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CONSTRUCTION OF SOME BRAVAIS TYPES OF POINT WISE LATTICES IN THE EUCLIDEAN FIVE-DIMENSIONAL SPACE

BY

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1. Definitions and necessary facts. Let $\{0, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ be a basis in an n-dimensional affine space. The set $\Re_n = \{M | \overrightarrow{OM} = \sum_{i=1}^{n} x_i \vec{a}_i, x_i \in Z\}$ is called a lattice $\Re_n$, i.e. a pointwise lattice is a set of points in an n-dimensional affine space which have integer coordinates in the basis that was built before. This basis is called the main basis of lattice. Let us further suppose that the affine space is an n-dimensional Euclidean space, $n = 2, 3, 4, 5$. A transformation of symmetry of the pointwise lattice is a motion of space that maps this lattice onto itself. The most simple transformations of a lattice are the parallel translations along the vectors of the lattice. The set of all the parallel translations of each lattice is a commutative group generated by the parallel translations determined by the vectors of a certain main basis. Other transformations of the symmetry of the lattice either keep invariant at least one point (these transformations are called pointwise transformations), or are compositions of pointwise transformations with parallel translations. The pointwise transformations keeping invariant one common point also form a group named pointwise group of the lattice. The pointwise group in certain main basis of the lattice is determined by a group of integer (elements of matrices are integer numbers) unimodular matrices. The set of all the transformations of symmetry of the given lattice forms its complete group of symmetry. Two lattices are related to the same Bravais type if their complete groups are isomorphic. An important problem is to construct Bravais types of pointwise lattices in the given space. So far all Bravais types of pointwise lattices in n-dimensional ($n = 2, 3, 4$) Euclidean spaces were

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built and described by means of geometrical or algebraical methods (see [4,5]). The comparison of lattices according to isomorphism of their complete groups of symmetry becomes simple because under such isomorphism every main basis of a lattice has a corresponding main basis of other lattice and the pointwise groups of these lattices are isomorphic too.

The plane of symmetry of a lattice $\Re_n$ corresponding to any transformation of symmetry g $(g(\Re_n) = \Re_n)$ is called the set of points from this space, which are respectively invariant under the g transformation and as a consequense under the cyclical group $< g >$ too. The properties of the planes of symmetry of any pointwise lattice in an Euclidean or Minkowsky space are proved in [1,2]. Namely:

a) If the lattice $\Re_n$ has a k-dimensional plane of symmetry Q $(k = 1, 2, \ldots, n-1)$, then a plane of symmetry passes through each point of this lattice, is parallel to Q and has the same dimension.

b) If Q is a k-dimensional plane of symmetry for the lattice $\Re_n$ $(k = 1, 2, \ldots, n-1)$ and $\wp$ is a plane passing through a point of the $\Re_n$ lattice absolutely perpendicular to Q $(\wp \perp Q)$, then $\wp$ containts a $(n-k)$-dimensional sublattice of points of the $\Re_n$ lattice.

c) Any k-dimensional plane of symmetry Q $(k = 1, 2, \ldots, n-1)$ passing through a point of the lattice contains a k-dimensional sublattice.

These properties allowed us to elaborate a method of constructing pointwise lattice in an n-dimensional space if Bravais types of lattices in $2, 3, \ldots, (n-1)$-dimensional spaces are already built. This method is described in detail in articles [5,6] with concrete examples of constructing pointwise lattices in three- and four-dimensional Minkowsky spaces. Briefly this method of elements of symmetry consists in the effectuation of following research works:

1) To establish the Bravais types of pointwise lattices in k-dimensional spaces $(k = 2, 3, 4, \ldots, n-1)$.

2) To define the centers of symmetry (points of intersection of all planes of symmetry) of lattices defined in number 1). It is sufficient to determine the centers of symmetry placed on a specially built parallelepiped (it is called Bravais parallelepiped), because other centers are obtained from these centers by parallel translations along the vectors of the lattice.

The Bravais parallelepiped of a pointwise lattice in the Euclidean space is a parallelepiped of the smallest volume built on the vectors of the lattice, its group of symmetry coincides with the pointwise group of the lattice. Such a parallelepiped can be built for any pointwise lattice.

3) To determine plane (or planes) of symmetry which the builded n-dimensional lattice has. The character of the plane of symmetry Q is defined by the pointwise group of the lattice in the plane $\wp \perp Q$.

4) To examine all the possible cases of intersection of the k-dimensional plane of symmetry and the $(n-k)$-dimensional plane of the given lattice. The point of intersection is a center of symmetry of the lattice $\Re_{n-k}$.

5) To compare the built lattices and to establish the Bravais types of these pointwise lattices in the n-dimensional space. This comparison is effected by means of the determination of groups of unimodular integer matrices corresponding to the transformations of symmetry of the pointwise group of built lattices and the examination of their equivalence (matrix A_1 is equivalent to the matrix A_2, if an integer non-degenerated matrix T exists so that $A_2 = T^{-1}A_1T$). If the groups of matrices are integer equivalent, then the lattices are related to the same Bravais type and inversely.

Let us show further the application of this method to the construction of lattices in the Euclidean spaces starting from the bidimensional lattices and finishing with the five-dimensional pointwise types of Bravais lattices.

Any basis $\{0, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ in the Euclidean space is characterized by the metric parameters $a_{ij} = \vec{a}_i \cdot \vec{a}_j$ which are called (in case of the lattices) the metric parameters of the lattice, too, when the given basis is a main basis of the respective lattice. In the pointwise lattices one can determine some main special bases, which are defined completely by Bravais parallelepipeds and vice-versa. The metric parameters of these bases verify some relations which are also specific to the given Bravais type of the lattices. Here as a rule, Bravais parallelepipeds of the given lattices and corresponding main bases are considered by means of the applied method. We shall also indicate the correspondence between the types constructed using the method of elements of symmetry and the types obtained in article [4].

2. Construction of Bravais types of pointwise lattices in the three-dimensional Euclidean space. There exist five Bravais types of pointwise lattices in the Euclidean plane. They are obtained by some elementary calculations. These lattices are built on the following Bravais parallelograms: an arbitrary parallelogram, a rectangle, a centered rectangle, a square and a rhombus with an acute angle equal to 60^0.

Let us apply the method of elements of symmetry in order to construct the Bravais types of pointwise lattices in the three-dimensional Euclidean space. We shall suppose that the lattice to be built has an axis of symmetry which can have the order 1, 2, 3, 4 and 6 (these parameters are obtained by the examination of pointwise groups of two-dimensional lattices). In case when we have an axis of the first order we obtain a three-dimensional lattice that is built on an arbitrary parallelepiped. If the Bravais parallelepiped is built on the vectors $\vec{a}, \vec{b}, \vec{c}$, then in the examined case the metric parameters verify the inequalities: $0 < 2k < a < b < c$, $0 < 2h < a$, $0 < 2g < b$, where $k = \vec{a} \cdot \vec{b}$, $h = \vec{a} \cdot \vec{c}$, $g = \vec{b} \cdot \vec{c}$. The second order axis may be perpendicular to the plane of a plane lattice of the type 1, 2, or 3. If, for example, the plane lattice is built on a rectangle, then considering $\vec{c}$ as a main vector of the lattice situated on the axis of rotation, we obtain three Bravais types of three-dimensional lattices, the main bases of which are $\{0, \vec{a}, \vec{b}, \vec{c}\}$, $\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{c}}{2}\}$ and $\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{b}+\vec{c}}{2}\}$ where 0 is a fixed point in the three-dimensional space (here $\vec{a}_1 = \vec{a}$, $\vec{a}_2 = \vec{b}$ and $\vec{a}_3 = \vec{c}$, or $\vec{a}_3 = \frac{\vec{a}+\vec{c}}{2}$, or $\vec{a}_3 = \frac{\vec{a}+\vec{b}+\vec{c}}{2}$, and then $c = \vec{c}^2$, or $c = \left(\frac{\vec{a}+\vec{c}}{2}\right)^2$, or $c = \left(\frac{\vec{a}+\vec{b}+\vec{c}}{2}\right)^2$, $h = \vec{a} \cdot \vec{c}$, or $h = \vec{a} \cdot \frac{\vec{a}+\vec{c}}{2}$, or $h = \vec{a} \cdot \frac{\vec{a}+\vec{b}+\vec{c}}{2}$, and $g = \vec{b} \cdot \vec{c}$, or $g = \vec{b} \cdot \frac{\vec{a}+\vec{c}}{2}$, or $g = \vec{b} \cdot \frac{\vec{a}+\vec{b}+\vec{c}}{2}$).

The results obtained by considering all the possible cases can be found in [4,5,6].

3. Construction of Bravais types of pointwise lattices in the four-dimensional Euclidean space. The construction of Bravais types of pointwise lattices in the four-dimensional Euclidean space by means of method of elements of symmetry is reduced to: 1) Examination of all possible cases when these lattices have at least one axis of symmetry, the order of which is defined by the pointwise groups of symmetry of the 14 built three-dimensional pointwise lattices. These orders are equal to 1, 2, 4, 8, 16, 64, 12, 24. 2) Examination of all possible cases when the searched lattices have at least one two-dimensional plane. As a plane of rotation this plane can have the order 2, 3, 4, 6. In the plane of symmetry as in the plane absolutely perpendicular to the given plane a lattice of the five possible Bravais types can be placed. It is necessary to study all the possibilities of combination of these types of lattices. 3) Comparison of obtained lattices in order to exclude those lattices that are related to the same Bravais types. We shall enumarate here only types obtained from triclinic, monoclinic and orthogonal singonies in the first case and types of lattices obtained from the plane lattices built on a parallelogram in the second case. As in the previous sections we shall suppose that $\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is a Bravais basis of the sought lattice, and $\{0, \vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$ is the main basis obtained from the Bravais basis. The metric parameters that characterize this main basis are denoted by: $a = \vec{a}_1^2$, $b = \vec{a}_2^2$, $c = \vec{a}_3^2$, $d = \vec{a}_4^2$, $k = \vec{a}_1 \cdot \vec{a}_2$, $h = \vec{a}_1 \cdot \vec{a}_3$, $p = \vec{a}_1 \cdot \vec{a}_4$, $g = \vec{a}_2 \cdot \vec{a}_3$, $q = \vec{a}_2 \cdot \vec{a}_4$, $r = \vec{a}_3 \cdot \vec{a}_4$. In two tables below we enumarate the results obtained by considering all possible cases, when the three-dimensional lattice belongs to the indicated holohedries and two-dimensional lattice is built on an arbitrary parallelogram.

Table 1

N	Name of lattice	Main basis	Number of parameters	Location of centeres of symmetry	Comparison with types from [4]
1	2	3	4	5	6
1	Hexa- -clinic	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	10	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}$, $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c})$, $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c})$, $\frac{1}{2}(\vec{b}+\vec{d}), \frac{1}{2}(\vec{c}+\vec{d})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{d})$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	I
2	Triclinic	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	7 $(p = q = r = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}$, $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c})$, $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	II/I
3		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{a}+\vec{d}}{2}\}$	7 $(p = \frac{1}{2}a, g = r = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}$, $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c})$, $\frac{1}{2}(\vec{b}+\vec{c}); \frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	II/II
4	Mono- -clinic	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	5 $(h = g = = p = q = = r = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}(\vec{a}+\vec{b})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}), \frac{1}{2}\vec{d}$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{d})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$; $\frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d})$, $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}); \frac{1}{2}(\vec{b}+\vec{c})$.	IV/I
5		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{a}+\vec{d}}{2}\}$	5 $(h = g = q = r = 0, p = \frac{1}{2}a)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}$, $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c})$, $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d}), \ldots$ $\frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d})$, $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c})$.	IV/III

1	2	3	4	5	6
6		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{c}+\vec{d}}{2}\}$	5 $(h = g =$ $p = q = 0,$ $r = \frac{1}{2}c)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d}), \ldots$ $\frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c}).$	IV/II
7		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{2}\}$	5 $(p = \frac{a}{2},$ $q = \frac{b}{2},$ $r = \frac{c}{2},$ $h = g = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d}), \ldots$ $\frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c}).$	IV/IV
8		$\{0, \vec{a}, \vec{b}, \frac{\vec{b}+\vec{c}}{2}, \frac{\vec{a}+\vec{d}}{2}\}$	5 $(h = q =$ $r = 0,$ $g = \frac{b}{2},$ $p = \frac{a}{2}$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d}), \ldots$ $\frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c}).$	IV/V
9	Ortho- -gonal	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	4 $(k = h =$ $g = p = q =$ $r = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}),$ $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/II
10		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{a}+\vec{d}}{2}\}$	4 $(k = h =$ $g = q =$ $r = 0,$ $p = \frac{1}{2}a)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}),$ $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/III

1	2	3	4	5	6
11		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{a}+\vec{b}+\vec{d}}{2}\}$	4 $(k = h =$ $g = r = 0,$ $p = \frac{1}{2}a,$ $q = \frac{1}{2}b)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}),$ $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/IV
12		$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{2}\}$	4 $(k = h = g = 0,$ $p = \frac{1}{2}a, q = \frac{1}{2}b,$ $r = \frac{1}{2}c)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}),$ $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/V
13		$\{0, \vec{a}, \frac{\vec{a}+\vec{b}}{2}, \vec{c}, \frac{\vec{a}+\vec{d}}{2}\}$	4 $(k = p = \frac{1}{2}a$ $h = r = 0,$ $q = \frac{1}{4}a)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}),$ $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/VII
14		$\{0, \vec{a}, \frac{\vec{a}+\vec{b}}{2}, \vec{c}, \frac{\vec{c}+\vec{d}}{2}\}$	4 $(k = \frac{1}{2}a,$ $h = g = p =$ $q = 0, r = \frac{1}{2}c)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},$ $\frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}),$ $\frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}),$ $\frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{b}+\vec{c}+\vec{d}),$ $\frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/VI

1	2	3	4	5	6
15		$\{0, \vec{a}, \frac{\vec{a}+\vec{b}}{2}, \vec{c}, \frac{\vec{a}+\vec{c}+\vec{d}}{2}\}$	4 $(k = p = \frac{1}{2}a, h = g = 0, q = \frac{1}{4}a, r = \frac{1}{2}c)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}), \frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/I
16		$\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{b}+\vec{c}}{2}, \frac{\vec{a}+\vec{b}+\vec{d}}{2}\}$	4 $(k = 0, h = p = \frac{1}{2}a, g = q = \frac{1}{2}b, r = \frac{a+b}{4})$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}), \frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/VIII
17		$\{0, \vec{a}, \frac{\vec{a}+\vec{b}}{2}, \frac{\vec{a}+\vec{c}}{2}, \frac{\vec{a}+\vec{d}}{2}\}$	4 $(k = h = p = \frac{1}{2}a, g = q = r = \frac{1}{4}a)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}(\vec{a}+\vec{c}), \frac{1}{2}(\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}), \frac{1}{2}(\vec{a}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{d}), \frac{1}{2}(\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{b}+\vec{c}+\vec{d}), \frac{1}{2}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$	V/IX

Table 2

N	Name of lattice from the plane of rotation	Main basis	Number of parameters	Location of centeres of symmetry	Comparison with types from [4]
1	Parallelogram	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	$6(h = g = p = q = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	III/I
2		$\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{c}}{2}, \vec{d}\}$	$6(h = \frac{1}{2}a, g = \frac{1}{2}k, p = q = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	III/II
3		$\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{2}, \vec{c}+\vec{d}\}$	$6(h = \frac{1}{2}a, g = \frac{1}{2}b, p = q = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	III/II
4	Rectangle	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	$5(h = g = p = q = r = 0)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	IV/I
5		$\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{c}}{2}, \vec{d}\}$	$5(g = p = q = r = 0, p = \frac{1}{2}a)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	IV/III
6	Centered rectangle	$\{0, \vec{a}, \vec{b}, \vec{c}, \frac{\vec{c}+\vec{d}}{2}\}$	$5(h = p = g = q = 0, r = \frac{1}{2}c)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	IV/II
7		$\{0, \vec{a}, \vec{b}, \frac{\vec{a}+\vec{c}}{2}, \frac{\vec{c}+\vec{d}}{2}\}$	$5(h = \frac{1}{2}a, g = p = q = 0, r = \frac{1}{2}c)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	IV/VI
8	Square	$\{0, \vec{a}, \vec{b}, \vec{c}, \vec{d}\}$	$4(h = g = p = q = r = 0, c = d)$	$0, \frac{1}{2}\vec{a}, \frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}), \frac{1}{2}\vec{c}, \frac{1}{2}\vec{d}, \frac{1}{2}(\vec{c}+\vec{d})$	VI/I

9		$\{0,\vec{a},\vec{b},\frac{\vec{a}+\vec{c}+\vec{d}}{2},\vec{d}\}$	$4(h=\frac{1}{2}a, p=q=0, g=\frac{1}{2}k, r=\frac{1}{2}d)$	$0,\frac{1}{2}\vec{a},\frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}),\frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},\frac{1}{2}(\vec{c}+\vec{d})$	VI/II
10	Equilateral triangle	$\{0,\vec{a},\vec{b},(\vec{c},\vec{d})\}$	$4(h=g=p=q=0, 2r=c=d)$	$0,\frac{1}{2}\vec{a},\frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}),\frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},\frac{1}{2}(\vec{c}+\vec{d})$	VII/II
11		$\{0,\vec{a},\vec{b},\frac{\vec{a}+\vec{c}}{2},\vec{d}\}$	$4(h=\frac{1}{2}a, g=\frac{1}{2}k, p=q=0, r=\frac{1}{4}c)$	$0,\frac{1}{2}\vec{a},\frac{1}{2}\vec{b}, \frac{1}{2}(\vec{a}+\vec{b}),\frac{1}{2}\vec{c}, \frac{1}{2}\vec{d},\frac{1}{2}(\vec{c}+\vec{d})$	VII/I

Remark. In table 2 there are lattices which repeat some lattices from table 1.

4. Construction of some Bravais types of pointwise lattices in the five-dimensional Euclidean space. Let us examine the construction of Bravais types of pointwise lattices in the five-dimensional Euclidean space, which have at least one axis of symmetry and they are obtained from the four-dimensional lattices belonging to hexaclinic, triclinic and monoclinic families. Let $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\vec{e}\}$ be the Bravais basis of the built lattices. The pointwise group of four-dimensional hexaclinic lattice has the order 2 and therefere the axis of lattice in the five-dimensional space that can be built on the base of this lattice has the order 1 or 2 and it intersects the four-dimensional lattice in one of the centers of symmetry indicated in the Table 1. As a result we obtain one lattice when the order of axis is equal to 1 and two lattices when the axis has the second order. These lattices are built on the bases: $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\vec{e}\}$ with 15 arbitrary parameters: $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\vec{e}\}$ $(\vec{e}\perp\vec{a},\ \vec{e}\perp\vec{b},\ \vec{e}\perp\vec{c},\ \vec{e}\perp\vec{d})$ and $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{a}+\vec{e}}{2}\}$ with 11 parameters (all other possible cases are reduced to these Bravais types). The pointwise group of the first lattice consists of two transformations (identical and the central symmetry), and the pointwise group of other two lattices contains four transformations (identical, the central symmetry, the rotation of the second order around an axis parallel to vector $\vec{e}$ and the reflection from hyperplane containing the basic four-dimensional lattice).

The pointwise group of four-dimensional lattices from the triclinic family has the order equal to 4 and therefore the axis of sought lattice has also

the fourth order. From the four-dimensional triclinic lattice built on the basis $\{0,\vec{a},\vec{b},\vec{c},\vec{d}\}$ by means of examination of all possible cases of intersection of axis with the basic lattice, the following pointwise five-dimensional lattices of different Bravais types can be built: 1) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\vec{e}\}$, 2) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{a}+\vec{e}}{2}\}$, 3) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{d}+\vec{e}}{2}\}$. From the four-dimensional triclinic lattice built on the basis $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2}\}$ by examination of all possible cases of crossing the axis with the basic lattice, other two five-dimensional pointwise lattices of different Bravais types can be built:
5) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{a}+\vec{e}}{2}\}$, 6) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{b}+\vec{e}}{2}\}$.

All lattices obtained from the triclinic family are characterized by 8 parameters $(\vec{d}\perp\vec{a},\ \vec{d}\perp\vec{b},\ \vec{d}\perp\vec{c},\ \vec{e}\perp\vec{a},\ \vec{e}\perp\vec{b},\ \vec{e}\perp\vec{c},\ \vec{e}\perp\vec{d})$, and their pointwise group of symmetry contains 8 transformations of symmetry (rotations around an axis parallel to the vector $\vec{e}$ and their products with the central symmetry).

From the four-dimensional monoclinic lattice (its pointwise group of symmetry contains 8 transformations) built on the main basis $\{0,\vec{a},\vec{b},\vec{c},\vec{d}\}$, the following five-dimensional lattices that have at least one axis of symmetry (the eighth order axis of rotation) can be built: 1) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\vec{e}\}$, 2) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{a}+\vec{e}}{2}\}$, 3) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{c}+\vec{e}}{2}\}$, 4) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{a}+\vec{c}+\vec{e}}{2}\}$.
5) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{a}+\vec{c}+\vec{d}+\vec{e}}{2}\}$, 6) $\{0,\vec{a},\vec{b},\vec{c},\vec{d},\frac{\vec{c}+\vec{d}+\vec{e}}{2}\}$. From the four-dimensional monoclinic lattice built on the basis $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2}\}$ by examination of all possible cases of intersection of axis with the basic lattice, some more five-dimensional pointwise lattices of different Bravais types can be built: 7) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{a}+\vec{e}}{2}\}$, 8) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{b}+\vec{e}}{2}\}$, 9) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{c}+\vec{e}}{2}\}$, 10) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{a}+\vec{c}+\vec{e}}{2}\}$, 11) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{b}+\vec{c}+\vec{e}}{2}\}$.

Analogously by examination of all possible cases of other lattices from the monoclinic family (table 1) we obtain the following new Bravais types of pointwise five-dimensional lattices:

From $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{c}+\vec{d}}{2}\}$ → 12) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{c}+\vec{d}}{2},\frac{\vec{a}+\vec{c}+\vec{e}}{2}\}$,

From $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{2}\}$ → 13) $\{0,\vec{a},\vec{b},\vec{c},\frac{\vec{a}+\vec{c}+\vec{d}}{2},\frac{\vec{b}+\vec{c}+\vec{e}}{2}\}$,

From $\{0,\vec{a},\vec{b},\frac{\vec{b}+\vec{c}}{2},\frac{\vec{a}+\vec{d}}{2}\}$ → 14) $\{0,\vec{a},\vec{b},\frac{\vec{b}+\vec{c}}{2},\frac{\vec{a}+\vec{d}}{2},\frac{\vec{a}+\vec{e}}{2}\}$.

All five-dimensional lattices obtained from the monoclinic family are characterized by 6 non-zero parameters and their pointwise groups consist of 16 transformations of symmetry. In order to satisfy ourself that the obtained lattices are related to different Bravais types, it is enough to determine the groups of corresponding matrices and to show that these groups are not integer equivalent, i.e. there doesn't exist an unimodular integer matrix T, so that the matrices $A_1, A_2, \dots, A_p$, from one group and the matrices $B_1, B_2, \dots, B_p$ from the other group verify the equality: $B_i T = T A_i$ for every $i = 1, 2, \dots, p$. For example, the group that characterizes the transformations of symmetry of the 7) type lattices is formed by the matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

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