(X)$, $SO(X)$, $\alpha(X)$). By $\beta(X, x)$ (resp. $SO(X, x)$, $\alpha(X, x)$), we denote the family of β-open (resp. semi-open, α-open) sets of X containing a point x of X. The union of all β-open (resp. preopen, semi-open) sets contained in A is called the β-interior [12] (resp. preinterior [1], semi-interior [3] and is denoted by $\beta\mathrm{Int}(A)$ (resp. $p\mathrm{Int}(A)$, $s\mathrm{Int}(A)$). The complement of a β-open (resp. preopen, semi-open) set is said to be β-closed (resp. preclosed, semi-closed). The intersection of all β-closed (resp. preclosed, semi-closed) sets containing A is called the β-closure [10] (resp. preclosure [6], semi-closure [5] and is denoted by $\beta\mathrm{cl}(A)$ (resp. $p\mathrm{cl}(A)$, $s\mathrm{cl}(A)$).

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