| type | characterization |
|---|---|
| V | $\overset{51}{T}=0,\ \overset{51}{R}=0,\ \overset{52}{R}=0,\ \underset{1}{\overset{51}{C}}=0,\ \underset{1}{\overset{51}{P}}=0,\ \underset{1}{\overset{52}{P}}=0$ <br> $\overset{52}{T}=0,\ \overset{53}{T}=0,\ \overset{54}{T}=0,\ \underset{1}{\overset{52}{C}}=0,\ \underset{2}{\overset{51}{P}}=0,\ \underset{2}{\overset{52}{P}}=0$ <br> $\underset{1}{\overset{52}{S}}=0,\ \underset{2}{\overset{52}{S}}=0,\ \overset{55}{T}=0,\ \underset{2}{\overset{51}{C}}=0,\ \underset{2}{\overset{52}{C}}=0$ |

REFERENCES


1. Atanasiu, Gh. – *The bundle of accelerations. The crossings and the equations of structure.* Proc. of the Eight Nat.Sem. on Finsler, Lagrange and Hamilton Spaces, Transilvania Univ., Braşov, 1994 (to appear).
2. Atanasiu, Gh., Păun, M., Stoica, E. – *Integrability of almost complex structures in the bundle of accelerations.* Memoriile Sect.St.Acad.Română (to appear).
3. Atanasiu, Gh., Miron, R. – *Structure equations of a N-linear connection in the bundle of higher order velocities.* Lecture Nat.Coll.Geom. Top., sept.1995, Univ."Al. I. Cuza" Iaşi (to appear).
4. Blănuţă, V., Gîrţu, M. – *N-linear connections compatible with almost $2-\pi$ structures on the k-osculator bundle.* Proc. of the Eight Nat.Sem. on Finsler, Lagrange and Hamilton Spaces, Transilvania Univ., Braşov, 1994 (to appear).
5. HSU, C.J. – *On some properties of $\pi$ -structures on differentiable manifold.* Tohoku Math.Journ., 12, 1960, 429–454.
6. Miron, R., Atanasiu, Gh. – *Lagrange Geometry of Second Order.* Math. Comput. Modelling, 20, 415, 1994, 41–56. & 7. Miron, R., Atanasiu, Gh. – *Differential geometry of the k-osculator bundle.* Rev. Roum. Mat Pures et Appl. (to appear).
8. Miron, R., Atanasiu, Gh. – *Prolongation of the Riemannian, Finslerian and Lagrangian structures.* Rev. Roum. Math. Pures et Appl. (to appear).
9. Miron, R., Atanasiu, Gh. – *Higher order Lagrange spaces.* Rev. Roum. Math. Pures et Appl. (to appear).
10. Stoica, E., Atanasiu, Gh. – *Integrability of almost product structure in the osculator bundle of second order.* Tensor N.S.Japonia (to appear).


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# SOME PROPERTIES OF $\beta$-CONTINUOUS MULTIFUNCTIONS

BY

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**1. Introduction.** As a generalization of semi-continuity [8] and precontinuity [9], Abd El-Monsef et al. [11] defined $\beta$-continuous functions. Quite recently, Przemski [20] has introduced the notion of pre-semi-continuous multifunctions. The purpose of the present paper is to introduce and investigate the notion of $\beta$-continuous multifunctions. We obtain several characterizations of such multifunctions and show that pre-semi-continuity of multifunctions due to Przemski [20] is equivalent to $\beta$-continuity of multifunctions.

**2. Preliminaries.** Let $X$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{cl}\,(A)$ and $\mathrm{Int}\,(A)$, respectively. A subset $A$ is said to be $\beta$-open [11] or presemiopen [1] (resp. preopen [9], semi-open [8], $\alpha$-open [13]) if $A \subset \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(A)))$ (resp. $A \subset \mathrm{Int}\,(\mathrm{cl}\,(A))$, $A \subset \mathrm{cl}\,(\mathrm{Int}\,(A))$, $A \subset \mathrm{Int}\,(\mathrm{cl}\,(\mathrm{Int}\,(A)))$). The family of $\beta$-open (resp. preopen, semi-open, $\alpha$-open) sets of $X$ is denoted by $\beta O(X)$ (resp. $PO(X)$, $SO(X)$, $\alpha(X)$). By $\beta(X, x)$ (resp. $SO(X, x)$, $\alpha(X, x)$), we denote the family of $\beta$-open (resp. semi-open, $\alpha$-open) sets of $X$ containing a point $x$ of $X$. The union of all $\beta$-open (resp. preopen, semi-open) sets contained in $A$ is called the $\beta$-interior [12] (resp. preinterior [1], semi-interior [3] and is denoted by $\beta\mathrm{Int}\,(A)$ (resp. $p\mathrm{Int}\,(A)$, $s\mathrm{Int}\,(A)$). The complement of a $\beta$-open (resp. preopen, semi-open) set is said to be $\beta$-closed (resp. preclosed, semi-closed). The intersection of all $\beta$-closed (resp. preclosed, semi-closed) sets containing $A$ is called the $\beta$-closure [10] (resp. preclosure [6], semi-closure [5] and is denoted by $\beta\mathrm{cl}\,(A)$ (resp. $p\mathrm{cl}\,(A)$, $s\mathrm{cl}\,(A)$).

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**Lemma 1.** (Andrijevic [1]) *Let $A$ be a subset of a topological space $X$. Then, the following hold:*

*(1)* $\beta\mathrm{Int}\,(A) = A \cap \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(A)))$.

*(2)* $\beta\mathrm{cl}\,(A) = A \cup \mathrm{Int}\,(\mathrm{cl}\,(\mathrm{Int}\,(A)))$.

Throughout this paper, $X$ and $Y$ always mean topological spaces and $F: X \to Y$ (resp. $f: X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F: X \to Y$, we shall denote the upper and lower inverse of a subset $B$ of $Y$ by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $\boldsymbol{A}(Y)$ be the collection of all nonempty subsets of $Y$. For an open set $O$ of $Y$, we denote

$$O^+ = \{A \in \boldsymbol{A}(Y) : A \subset O\} \text{ and } O^- = \{A \in \boldsymbol{A}(Y) : A \cap O \neq \emptyset\}.$$

**Definition 1.** *A multifunction $F: X \to Y$ is said to be pre-semi-continuous [20] at a point $x \in X$ if for each open sets $U, V$ of $Y$ such that $F(x) \in U^+ \cap V^-$, $x \in \beta\mathrm{Int}\,(F^+(U) \cap F^-(V))$. $F: X \to Y$ is said to be pre-semi-continuous if it has this property at any point $x \in X$.*

## 3. $\beta$-continuous multifunctions.

**Definition 2.** *A multifunction $F: X \to Y$ is said to be $\beta$-continuous at $x \in X$ if for each open sets $V_1, V_2$ of $Y$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$, there exists $U \in \beta(X, x)$ such that $F(u) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for every $u \in U$. $F: X \to Y$ is said to be $\beta$-continuous if it has this property at any point $x \in X$.*

**Theorem 1.** *The following are equivalent for a multifunction $F: X \to Y$ :*

*(1)* $F$ *is pre-semi-continuous at* $x$ ;

*(2)* $F$ *is* $\beta$*-continuous at* $x$ ;

*(3) for any open sets $G_1, G_2$ of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$, $x \in \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))))$ ;*

*(4) for each neighborhood $U$ of $x$ and any open sets $G_1, G_2$ of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$, there exists an open set $G$ of $X$ such that*

$$\emptyset \neq G \subset U \text{ and } G \subset \mathrm{cl}\,(F^+(G_1) \cap F^-(G_2));$$

*(5) for any open sets $G_1, G_2$ of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$, there exists $U \in SO(X, x)$ such that $U \subset \mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))$.*



**Proof.** (1) $\Rightarrow$ (2): Let $F$ be pre-semi- continuous at $x$ and $G_1, G_2$ any open sets of $Y$ such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. Then, we have $F(x) \in G_1^+ \cap G_2^-$ and hence $x \in \beta\mathrm{Int}\,(F^+(G_1) \cap F^-(G_2))$. Put $U = \beta\mathrm{Int}\,(F^+(G_1) \cap F^-(G_2))$. Then $U \in \beta(X, x)$ and $F(u) \subset G_1$, and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Therefore, $F$ is $\beta$-continuous at $x$.

(2) $\Rightarrow$ (3): Let $G_1, G_2$ be any open sets of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$. By (2), there exists $U \in \beta(X, x)$ such that $F(u) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for every $u \in U$. Thus, we have $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Since $U$ is $\beta$-open, we obtain

$$x \in U \subset \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(U))) \subset \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(F^+(G_1) \cap F^-(G_2)))).$$

(3) $\Rightarrow$ (4): Let $G_1, G_2$ be any open sets of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$ and $U$ an open neighborhood of $x$. Then, by (3) we have $x \in \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))))$. Put $G = U \cap \mathrm{Int}\,(\mathrm{cl}\,(F^+(G_1) \cap F^-(G_2)))$. Then $G$ is open in $X$, $\emptyset \neq G \subset U$ and $G \subset \mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))$.

(4) $\Rightarrow$ (5): Let $G_1, G_2$ be any open sets of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$. By $\boldsymbol{U}(x)$ we denote the family of all open neighborhoods of $x$. For each $U \in \boldsymbol{U}(x)$, there exists an open set $G_U$ of $X$ such that $\emptyset \neq G_U \subset U$ and $G_U \subset \mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))$. Put $W = \bigcup\{G_U : U \in \boldsymbol{U}(x)\}$. Then $W$ is open in $X$, $x \in \mathrm{cl}\,(W)$ and $W \subset \mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))$.

(5) $\Rightarrow$ (1): Let $G_1, G_2$ be any open sets of $Y$ such that $F(x) \in G_1^+ \cap G_2^-$. There exists $U \in SO(X, x)$ such that $U \subset \mathrm{cl}\,(F^+(G_1) \cap F^-(G_2))$. Therefore, we have

$$x \in U \subset \mathrm{cl}\,(\mathrm{Int}\,(U)) \subset \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(F^+(G_1) \cap F^-(G_2)))).$$

Thus by Lemma 1 we have

$$x \in [F^+(G_1) \cap F^-(G_2)] \cap \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,[F^+(G_1) \cap F^-(G_2)])) = \beta\mathrm{Int}\,[F^+(G_1) \cap F^-(G_2)].$$

This shows that $F$ is pre-semi-continuous at $x$.

**Corollary 1.** (Popa and Noiri [18]) *The following properties are equivalent for a function $f: X \to Y$:*

*(1)* $f$ *is pre-semi-continuous at* $x$;

*(2)* $f$ *is* $\beta$*-continuous at* $x$;

*(3) for any open set $G$ of $Y$ containing $f(x)$, $x \in \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(f^{-1}(G))))$;*

(4) *for each neighborhood $V$ of $f(x)$ and each neighborhood $U$ of $x$ there exists a nonempty open set $G$ of $X$ such that $G \subset U$ and $G \subset \mathrm{cl}\,(f^{-1}(V))$;*
(5) *for each neighborhood $V$ of $f(x)$, there exists $U \in SO(X, x)$ such that $U \subset \mathrm{cl}\,(f^{-1}(V))$.*

**Theorem 2.** *The following are equivalent for a multifunction $F\colon X \to Y$:*
(1) *$F$ is $\beta$-continuous;*
(2) *$F^+(G_1) \cap F^-(G_2) \in \beta(X)$ for any open sets $G_1, G_2$ of $Y$;*
(3) *$F^-(K_1) \cup F^+(K_2)$ is $\beta$-closed in $X$ for any closed sets $K_1, K_2$ of $Y$;*
(4) *$\beta\mathrm{cl}\,(F^-(B_1) \cup F^+(B_2)) \subset F^-(\mathrm{cl}\,(B_1)) \cup F^+(\mathrm{cl}\,(B_2))$ for any subsets $B_1, B_2$ of $Y$;*
(5) *$\mathrm{Int}\,(\mathrm{cl}\,(\mathrm{Int}\,(F^-(B_1) \cup F^+(B_2)))) \subset F^-(\mathrm{cl}\,(B_1)) \cup F^+(\mathrm{cl}\,(B_2))$ for any subsets $B_1, B_2$ of $Y$;*
(6) *$F^-(\mathrm{Int}\,(B_1)) \cap F^+(\mathrm{Int}\,(B_2)) \subset \beta\mathrm{Int}\,(F^-(B_1) \cap F^+(B_2))$ for any subsets $B_1, B_2$ of $Y$.*

**Proof.** (1) $\Rightarrow$ (2): Let $G_1, G_2$ be any open sets of $Y$ and $x \in F^+(G_1) \cap F^-(G_2)$. Then we have $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$ and hence $F(x) \in G_1^+ \cap G_2^-$. Therefore, we have $x \in \beta\mathrm{Int}\,(F^+(G_1) \cap F^-(G_2))$ by Theorem 1. Thus, we obtain $F^+(G_1) \cap F^-(G_2) \subset \beta\mathrm{Int}\,(F^+(G_1) \cap F^-(G_2))$. This shows that $F^+(G_1) \cap F^-(G_2) \in \beta(X)$.

(2) $\Rightarrow$ (3): This easily follows from the fact that $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for every subset $B$ of $Y$.

(3) $\Rightarrow$ (4): Let $B_1, B_2$ be any subsets of $Y$.
Then $F^-(\mathrm{cl}\,(B_1)) \cup F^+(\mathrm{cl}\,(B_2))$ is a $\beta$-closed set of $X$. Therefore, we obtain

$$\begin{aligned}&\beta\mathrm{cl}\,(F^-(B_1) \cup F^+(B_2)) \subset \beta\mathrm{cl}\,(F^-(\mathrm{cl}\,(B_1)) \cup F^+(\mathrm{cl}\,(B_2))) =\\ &= F^-(\mathrm{cl}\,(B_1)) \cup F^+(\mathrm{cl}\,(B_2)).\end{aligned}$$

(4) $\Rightarrow$ (5): Let $B_1, B_2$ be any subsets of $Y$. It follows from Lemma 1 that

$$\begin{aligned}&\mathrm{Int}\,(\mathrm{cl}\,(\mathrm{Int}\,(F^-(B_1) \cup F^+(B_2)))) \subset \beta\mathrm{cl}\,(F^-(B_1) \cup F^+(B_2)) \subset\\ &\subset F^-(\mathrm{cl}\,(B_1)) \cup F^+(\mathrm{cl}\,(B_2)).\end{aligned}$$

(5) $\Rightarrow$ (6): Let $B_1, B_2$ be any subsets of $Y$. Then, by using Lemma 1 we have



$$\begin{aligned}&X - \beta\mathrm{Int}\,(F^-(B_1) \cap F^+(B_2)) = \beta\mathrm{cl}\,(X - [F^-(B_1) \cap F^+(B_2)]) =\\ &= \beta\mathrm{cl}\,[F^+(Y - B_1) \cup F^-(Y - B_2)] = [F^+(Y - B_1) \cup F^-(Y - B_2)] \cup\\ &\cup \mathrm{Int}\,(\mathrm{cl}\,(\mathrm{Int}\,[F^+(Y - B_1) \cup F^-(Y - B_2)] \subset F^+(\mathrm{cl}\,(Y - B_1)) \cup\\ &\cup F^-(\mathrm{cl}\,(Y - B_2)) = F^+(Y - \mathrm{Int}\,(B_1)) \cup\\ &\cup F^-(Y - \mathrm{Int}\,(B_2)) = [X - F^-(\mathrm{Int}\,(B_1))] \cup\\ &\cup [X - F^+(\mathrm{Int}\,(B_2))] = X - [F^-(\mathrm{Int}\,(B_1)) \cap F^+(\mathrm{Int}\,(B_2))].\end{aligned}$$

Therefore, we obtain $F^-(\mathrm{Int}\,(B_1)) \cap F^+(\mathrm{Int}\,(B_2))) \subset \beta\mathrm{Int}\,(F^-(B_1) \cap F^+(B_2))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and $G_1, G_2$ be any open sets of $Y$ such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \emptyset$. Then, we have $F^+(G_1) \cap F^-(G_2) \subset \beta\mathrm{Int}\,(F^+(G_1) \cap F^-(G_2))$. Now, put $U = F^+(G_1) \cap F^-(G_2)$. Then we obtain $U \in \beta(X, x)$, $F(u) \subset G_1$ and $F(u) \cap G_2 \neq \emptyset$ for each $u \in U$. This shows that $F$ is $\beta$-continuous.

**Corollary 2.** (Borsik [3], Abd El-Monsef et al.[11] and Popa and Noiri [18]) *The following are equivalent for a function $f\colon X \to Y$:*
(1) *is $\beta$-continuous;*
(2) *$f^{-1}(G)$ is $\beta$-open in $X$ for every open set $G$ of $Y$;*
(3) *$f^{-1}(K)$ is $\beta$-closed in $X$ for every closed set $K$ of $Y$;*
(4) *$\mathrm{Int}\,(\mathrm{cl}\,(\mathrm{Int}\,(f^{-1}(B)))) \subset f^{-1}(\mathrm{cl}\,(B))$ for every subset $B$ of $Y$;*
(5) *$\beta\mathrm{cl}\,(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}\,(B))$ for every subset $B$ of $Y$;*
(6) *$f^{-1}(\mathrm{Int}\,(B)) \subset \beta\mathrm{Int}\,(f^{-1}(B))$ for every subset $B$ of $Y$.*

**Definition 3.** *A subset $A$ of a topological space $X$ is said to be $\alpha$-regular [7] if for any point $x \in A$ and any open set $U$ of $X$ containing $x$, there exists an open set $G$ of $X$ such that $x \in G \subset \mathrm{cl}\,(G) \subset U$.*

**Definition 4.** *A subset $A$ of a topological space $X$ is said to be $\alpha$-paracompact [21] if every cover of $A$ by open sets of $X$ is refined by a cover of $A$ which consists of open sets of $X$ and is locally finite in $X$.*

**Lemma 2.** (Kovacevic [7]) *If $A$ is an $\alpha$-regular $\alpha$-paracompact subset of a topological space $X$ and $U$ is an open neighborhood of $A$, then there exists an open set $G$ of $X$ such that $A \subset G \subset \mathrm{cl}\,(G) \subset U$.*

For a multifunction $F\colon X \to Y$, a multifunction $\beta\mathrm{cl}\,F\colon X \to Y$ is defined as follows: $(\beta\mathrm{cl}\,F)(x) = \beta\mathrm{cl}\,(F(x))$ for each $x \in X$. Similarly, $\mathrm{cl}\,F$ (resp. $p\mathrm{cl}\,F$, $s\mathrm{cl}\,F$, $\alpha\mathrm{cl}\,F$) is defined in [2] ,resp. [17], [16], [19].

**Lemma 3.** *If $F: X \to Y$ is a multifunction such that $F(x)$ is $\alpha$-regular $\alpha$-paracompact for each $x \in X$, then $G^+(V) = F^+(V)$ for each open set $V$ of $Y$, where $G$ denotes $\mathrm{cl}\,F, \mathrm{pcl}\,F, \mathrm{scl}\,F$, and $\alpha\mathrm{cl}\,F$.*

**Proof.** Let $V$ be any open set of $Y$ and $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$ and hence $G^+(V) \subset F^+(V)$. Conversely, let $x \in F^+(V)$. Then, we have $F(x) \subset V$ and by Lemma 2 there exists an open set $H$ of $Y$ such that $F(x) \subset H \subset \mathrm{cl}(H) \subset V$. Since $G(x) \subset \mathrm{cl}(F(x))$, $G(x) \subset V$ and hence $x \in G^+(V)$. Thus, we obtain $F^+(V) \subset G^+(V)$.

**Lemma 4.** *For a multifunction $F: X \to Y$, it follows that $G^-(V) = F^-(V)$ for every open set $V$ of $Y$, where $G$ denotes $\mathrm{cl}\,F, \mathrm{pcl}\,F, \mathrm{scl}\,F$, and $\alpha\mathrm{cl}\,F$.*

**Proof.** Let $V$ be any open set of $Y$ and $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$ since $V$ is open. Thus, we obtain $x \in F^-(V)$ and hence $G^-(V) \subset F^+(V)$. Conversely, let $x \in F^-(V)$. Then, we have $\emptyset \neq F(x) \cap V \subset G(x) \cap V$ and $x \in G^-(V)$. Consequently, we obtain $G^-(V) = F^+(V)$.

**Theorem 3.** *Let $F: X \to Y$ be a multifunction such that $F(x)$ is $\alpha$-paracompact $\alpha$-regular for each $x \in X$. Then, $F$ is $\beta$-continuous if and only if $G: X \to Y$ is $\beta$-continuous, where $G$ denotes $\mathrm{cl}\,F, \mathrm{pcl}\,F, \mathrm{scl}\,F$, and $\alpha\mathrm{cl}\,F$.*

**Proof.** Necessity. Suppose that $F$ is $\beta$-continuous. Let $x \in X$ and $V_1, V_2$ be any open sets of $Y$ such that $G(x) \in V_1^+ \cap V_2^-$. By Lemmas 3 and 4, we have $x \in G^+(V_1) = F^+(V_1)$ and $x \in G^-(V_2) = F^-(V_2)$. Since $F$ is $\beta$-continuous, there exists $U \in \beta(X, x)$ such that $F(u) \in V_1^+ \cap V_2^-$ for every $u \in U$. By Lemma 2, there exists an open set $H$ of $Y$ such that $F(u) \subset H \subset \mathrm{cl}(H) \subset V_1$ for every $u \in U$. Since $G(u) \subset \mathrm{cl}(F(u))$, we obtain $G(u) \subset V_1$ and hence $G(u) \in V_1^+$ for every $u \in U$. Since $u \in F^-(V_2)$, it follows from Lemma 4 that $u \in F^-(V_2) = G^-(V_2)$ for every $u \in U$. Therefore, we obtain $G(u) \in V_2^-$ for every $u \in U$. Consequently, we obtain $G(u) \in V_1^+ \cap V_2^-$ for every $u \in U$. This shows that $G$ is $\beta$-continuous.

Sufficiency. Suppose that $G$ is $\beta$-continuous. Let $x \in X$ and $V_1, V_2$ be any open sets of $Y$ such that $F(x) \in V_1^+ \cap V_2^-$. By Lemmas 3 and 4, we have $x \in F^+(V_1) = G^+(V_1)$ and $x \in F^-(V_2) = G^-(V_2)$ and hence $G(x) \in V_1^+ \cap V_2^-$. Since $G$ is $\beta$-continuous, there exists $U \in \beta(X, x)$ such that $G(u) \in V_1^+ \cap V_2^-$ for every $u \in U$. Therefore, we obtain $F(u) \subset G(u) \subset V_1$ and hence $F(u) \in V_1^+$ for every $u \in U$. Since $u \in G^-(V_2)$ it follows from Lemma 4 that $u \in G^-(V_2) = F^-(V_2)$ for every $u \in U$. Therefore, we obtain $F(u) \in V_2^-$ for every $u \in U$. This shows that $F$ is $\beta$-continuous.



**Lemma 5.** (Abd El-Monsef et al.[11]) *Let $U$ and $X_0$ be subsets of a topological space $X$. The following properties hold:*

*(1) If $U \in \alpha(X)$ and $X_0 \in \beta(X)$, then $U \cap X_0 \in \beta(U)$,*

*(2) If $U \subset X_0 \subset X$, $U \in \beta(X_0)$ and $X_0 \in \beta(X)$, then $U \in \beta(X)$.*

**Theorem 4.** *If a multifunction $F: X \to Y$ is $\beta$-continuous and $X_0 \in \alpha(X)$, then the restriction $F/X_0: X_0 \to Y$ is $\beta$-continuous.*

**Proof.** Let $x \in X_0$ and $V_1, V_2$ be any open sets of $Y$ such that $(F/X_0)(x) \subset V_1$ and $(F/X_0)(x) \cap V_2 \neq \emptyset$. Since $(F/X_0)(x) = F(x)$ and $F$ is $\beta$-continuous, there exists $U \in \beta(X, x)$ such that $F(u) \subset V_1$ and $F(u) \cap V_2 \neq \emptyset$ for each $u \in U$. Let $U_0 = U \cap X_0$, then $U_0 \in \beta(X_0, x)$ by Lemma 5 and $(F/X_0)(u) = F(u) \subset V_1$ and $(F/X_0)(u) \cap V_2 \neq \emptyset$ for each $u \in U_0$. This shows that $F/X_0$ is $\beta$-continuous.

**Corollary 3.** (Abd El-Monsef et al.[11]) *If $f: X \to Y$ is a $\beta$-continuous function and $X_0 \in \alpha(X)$, then the restriction $f/X_0: X_0 \to Y$ is $\beta$-continuous.*

**Theorem 5.** *A multifunction $F: X \to Y$ is $\beta$-continuous if for each $x \in X$ there exists $X_0 \in \beta(X, x)$ such that the restriction $F/X_0: X_0 \to Y$ is $\beta$-continuous.*

**Proof.** Let $x \in X$ and $V_1, V_2$ be any open sets of $Y$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$. There exists $X_0 \in \beta(X, x)$ such that $F/X_0: X_0 \to Y$ is $\beta$-continuous. Therefore, there exists $U_0 \in \beta(X_0, x)$ such that $(F/X_0)(u) \subset V_1$ and $(F/X_0)(u) \cap V_2 \neq \emptyset$ for each $u \in U_0$. By Lemma 5, $U_0 \in \beta(X, x)$ and $F(u) = (F/X_0)(u) \subset V_1$ and $F(u) \cap V_2 = (F/X_0)(u) \cap V_2 \neq \emptyset$ for each $u \in U_0$. Therefore, $F$ is $\beta$-continuous.

It is shown in [14, Lemma 3.1] that a subset is an $\alpha$-open if and only it is semi-open and preopen.

**Corollary 4.** *Let $\{U_\alpha : \alpha \in \nabla\}$ be a cover of $X$ by $\alpha$-open sets of $X$. Then, a multifunction $F: X \to Y$ is $\beta$-continuous if and only if the restriction $F/X_\alpha : X_\alpha \to Y$ is $\beta$-continuous for each $\alpha \in \nabla$.*

**Proof.** This is an immediate consequence of Theorems 4 and 5.

## 4. Sufficient conditions for $\beta$-continuity.

**Definition 5.** *A multifunction $F: X \to Y$ is said to be*

*(a) upper $\beta$-continuous [19] (resp. $\alpha$- continuous [19]) if for each point $x \in X$ and each open set $V$ containing $F(x)$, there exists $U \in \beta(X, x)$ (resp. $\alpha(X, x)$) such that $F(U) \subset V$,*

(b) *lower $\beta$-continuous* [19] *(resp. $\alpha$- continuous* [19]*) if for each point $x \in X$ and each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(X, x)$ (resp. $\alpha(X, x)$) such that $U \subset F^-(V)$.*

**Remark.** If $F: X \to Y$ is $\beta$-continuous, then it is upper and lower $\beta$-continuous.

**Lemma 6.** *A multifunction $F: X \to Y$ is upper $\beta$-continuous (resp. lower $\beta$-continuous), if and only if $F^+(G) \in \beta(X)$ (resp. $F^-(G) \in \beta(X)$) for every open set $G$ of $Y$.*

**Proof.** We prove only the first case, the proof of the second being analogous. Let $G$ be any open set of $Y$ and $x \in F^+(G)$. There exists $U \in \beta(X, x)$ sucht that $F(U) \subset V$. Therefore, we obtain $U \in \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(U))) \subset \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(F^+(G))))$. We have $F^+(G) \subset \mathrm{cl}\,(\mathrm{Int}\,(\mathrm{cl}\,(F^+(G))))$ and hence $F^+(G) \in \beta(X)$. Conversely, $x \in X$ and $G$ be any open set of $Y$ such that $F(x) \subset G$. By hypothesis, we have $F^+(G) \in \beta(X)$. Let $U = F^+(G)$, then $U \in \beta(X, x)$ and $F(U) \subset G$. This shows that $F$ is upper $\beta$-continuous at $x$.

**Theorem 6.** *If a multifunction $F: X \to Y$ is upper $\beta$-continuous and lower $\alpha$-continuous, then $F$ is $\beta$-continuous.*

**Proof.** Let $V_1, V_2$ be any open sets of $Y$. Since $F$ is upper $\beta$-continuous, by Lemma 6 $F^+(V_1) \in \beta(X)$. Since $F$ is lower $\alpha$-continuous, $F^-(V_2)$ is $\alpha$- open in $X$ [19, Theorem 3.4]. Therefore, it follows from [1, Corollary 2.14] that $F^+(V_1) \cap F^-(V_2) \in \beta(X)$. Therefore, $F$ is $\beta$-continuous by Theorem 2.

**Theorem 7.** *If a multifunction $F: X \to Y$ is lower $\beta$-continuous and upper $\alpha$-continuous, then $F$ is $\beta$-continuous.*

**Proof.** The proof is similar to that of Theorem 6.

REFERENCES

1. Andrijević, D. - *Semi-preopen sets*, Mat.Vesnik 38(1986), 24-32.
2. Banzaru, T. - *Multifunctions and $M$-product spaces* (Romanian), Bul. St. Tehn. Inst. Politeh. "T. Vuia" Timişoara Ser. Mat. Fiz. Mec. Teor. Apl. 17(31) (1972), 17-23.
3. Borsik, J. - *On almost quasicontinuous functions*, Math.Bohemica 118(1993), 241-248.
4. Borsik, J. and Doboš, J. - *On decompositions of quasicontinuity*, Real Anal.Exchange 16 (1990/1991), 292-305.



5. Crossley, S.G. and Hildebrand, S.K. - *Semi-closure*, Texas J.Sci. 22(1971), 99-112.
6. El-Deeb, N., Hasanein, I.A., Mashhour, A.S. and Noiri, T. - *On $p$-regular spaces*, Bull. Math. Soc. Sci. Math. R.S.Roumanie 27(75) (1983), 311-315.
7. Kovačević, I. - *Subsets and paracompactness*, Univ. u Novom Sadu, Zb.Rad.Prirod.-Mat.Fak.Ser.Mat. 14(1984), 79-87.
8. Levine, N. - *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly 70 (1963), 36-41.
9. Mashhour, A.S., Abd El-Monsef, M.E. and El-Deeb, S.N. - *On precontinuous and weak precontinuous mappings*, Proc.Math.Phys.Soc Egypt 53 (1982), 47-53.
10. Mashhour, A.S., Abd El-Monsef, M.E. and Husain, J.H. - *On pretopological spaces*, Bull. Math. Soc. Sci. Math. R.S.Roumanie 28(76) (1984), 39-45.
11. Abd El Monsef, M.E., El-Deeb, S.N. and Mohmoud, R.A. - *$\beta$-open sets and $\beta$-continuous mappings*, Bull.Fac.Sci. Assiut Univ. 12(1983), 77-90
12. Abd El-Monsef, M.E., Mahmoud, R.A. and Lashin, E.R. - *$\beta$-closure and $\beta$-interior*, J.Fac.Ed.Ain Shams Univ. 10(1986), 235-245.
13. Njåstad, O. - *On some classes of nearly open sets*, Pacific J.Math. 15(1965), 961-970.
14. Noiri, T. - *On $\alpha$-continuous functions*, Časopis Pest.Math. 109(1984), 118-126.
15. Noiri, T. and Popa, V. - *On some forms of continuity for multifunctions*, Res.Rep. Yatsushiro Nat.Coll.Tech.12(1990), 65-70.
16. Popa, V. - *Multifunctions semi-continues*, Rev.Roumaine Math. Pure Appl. 27 (1982), 807-815.
17. Popa, V. - *Some properties of $H$-almost continuous multifunctions*, Problemy Mat. 10 (1988), 9-26.
18. Popa, V. and Noiri, T. - *On $\beta$-continuous functions*, Real Anal.Exchange 18 (1992/1993), 544-548.
19. Popa, V. and Noiri, T. - *On upper and lower $\alpha$-continuous multifunctions*, Math.Slovaca 43(1993), 477-491.
20. Przemski, M. - *Some generalizations of continuity and quasicontinuity of multivalued maps*, Demonstratio Math. 26(1993), 381-400.
21. Wine, D. - *Locally paracompact spaces*, Glasnik Mat. 10(30) (1975), 351-357.

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