science ne peut pas omettre le fait que pendant la période 1958–1960, Octav Mayer a été le Président de la Filiale de Jassy de l'Académie, et que pour ses mérites exceptionnels dans la recherche et l'enseignement on lui a accordé le titre d'Homme de science émérite, et il a reçu de hautes décorations et ordres.

Après sept décennies de vie agitée, pleine de réalisations, mais aussi de déceptions, souffrant d'une maladie chronique, Octav Mayer nous a quittés, le 9 septembre 1966, en laissant une œuvre scientifique de grande valeur, la tradition d'un enseignement mathématique d'un haut niveau et le souvenir de la noblesse d'un Homme d'un comportement éthique et civique exceptionnel.

L'Académie Roumaine gardera toujours le souvenir de sa figure lumineuse dans la section d'or de ses grands savants.

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GEOMETRIZATION OF LOCALLY VARYING SPEEDS OF LIGHT VIA THE ISORIEMANNIAN GEOMETRY

BY

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1. Introduction. Since the pioneering work by Riemann [7], the **geometrization of the space-time in vacuum**, historically called **the exterior problem**, is well known (for a contemporary account see, e.g., [3] and references quoted therein).

However, such a geometrization is inapplicable for systems within physical media, historically called *the interior problem*, as established by physical evidence. For example, the speed of light in vacuum c_0 is *altered* within (transparent) media into the value $c = c_0/n$, where n is the familiar index of refraction, and assumes different values depending on whether the electromagnetic waves propagate in atmosphere, water, glass etc. By recalling that the speed of light is a primitive geometric characteristic of space-time, its alteration has rather profound geometric implications which have to be addressed.

Similarly, consider the *deviation* from a geodesic on a Riemannian space occurring in free motion within physical media, such as the free fall of a leaf from the Pisa tower. The deviation of the trajectory of such a leaf from the geodesic trajectory in the Riemannian space is evidence of a *necessary* alteration of the geometry for the following reasons:

I) An effective interior geometry must represent the fall of the leaf as a geodesic evidently because the fall is free, which task cannot be achieved via the Riemannian geometry.

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II) An effective representation of interior systems requires *arbitrary non linearity in the velocities* $v = dx/dt$ and at times also *in the accelerations* $a = dv/dt$ and other quantities. This implies again the lack of applicability of the Riemannian geometry for interior problem because this geometry is arbitrarily nonlinear only *in the coordinates*, and can represent only a limited class of systems nonlinear in the velocities (e.g., those quadratic in v). As a specific example, missiles in atmosphere have nowadays reached such speeds that their resistive force can be proportional up to the *tenth power of the velocity*, thus being outside any realistic capability of the Riemannian geometry.

III) An effective interior geometry must represent *variationally nonselfadjoint systems* [10], that is, systems which generally violate the integrability conditions for the existence of a first-order Lagrangian in the coordinates of the observer, thus establishing the inapplicability of the Riemannian geometry on dynamical grounds.

IV) An effective interior geometry must also be able to represent *nonlocal- integral forces* due to the motion of extended bodies within physical media which evidently require surface integrals. This establishes the inapplicability of the Riemannian geometry on topological grounds because the latter geometry is strictly local-differential, while the systems considered are nonlocal- integral.

V) An effective interior geometry must represent the *inhomogenity and anisotropy* of physical media (such as that in the interior of Jupiter), thus establishing the inapplicability of the Riemannian geometry owing to its notorious *homogenity and isotropy*. Moreover,

VI) The interior geometry must contain the exterior one as a particular case, thus being a *generalization-covering* of the Riemannian geometry, and finally,

VII) The interior geometry must be able to separate nonlocal-integral and non-(first-order)-Lagrangian effects from conventional local-differential and Lagrangian terms, as requested for the experimental verification of expected novel internal effects.

When exposed to the above occurrences, a rather general contemporary trend is that of reducing an interior system to a collection of exterior ones, such as the reduction of a space-ship during re-entry to its elementary constituents. Even though certainly possible as a *first approximation*, certain *No-Reduction Theorems* [16b] have established that the reduction is not possible in an exact form. For instance, the space-ship during re-entry in our atmosphere with *continuously decaying angular momentum* simply cannot be reduced in a mathematically consistent way to a finite collection of elementary particles *each having conserved angular momentum*, as necessary

to recover exterior conditions.

In summary, we can say that clear physical evidence establishes the *geometric inequivalence of the exterior and interior problems*. Moreover, simple mathematical studies establish the *irreducibility of interior to exterior systems*.

We can therefore state that the conventional Riemannian geometry is indeed exactly valid for the geometrization of space-time in vacuum (exterior problem), but the same geometry is "inapplicable" for the corresponding geometrization within physical media (interior problem) for a number of geometric, topological and dynamical reasons. Note that the term "violation" would not be appropriate because the Riemannian geometry simply was not conceived for interior problems.

On historical grounds, we note that the distinction between exterior and interior problems was well known up to the early part of this century and thereafter ignored (see [16b] for historical notes and references). In fact, the distinction was introduced by the founders of Analytic Dynamics, such as Lagrange, Hamilton, Jacobi and others. The distinction then persisted in the early studies of gravitations via the Riemannian geometry. For instance, Schwarzschild wrote *two* papers with the clear distinction between exterior and interior problems: the first paper [21] on the celebrated *exact* solution for the exterior problem and the virtually unknown second paper [21] on the *approximate* solution for the interior problem. The same distinction can be found in the *titles* of chapters of well written treatises in gravitation of the first part of this century. Subsequently, the distinction between exterior and interior problems was progressively abandoned, up to the current virtual complete silence in the contemporary literature in the field. The latter silence is unfortunate owing to the geometric inequivalence and irreducibility of the exterior and interior problems indicated above.

The above occurrence has created the need for constructing a new geometry specifically conceived for the interior problem which satisfies conditions I-VII above, beginning with the characterization of the free fall within physical media via geodesic trajectories. This problem can be reduced to the primitive task of *constructing a covering of the Riemannian geometry for the geometrization of the locally varying speed of light* $c = c_0/n(x, v, a, \dots)$ *while admitting the constant value in vacuum* c_0 *and related Riemannian geometrization as a particular case.*

Among the various studies in geometry conducted until now, we here quote the *higher-order Lagrangian representation* by R.Miron and his associates (see. e.g., [4] and references quoted therein) which is indeed arbitrarily nonlinear in the velocities and other quantities, thus constituting one of the few geometries in scientific record which is directly applicable to interior problems.

Another significant new geometry is that constructed by Papuc (see, e.g., [5] and references quoted therein) which is structurally anisotropic, thus significant for applications, e.g., to represent the irreversibility of interior trajectories.

In this note we present the latest advances on a generalized geometry satisfying conditions I–VII above submitted by Santilli in memoir [13] of 1988 (see [16] for a review) under the name of the *isotopies of the Riemannian geometry*, or *isoriemannian geometry* for short.

The isotopies were first introduced in ref.[9] of 1978 in the form used in this note, and are today referred to *axiom-preserving liftings of linear, local and Lagrangian systems into their most general possible nonlinear, nonlocal and nonlagrangian forms which are however capable of restoring the linear, local and Lagrangian character on certain isotopic spaces over isotopic fields.*

The isotopies are here selected for a number of reasons, such as the verification of conditions I–VII above, and their *axiom-preserving character.* This permits the achievement of a unified geometric treatment in which both the exterior and interior descriptions are particular *realizations* of the same abstract axioms.

The isotopies therefore permit the preservation of Einstein's axioms also in interior conditions, which axioms are generally lost for non-axiom-preserving approaches, thus creating the additional, rather difficult problem of identifying their replacement.

Finally, the isotopies have been selected because, even though not unique, the isoriemannian geometry is "directly universal" in the sense of containing as particular cases all infinitely possible well behaved and signature-preserving modifications of the Riemannian metric (universality) directly in the frame of the observer (direct universality).

In this note we show that the isoriemannian geometry admits a rather simple, yet effective formulation via the recently achieved *isodifferential calculus* [19]. To render note self-sufficient, we first review the elements of isotopies and then study the new formulation of the isoriemannian geometry. It should be stressed that this note has been written by a theoretical physicist and, in any case, the studies are in their first infancy, thus requiring comprehensive mathematical reformulating by interested mathematicians.

2. Elements of isotopic methods.

2.1. Isotopies of the unit. The fundamental isotopies from which all others can be uniquely derived are those of the *unit* [9], i.e., the liftings of the n-dimensional unit $I = \text{diag.}(1, 1, 1, \dots)$ of the Riemannian geometry (in the same dimension) into real-valued and symmetric $n \times n$ matrices $\hat{I} = (\hat{I}^i_j) = \hat{I}^t$ whose elements $\hat{I}^i_j$ have an unrestricted functional dependence in the coordinates x, the velocities $v = dx/dt$, the accelerations $a = dv/dt$, the local density μ, the local temperature τ, and any needed other characteristics of the interior problem,

$$I \to \hat{I} = \hat{I}(x, v, a, \mu, \tau, \dots) = \hat{I}^t. \tag{2.1}$$

The above liftings were classified by Kadeisvili [2] into: **Class I** (generalized units that are nondegenerate, Hermitean and positive-definite, characterizing the *isotopies* properly speaking); **Class II** (the same as Class I although $\hat{I}$ is negative-definite, characterizing the so-called *isodualities*); **Class III** (the union of Class I and II); **Class IV** (Class III plus the zeros of the generalized unit, $\hat{I} = 0$); and **Class V** (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices etc.).

All isotopic structures identified below also admit the same classification which will be omitted for brevity. In this note we shall study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. The isotopies of Classes IV and V are vastly unexplored at this writing.

2.2. Isofields. The first significant application of the isotopies of the unit is that for the liftings of conventional numbers and fields, which was presented by Santilli at the 1980 Clausthal meeting on *Differential Geometric Methods in Mathematical Physics* (see the latest study [15] and the general presentation in [16]).

Let $F = F(a, +, \times)$ be a field (hereon assumed to have characteristic zero) with elements $a, b, \dots$, sum $a + b$, multiplication $a \times b = ab$, additive unit 0, multiplicative unit 1, and familiar properties $a+0 = 0+a = a, a \times 1 = 1 \times a = a, \forall a \in F$, etc. We have in particular the field $R(n, +, \times)$ of *real numbers* n, the field $C(c, +, \times)$ of *complex numbers* c, and the field $Q(q, +, \times)$ of *quaternions* q.

Definition 2.1. ([15]) *An "isofield" $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times})$ is a ring with elements $\hat{a} = a \times \hat{I}$, called "isonumbers", where $a \in F$, and $\hat{I}$ is a Class III quantity generally outside F, equipped with two operations $(+, \hat{\times})$, where $\hat{+} \equiv +$ is the conventional sum of F with conventional additive unit $\hat{0} \equiv 0$ and $\hat{\times}$ is a new multiplication* [9]

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \times \hat{T} \times \hat{b}, \quad \hat{I} = \hat{T}^{-1} \tag{2.2}$$

called "isomultiplication", with $\hat{I}$ the left and right unit of $\hat{F}$,

$$\hat{I} \hat{\times} \hat{a} = \hat{a} \hat{\times} \hat{I} \equiv \hat{a}, \ \forall \hat{a} \in \hat{F}, \tag{2.3}$$

in which case (only). $\hat{I}$ is called the "isounit" and $\hat{T}$ is called the "isotopic element". Under these assumptions $\hat{F}$ is a field, i.e., it satisfies all properties of F in their isotopic form:

1. The set $\hat{F}$ is closed under addition, $\hat{a}+\hat{b} \in \hat{F}, \forall \hat{a}, \hat{b} \in \hat{F}$,
2. The addition is commutative, $\hat{a}+\hat{b}=\hat{b}+\hat{a}, \forall \hat{a}, \hat{b} \in \hat{F}$,
3. The addition is associative, $\hat{a}+(\hat{b}+\hat{c})=(\hat{a}+\hat{b})+\hat{c}, \forall \hat{a}, \hat{b}, \hat{c}, \in \hat{F}$,
4. There is an element $\hat{0} \equiv 0$, the "additive unit", such that $\hat{a}+\hat{0}=\hat{0}+\hat{a}=\hat{a}, \forall \hat{a} \in \hat{F}$,
5. For each element $\hat{a} \in \hat{F}$, there is an element $-\hat{a} \in \hat{F}$, called the "opposite of $\hat{a}$", which is such that $\hat{a}+(-\hat{a})=\hat{0}$,
6. The set $\hat{F}$ is closed under isomultiplication, $\hat{a} \hat{\times} \hat{b} \in \hat{F}, \forall \hat{a}, \hat{b} \in \hat{F}$,
7. The multiplication is generally non-isocommutive, $\hat{a} \hat{\times} \hat{b} \neq \hat{b} \hat{\times} \hat{a}$, but "isoassociative", $\hat{a} \hat{\times}(\hat{b} \hat{\times} \hat{c})=(\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{c}, \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}$,
8. The quantity $\hat{I}$ in the factorization $\hat{a}=a \times \hat{I}$ is the "multiplicative isounit" of $\hat{F}$ as per Eq.s (2.3),
9. For each element $\hat{a} \in \hat{F}$, there is an element $\hat{a}^{-\hat{I}} \in \hat{F}$, called the "isoinverse", which is such that $\hat{a} \hat{\times}(\hat{a}^{-\hat{I}})=(\hat{a}^{-\hat{I}}) \hat{\times} \hat{a}=\hat{I}$.
10. The set $\hat{F}$ is closed under joint isomultiplication and addition,

$$\hat{a} \hat{\times}(\hat{b}+\hat{c}) \in \hat{F}, \quad (\hat{a}+\hat{b}) \hat{\times} \hat{c} \in \hat{F}, \quad \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}. \tag{2.4}$$

11. All elements $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ verify the right and left "isodistributive laws"

$$\hat{a} \hat{\times}(\hat{b}+\hat{c})=\hat{a} \hat{\times} \hat{b}+\hat{a} \hat{\times} \hat{c}, \qquad (\hat{a}+\hat{b}) \hat{\times} \hat{c}=\hat{a} \hat{\times} \hat{c}+\hat{b} \hat{\times} \hat{c}. \tag{2.5}$$

When there exists a least positive isointeger $\hat{p}$ such that the equation $\hat{p} \hat{\times} \hat{a}=\hat{0}$ admits solution for all elements $\hat{a} \in \hat{F}$, then $\hat{F}$ is said to have "isocharacteristic p". Otherwise, $\hat{F}$ is said to have "isocharacteristic zero". Unless otherwise stated, all isofields considered hereon shall be Class III isofields of isocharacteristic zero.

We then have the isofield $\hat{R}(\hat{n},+,\hat{\times})$ of isoreal numbers $\hat{n}$; the isofield $\hat{C}(\hat{c},+,\hat{\times})$ of isocomplex isonumbers $\hat{c}$; and the isofield $\hat{Q}(\hat{q},+,\hat{\times})$ of isoquaternions $\hat{q}$ (see [15] for the isooctonions). Since all infinitely possible $\hat{F}$ preserves by construction all axioms of F, they are called *isotopes* of F and the liftings $F \rightarrow \hat{F}$ are called *isotopies*.

All conventional operations dependent on the multiplication on F are generalized on $\hat{F}(\hat{a},+,\hat{\times})$, thus yielding *isopowers*, *isosquare roots*, *isoquotients* etc.

$$\hat{a}^{\hat{2}}=\hat{a} \hat{\times} \hat{a}=(a \times a) \times \hat{I}, \quad \hat{A}^{\frac{1}{2}}=a^{\frac{1}{2}} \times \hat{I}^{\frac{1}{2}}, \quad \hat{/}=/ \times \hat{I}, \ etc. \tag{2.6}$$

It is then easy to see that the *isounit verifies all axiomatic properties of the conventional unit*, e.g.,

$$\hat{I}^{\hat{n}}=\hat{I} \hat{\times} \hat{I} \hat{\times} \ldots \hat{\times} \hat{I} \equiv \hat{I}, \quad \hat{I}^{\frac{\hat{1}}{2}} \equiv \hat{I}, \quad \hat{I} \hat{/} \hat{I} \equiv \hat{I}, \ etc. \tag{2.7}$$

Despite its simplicity, the liftings $F \rightarrow \hat{F}$ have significant implications in number theory itself. For instance, real numbers which are conventionally prime (under the *tacit* assumption of the unit I) are not necessarily prime with respect to a different unit [15]. This illustrates that most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit simple, yet intriguing and significant isotopies.

It is important to understand that an isofield of Class III, $\hat{F}_{III}(\hat{a},+,\hat{\times})$, is the union of two isofields, one of Class I, $\hat{F}_I(\hat{a},+,\hat{\times})$, in which the unit is positive-definite, and one of class II, $\hat{F}_{II}(\hat{a},+,\hat{\times})$, in which the isounit is negative-definite,

$$\hat{F}_{III}(\hat{a},+,\hat{\times})=\{\hat{F}_I(\hat{a},+,\hat{\times}), \hat{I}>0\} \cup \{\hat{F}_{II}(\hat{a},+,\hat{\times}), \hat{I}<0\}, \tag{2.8}$$

with interconnecting map called *isoduality* [16a]

$$\hat{I}>0 \rightarrow \hat{I}^d=-\hat{I}<0, \tag{2.9}$$

The Class II isofields are also written in the literature $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ and called *isodual isofields* with *isodual isonumbers* $\hat{a}^d=a \times \hat{I}^d=-\hat{a}$, isodual isoproduct $\hat{\times}^d=\times \hat{T}^d \times=-\hat{\times}$ etc.

Note that isoduality also applies to *ordinary* numbers and fields, in which case, $I^d=-1, n^d=-n, c^d=-\bar{c}$ and $q^d=-q^{\dagger}$, where $^{-}(\dagger^{\dagger})$ represents complex (Hermitean) conjugation. Note also that the imaginary quantity i is *isoselfdual*, i.e., invariant under isoduality (because $i^d=-\bar{i} \equiv i$). Note that the set of imaginary numbers *is not* a field, but the isoreal field with $\hat{I}=i$ is indeed a field. Note finally that the sum is not changed under isotopies otherwise there is the loss of the axioms of a field. For these and other aspects, one can inspect refs[15,16a].

A most significant aspect of isoduality is that it is an *antiautomorphic map*. As such, it has permitted a novel representation of *antimatter* which begins at the *classical* level and then persists at the operator level where it results to be equivalent to charge conjugation [16b]. All isotopies of class I shall therefore be restricted hereon to represent *matter*, and those of Class II to represent *antimatter*. The *isonorm* of an isofield (of Class III) is defined by

$$\uparrow \hat{a} \uparrow = |a| \times \hat{I}, \tag{2.10}$$

where $|a|$ is the conventional norm. It is therefore easy to see that *the isonorm of isofields of Class I is positive-definite, while that of Class II is negative definite.* This implies that all characteristics which are conventionally positive for matter, becomes negative-definite for antimatter, thus including energy, time, curvature etc.

To understand the latter occurrence, one should keep in mind the equivalence of *positive characteristics referred to a positive unit* and *negative characteristics referred to a negative unit.* This mathematically elementary property has rather intriguing applications, such as the prediction of *antigravity for elementary antiparticles in the field of matter*, and others [16b].

It may be of some guidance to present since these introductory lines a few examples of isounits used in applications (see [16b] for all details). One of the simplest possible example is *the use of the isounit for the representation of extended, nonspherical and deformable shapes.* For instance, a spheroidal ellipsoid in three dimensions can be represented via the isounit

$$\hat{I} = \text{diag.}(n_1^2, n_2^2, n_3^2). \tag{2.11}$$

where the quantities n_k^2 are sufficiently well behaved, real valued and positive-definite functions of local quantities. In fact, the isotopies of the Euclidean space with the above isounit characterize all infinitely possible ellipsoids.

The next simple example is the representation of nonconservative systems with isounits of the type

$$\hat{I} = e^{f(t, x, v, \ldots)}, \quad \hat{I}_{|f=0} = I. \tag{2.12}$$

which permit the representation of continuously decaying angular momenta; extended, nonspherical and deformable particles moving within resistive media under nonhamiltonian but local-differential forces; and other systems.

which experience an *attractive* interaction against their *repulsive* Coulomb force. The use of the quantum mechanical Coulomb law with conventional unit $I = \text{diag.}(1, 1, 1)$ leads to repulsion. On the contrary, the use of the following isounit, called *Animalu isounit*,

$$\hat{I} = e^{\int d^3 x \phi^{\dagger}{}_{\downarrow}(x) \phi_{\downarrow}(x)} \text{diag.}(1, 1, 1,), \tag{2.13}$$

permits a quantitative interpretation of the attraction among the two identical electrons in a way conform with experimental evidence, where $\phi_{\uparrow}$ and $\phi_{\downarrow}$ are the wavefunctions of the two electrons with related spin orientation $\uparrow$ and $\downarrow$. The exponent then illustrates the type of nonlocality considered in this note.

Note that all isounits are restricted by the condition of admitting the conventional unit as a particular case. For instance, when the overlapping of the two wavepackets $\phi_{\uparrow}$ and $\phi_{\downarrow}$ is no longer appreciable, the integral in the exponent of isounit (2.13) is null and $\hat{I}$ recovers the conventional unit I.

Note also that all the above isounits are outside the original field F, as it is generally the case for applications [16b].

The reader is suggested to meditate an instant on the fact that the entire contemporary mathematical knowledge is based on the simplest possible unit +1 which has essentially remained unchanged from its inception during biblical times. One can therefore see the horizon of new possibilities permitted by the generalization of such a fundamental notion.

2.3. Isospaces. The second significant application of the isotopies is the lifting of the conventional notions of vector and metric spaces, first presented in paper [13] of 1983 (see monographs [16b] for detailed treatment). In this section we shall review the main lines of the isotopies of Riemannian spaces.

Let $\Re(x, g, R)$ be an n-dimensional Riemannian space [3], with local chart $x = \{x^k\}, k = 1, 2, \ldots, n$, n-dimensional metric $g = (g_{ij}), g_{ij} = g_{ij}(x)$, and interval between two points $x, y \in \Re$,

$$(x - y)^2 = (x^i - y^i) g_{ij} (x^j - y^j) \in R(n, +, \times), \tag{2.14}$$

where the convention on the sum of repeated indices is assumed hereon. It is evident that the liftings of the field require, for consistency, a corresponding liftings of the spaces, and we have the following:

Definition 2.2. ([13,16b]) *The "isoriemannian spaces" $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ of Class III are n-dimensional metric spaces defined over an isoreal isofield $\hat{R}(\hat{n}, +, \hat{\times})$ with an $n \times n$-dimensional real-valued and symmetric isounit $\hat{I} = \hat{I}^t$ of Class III, equipped with the "isometric"*

$$(2.15) \qquad \hat{g}(t, x, v, a, \mu, \tau, \dots) = (\hat{g}_{ij}) = \hat{T}(t, x, v, a, \mu, \tau, \dots) \times g(x) = \hat{g}^t,$$

$$\hat{I} = \hat{T}^{-1} = \hat{I}^t,$$

local chart in contravariant and covariant forms

$$(2.16) \qquad \hat{x} = \{\hat{x}^k\} = \{x^k \times \hat{I}\}, \quad \hat{x}_k = \hat{g}_{ki}\hat{x}^i = \hat{T}_k^r g_{ri} x^i \times \hat{I}, \quad x^k, x_k \in \Re;$$

and "isoseparation" among two points $\hat{x}, \hat{y} \in \hat{\Re}$

$$(2.17) \qquad (\hat{x} - \hat{y})^{\hat{2}} = [(\hat{x}^i - \hat{y}^i) \times \hat{g}_{ij} \times (\hat{x}^j - \hat{y}^j)] \times \hat{I} \in \hat{R}.$$

The same rules apply for the definition of the isoeuclidean and isominkowskian, isofinslerian and other isospaces.

The primary property of the liftings $\Re(x, g, R) \to \hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ is the preservation of the original geometric axioms, as a necessary condition for isotopies. In actuality, $\Re(x, g, R)$ and $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ coincide at the abstract level for Class I isotopies by construction. This is due to the construction of the isospaces via the deformation of the metric g into the isomorphic $\hat{g} = \hat{T} \times g$ while jointly the original unit I is deformed in the amount *inverse* of the deformation of g, $\hat{I} = \hat{T}^{-1}$. This mechanism then ensures the preservation of all original geometric properties, as we shall see in Sect. 3.

The isoriemannian spaces of class II are instead *antiautomorphic* to the original spaces, and this renders them particularly suited to represent *antimatter* [16b].

Note that in the conventional space $\Re(x, g, R)$ the unit of the field is the *number* +1, while the unit of the space is the *matrix* I = diag.(1, 1, 1, ...). For $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ the isounits of the isofield and that of the isospaces coincide and are given by $\hat{I}$.

Note also that the isoseparation $\hat{x}^{\hat{2}}$, for consistency, must be an element of the isofield, that is, must have the structure of a number n multiplied by the isounit $\hat{I}$. This isoscalar character is expressed by the isomultiplication

$$(2.18) \qquad \hat{x}^{\hat{2}} = \hat{x}^k \hat{\times} \hat{x}_k = (\hat{x}^k \times \hat{I}) \times \hat{T} \times (x_k \times \hat{I}) = (x^k \times x_k) \times \hat{I} = n \times \hat{I}.$$

But the contraction over the repeated index k is in isospace. We recover in this way the isoseparation of Def.2.2, as one can see.

Because of the above occurrences, isospaces can be practically treated via the conventional coordinates x^k rather than the isotopic ones $\hat{x}^k = x^k \times \hat{I}$ whenever no confusion arises. The symbol x will be used for the coordinates of conventional spaces, while the symbol $\hat{x}$ will be used for the coordinates of isospaces. When writing $\hat{g}(x, \dots)$ we refer to the *projection* of the isometric $\hat{g}$ in the original space.

Despite their simplicity, the liftings $\Re(x, g, R) \to \hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ have significant implications. In fact, the functional dependence of the isounit $\hat{I}$ remains unrestricted under isotopies. Besides being well behaved, real valued and symmetric, the isometrics $\hat{g}$ therefore have an unrestricted functional dependence indicated earlier, $\hat{g} = \hat{g}(x, v, a, \mu, \tau, \dots)$. This establishes the "direct universality" of the isoriemannian spaces indicated in Sect.1.

Isospaces $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ therefore satisfy all conditions I-VII of the Sect.1. Condition I on the geodesic representation of free trajectories within phisical media will be studied in Sect.3.5. The isometric $\hat{g}$ is arbitrarily nonlinear in the velocities and other quantities, thus verifying condition II. Isolagrangians on $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ are universal for all possible resistive forces, as proved in ref.[16b], thus satisfying condition III. Isospaces $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ can indeed represent nonlocal-integral effects as shown in Sect.2.6, thus verifying condition IV. The isometric $\hat{g}$ is manifestly inhomogeneous and anisotropic, thus satisfying condition V. The generalized/covering character of $\hat{\Re}$ over $\Re$ is evident with the latter admitted for the particular case $\hat{I} = I$, thus verifying condition VI. Finally, the nonlocal-nonlagrangian effects are represented with the isotopic element $\hat{T}$ while the local-Lagrangian effects are presented with the conventional metric g. The two effects are then separated in the very structure of the isometric $\hat{g} = \hat{T}g$, thus verifying condition VII.

2.4. Isodifferential calculus. At the *international Workshop on Differential Geometry and Lie Algebras*, held in Dec. 1994 at Aristotie University in Thessaloniki, Greece, Santilli submitted the *isotopies of differential calculus*, or *isodifferential calculus* for short, subsequently published in ref.[19], which permits a simple, yet unique and significant formulation of the isogeometries.

For the reader's convenience, we review below the isodifferential calculus on a flat isoeuclidean space. Topological aspects are studied in Sect.2.5 and 2.6. The formulation on curved isospaces is studied in Sect.3.

Let $E(x, \delta, R)$ be the ordinary n–dimensional Euclidean space with local coordinates $x = \{x^k\}, k = 1, 2, \dots, n$, and metric δ = diag.(1, 1, 1, ...) over the reals $R(n, +, \times)$. Let $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ denote isotopic images with local coordinates $\hat{x} = \{\hat{x}^k\}$, isometric $\hat{\delta} = \hat{T}\delta$ over the isoreals $\hat{R}(\hat{n}, +, \hat{\times})$ with real-valued and symmetric isounit of Class III, $\hat{I} = (\hat{I}_i^j) = \hat{T}^{-1} = (T_i^j)^{-1} =$

$\hat{I}^t$ whose elements have a smooth but otherwise arbitrary dependence on the local coordinates, their derivatives with respect to an independent variable and any needed additional quantity, $\hat{I} = \hat{I}(\hat{x}, \hat{v}, \hat{a}, \ldots)$.

Definition 2.3. ([19]) *The "first-order isodifferentials" of Class III of the contravariant and covariant coordinates $\hat{x}^k$ and $\hat{x}_k$, on an isoeuclidean space $\hat{E}$ are given by*

$$\hat{d}\hat{x}^k = \hat{I}^k{}_i(x, \ldots)dx^i, \quad \hat{d}\hat{x}_k = \hat{T}_k{}^i(x, \ldots)dx_i, \tag{2.19}$$

where the expresions $\hat{d}\hat{x}^k$ and $\hat{d}\hat{x}_k$ are definided on $\hat{E}$ while the corresponding expressions $\hat{I}^k{}_i dx^i$ and $\hat{T}_k{}^i dx_i$ are the projections on the conventional Euclidean space E. Let $\hat{f}(\hat{x})$ be a suficiently smooth isofunction on a closed domain $\hat{D}(\hat{x}^k)$ of contravariant coordinates $\hat{x}^k$ on $\hat{E}$. Then the "isoderivative" at a point $\hat{q}^k \in \hat{D}(\hat{x}^k)$ is given by

(2.20)
$$\hat{f}'(\hat{q}^k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}|_{\hat{x}^k=\hat{q}^k} = \hat{T}_k{}^i\frac{\partial f(x)}{\partial x^i}|_{x^k=\hat{q}^k} = \operatorname*{Lim}_{\hat{d}\hat{x}^k\to\hat{0}^k}\frac{\hat{f}(\hat{q}^k+\hat{d}\hat{x}^k)-\hat{f}(\hat{q}^k)}{\hat{d}\hat{x}^k},$$

where $\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x}^k$ is computed on $\hat{E}$ and $\hat{T}_k{}^i\partial f(x)/\partial x^i$ is the projection in E. *The "isoderivative" of a smooth isofunction $\hat{f}(\hat{x})$ of the covariant variable $\hat{x}_k$ at the point $\hat{q}_k \in \hat{D}(\hat{x}_k)$ is given by*

(2.21)
$$\hat{f}'(\hat{q}_k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_k}|_{\hat{x}_k=\hat{q}_k} = \hat{I}_k{}^i\frac{\partial f(x)}{\partial x_i}|_{\hat{x}_k=\hat{q}_k} = \operatorname*{Lim}_{\hat{d}\hat{x}_k\to\hat{0}_k}\frac{\hat{f}(\hat{q}_k+\hat{d}\hat{x}_k)-\hat{f}(\hat{q}_k)}{\hat{d}\hat{x}_k}.$$

The above definition and the axiom-preserving character of the isotopies then permit the lifting of the various aspects of the conventional differential calculus. We here mention for brevity the following isotopies: the *isodifferentials of an isofunction* of contravariant (covariant) coordinates $\hat{x}^k$ ($\hat{x}_k$) on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ are defined via the isoderivatives according to the respectives rules

$$\hat{d}\hat{f}(\hat{x})|_{\text{contrav}} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\hat{d}\hat{x}^k = \hat{T}_k{}^i\frac{\partial f}{\partial x^i}\hat{I}^k{}_j dx^j = \frac{\partial f}{\partial x^k}dx^k = \frac{\partial f}{\partial x^i}\hat{T}_i{}^j dx^j, \tag{2.22}$$

$$\hat{d}\hat{f}(\hat{x})|_{\text{covar.}} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k}\hat{d}\hat{x}_k = \hat{I}^k{}_i\frac{\partial f}{\partial x_i}\hat{T}_k{}^j dx_j = \frac{\partial f}{\partial x_k}dx_k = \frac{\partial f}{\partial x_j}\hat{I}^i{}_j dx_i$$

where the last expression originate from the fact that the contraction is in space, thus with respect to the isometric; an iteration of the notion of isoderivative leads to the *second-order isoderivatives*

(2.23)
$$\frac{\hat{\partial}^2\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{k2}} = \hat{T}_k{}^i\hat{T}_k{}^j\frac{\partial^2 f(x)}{\partial x^i\partial x^j}, \quad \frac{\hat{\partial}^2\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_k{}^2} = \hat{I}^k{}_i\hat{I}^k{}_j\frac{\partial^2 f(x)}{\partial x_i\partial x_j} \text{ (no sums on } k)$$

and similarly for isoderivatives of higher order; the *isolaplacian* on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ is given

$$\hat{\Delta} = \hat{\partial}_k\hat{\partial}^k = \hat{\partial}^i\hat{\delta}_{ij}\hat{\partial}^j = \hat{I}^i{}_k\partial^k\delta_{ij}\partial^j, \quad \hat{\partial}_k = \hat{\partial}/\hat{\partial}\hat{x}^k, \partial_k = \partial/\partial x^k, \tag{2.24}$$

and results to be different than the corresponding expression on a Riemannian space $\Re(x, g, R)$ with metric $g(x) = \hat{\delta}$, $\Delta = \hat{\delta}^{-1/2}\partial_i\hat{\delta}^{1/2}\hat{\delta}^{ij}\partial_j$, *even though the isoeuclidean metric $\hat{\delta}(x, v, a, \ldots)$ is more general than the Riemannian metric $g(x)$.*

The following properties then follow,

$$\hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}^j = \delta^i{}_j, \; \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}_j = \delta_i{}^j, \; \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}^j = \hat{T}_i{}^j, \; \hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}_j = \hat{I}^i{}_j. \tag{2.25}$$

Other properties can be derived by the interesed reader.

For completeness we mention the (indefinite) *isointegration* which is the inverse of the isodifferential, e.g.,

$$\hat{\int}\hat{d}\hat{x} = \int\hat{T}\hat{I}dx = \int dx = x, \tag{2.26}$$

namely, $\hat{\int} = \int\hat{T}$. Definite isointegrals are formulated accordingly. Due to its simplicity we shall tacitly assume the isotopies of integration hereon.

The above basic notions are sufficient for our limited needs at this time. The class of isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$.

An important property is that the isodifferentials and isoderivatives preserve the basic isounit $\hat{I}$. Mathematically , this conditions is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x}), \hat{g}(\hat{x}), \ldots,$ on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ over the isofield $\hat{R}(\hat{n}, +, \hat{\times})$ with isounit $\hat{I}$ are mapped via isoderivatives into a set of isofunctions $\hat{f}'(\hat{x}), \hat{g}'(\hat{x}), \ldots,$ defined over a *different* field because of the alteration of the isounit. Physically, the condition is also *necessary* because the unit is a pre-requisite for measurements. The lack of conservation of the unit therefore implies the lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^k = d(\hat{I}^k{}_i x^i) = [(\partial_i \hat{I}^k{}_r)x^r + \hat{I}^k{}_i]dx^i = \hat{W}^k{}_i dx^i, \tag{2.27}$$

would imply the alteration of the isounit, $\hat{I} \to \hat{W} \neq \hat{I}$, thus being mathematically and physically unacceptable.

Nevertheless, when using isoderivatives of independent variables, say, isoderivatives on coordinates and time, the above rules does not apply and we have

$$\hat{\partial}_t \hat{\partial}_k \hat{f}(\hat{t}, \hat{x}) = \hat{\partial}_t[\hat{\partial}_k \hat{f}(\hat{t}, \hat{x})] = \hat{\partial}_t[T_k{}^i(t, x, \ldots)\partial_i f(t, x)]. \tag{2.28}$$

Additional properties of the isodifferential calculus will be identified during the course of our analysis.

Note that the ordinary differential calculus is local-diffential on E. The isodifferential calculus is instead local-differential on $\hat{E}$ but, when projected on E, it becomes *integro-differential* because it incorporates integral terms in the isounit. The above isocalculus admits a natural extensions to the *genodifferential and hiperdifferential calculus* [18].

Note finally that the isodifferential calculus is a particular case of the broader *genodifferential calculus* in which the isounit is no longer symmetric, in which case all products must be ordered to the right or, separately, to the left. In turn, the genodifferential calculus is expected to be a particular case of the still broader *hyperdifferential calculus* in which , besides the ordering of all multiplications, the individual right and left isounit and isotopic element are multivalued [18].

2.5. Kadeisvili's isocontinuity. The notion of *isocontinuity* on an isospace was first studied by Kadeisvili [2] and resulted to be easily reducible to that of conventional continuity for the isotopies of Class III because the *isomodulus* $\hat{|}\hat{f}(\hat{x})\hat{|}$ of a function $\hat{f}(\hat{x})$ on isospace $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ over the isofield $\hat{R}(\hat{n}, +, \hat{\times})$ is given by the conventional modulus $|\hat{f}(\hat{x})|$ multiplied by the a well behaved isounit $\hat{I}$,

$$\hat{|}\hat{f}(\hat{x})\hat{|} = |\hat{f}(\hat{x})| \times \hat{I}. \tag{2.29}$$

Definition 2.4. ([2]) *An infinite sequence* $\hat{f}_1, \hat{f}_2, \ldots$ *is said to be "strongly isoconvergent" to* $\hat{f}$ *when*

$$\operatorname*{Lim}_{k \to \infty} \hat{|}\hat{f}_k - \hat{f}\hat{|} = 0. \tag{2.30}$$

while the "isocauchy condition" can be expressed by

$$\hat{|}\hat{f}_m - \hat{f}_n\hat{|} < \hat{\delta} = \delta \times \hat{I}, \tag{2.31}$$

where δ is real and m and n are greater than a suitably chosen $N(\delta)$.

The isotopies of other notions of continuity, limits, series, etc. can be easily constructed [2].

Note that functions which are conventionally continuous are also isocontinuous. Similarly, a series which is strongly convergent is also strongly isoconvergent. However, a *series which is strongly isoconvergent is not necessarily strongly convergent* ([16b], p.271). As a result, a *series which is conventionally divergent can be turned into a convergent form under a suitable isotopy*. This mathematically trivial property has rather important applications, e.g., for the reconstruction of convergent perturbative series for *strong* interactions, which are conventionally divergent.

Similarly, the reader may be interested in knowing that, given a function which is not square-integrable in a given interval, there always exists an isotopy which turns the function into a square-integrable form [16a]. The novelty is due to the fact that the underlying mechanism *is not* that of the use of q weight function, but that of *altering the underlying field*.

2.6. Tsagas-Sourlas isotopology. The notion of n-dimensional *isomanifold* was first studied by Tsagas and Sourlas [22]. The main lines can be summarized as follows. All isounits of Class I can always be diagonalized into the form

$$\hat{I} = \text{diag.}(B_1, B_2, \ldots, B_n), \quad B_k(x, \ldots) \neq 0, \ k = 1, 2, \ldots, n. \tag{2.32}$$

Consider then n isoreal isofields $\hat{R}_k(\hat{n}, +, \hat{\times})$ each characterized by isounit $\hat{I}_k = B_k$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_n. \tag{2.33}$$

Since $\hat{R}_k \approx R$, it is evident that $\hat{R}^n \approx R^n$, where R^n is the Cartesian product of n conventional fields $R(n, +, \times)$. But the total unit of $\hat{R}^n$ is expression (2.32). Therefore, one can introduce a topology on $\hat{R}^n$ via simple isotopy of the conventional topology on R^n,

$$\hat{\tau} = \{\emptyset, \hat{R}^n, \hat{K}_i\}, \tag{2.34}$$

where $\hat{K}_i$ represents the subset of $\hat{R}^n$ defined by

$$\hat{K}_i = \{\hat{P} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)/\hat{n}_i < \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n < \hat{m}_i, \hat{n}_i, \hat{m}_i, \hat{a}_k \in \hat{R}\}. \tag{2.35}$$

As one can see, the above topology coincides everywhere with the conventional topology τ of R^n *except at the isounit* $\hat{I}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{I}$ which can incorporate integral terms. The above structure is called the *Tsagas-Sourlas isotopology* or an *integro-differential topology*.

Definition 2.5. ([22]) *A "topological isospace" $\hat{\tau}(\hat{R}^n)$ of Class I is the isospace $\hat{R}^n$ of the same class equipped with the isotopology $\hat{\tau}$ of Eq.(2.34). An "isocartesian isomanifold" $\hat{M}(\hat{R}^N)$ of Class I is the topological isospace $\hat{\tau}(R^n)$ of the same class equipped with a vector structure, an affine structure and the mapping*

$$\hat{f} : \hat{R}^n \to \hat{R}^n, \; \hat{f} : \hat{a} \to \hat{f}(\hat{a}) = \hat{a}, \; \forall \hat{a} \in \hat{R}. \tag{2.36}$$

An "isoeuclidean isomanifold" $\hat{M}(\hat{E}(\hat{x}, \hat{\delta}, \hat{R}))$ occurs when the $n-$dimensional isospace $\hat{E}$ is realized as the Cartesian product

$$\hat{E}(\hat{x}, \hat{\delta}, \hat{R}) \approx \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_n, \tag{2.37}$$

and equipped with the isotopology τ with isounit (2.31). The "isodual topological isospaces", "isodual isocartesian isomanifolds" and "isodual isoeuclidean isomanifolds" are the images of their corresponding structures under isoduality.

For all additional aspects of isomanifolds and related topological properties we refer the interesed reader to Tsagas and Sourlas [22,23]. It should be noted that their study is referred to $M(\hat{R}^n)$, and that the structure $\hat{M}(\hat{R}^n)$ was introduced in [19].

2.7. Isotopies of functional analysis. The isotopies imply simple, yet nontrivial generalizations of *all* conventional mathematical structures, with no exception known to this author. This implies also a compatible lifting of functional analysis whose study was initiated by Kadeisvili in ref.[22] under the name of *functional isoanalysis*. The latter discipline begins with the isotopies of continuity outlined in Sect.2.5, and includes the isotopies of conventional square-integrable, Banach and other spaces; special functions and transforms; etc. (see [16a] for brevity).

The reader should be alerted that the use of *conventional* functions, such as the trigonometric or hyperbolic functions, within the context of *isotopic* spaces leads to a number of inconsistencies which may remain undetected by non-experts in the fields.

3. Isoriemannian geometry

3.1. Background notions. In memoir [13] of 1988 we presented, apparently for the first time, the *isotopies of the Riemannian geometry* or *isoriemannian geometry* for short, which was constructed via the isotopies of the *product*. In this section we present the isoriemannian geometry constructed via the *isotopies of the differential calculus* and show that the latter formulation is more conductive to a single, unified, abstract formulation of the conventional and isotopic geometries, that is, of the exterior and interior problems.

Our study is in local coordinates representing the fixed frame of the observer, and all abstract, coordinate-free treatments are left to the interested mathematician. For the conventional geometry we assume all topological properties of Lovelock and Rund [3] of which we preserve the symbols for clarity in the comparison of the results. For the isotopic geometry we assume the topological properties by Tsagas and Sourlas [22] as outlined of Def.2.6. Our presentation is made, specifically, for the (3+1)-dimensional space-time, with the understanding that the extension to arbitrary dimensions and signatures is elementary. In particular, for clarity we shall first study the isoriemannian geometry of Class I for the characterization of *matter* an then study its isodual image (Class II) for the characterization of *antimatter*.

Let $\Re = \Re(x, g, R)$ be a (3+1)-dimensional Riemannian space over the reals $R(n, +, \times)$ [3] with : local coordinates $x = \{x^\mu\} = \{r, x^4\}$, $x^4 = c_0 t$, $\mu = 1, 2, 3, 4$, where c_0 is the speed of light in vacuum; nowhere singular, symmetric and real-valued metric $g(x) = (g_{\mu\nu}) = g^t$; and tangent Minkowski space $M(x, \eta, R)$ with metric $\eta = \mathrm{diag.}(1, 1, 1, -1)$ over the reals R. Let the interval be written in the familiar expression $x^2 = x^\mu g_{\mu\nu}(x) x^\nu \in R$ with infinitesimal line element $ds^2 = dx^\mu g_{\mu\nu}(x) dx^\nu$ and related formalism (covariant derivative, Christoffel's symbols, etc.[3]).

Let $\hat{\Re} = \hat{\Re}(\hat{x}, \hat{g}, \hat{R})$ be the *isoriemannian space* of Definition 2.2 with local coordinates $\hat{x} = \{\hat{x}^\mu\} = \{x^\mu\}$ and *isometric* $\hat{g}(x, v, a, \mu, \tau, \ldots) = \hat{T}(x, v, a, \mu, \tau, \ldots) g(x)$, where $\hat{T} = (\hat{T}_\mu{}^\nu)$ is a nonwhere singular, real valued and symmetric matrix of Class III with C^∞ elements. The isospace $\hat{\Re}$ is defined over the isoreals $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$ with *common* isounit $\hat{I} = (\hat{I}^\mu{}_\nu) = \hat{T}^{-1}$. We then have the *isoline element*

$$\hat{x}^{\hat{2}} = [\hat{x}^\mu \hat{g}_{\mu\nu}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) \hat{x}^\nu] \times \hat{I} \in \hat{R}. \tag{3.1}$$

with infinitesimal version $\hat{d}\hat{s}^{\hat{2}} = (\hat{d}\hat{x}^\alpha \hat{g}_{\alpha\beta} \hat{d}\hat{x}^\beta) \times \hat{I} \in \hat{R}$.

The *isonormal coordinates* $\hat{y}$ occur when the isometric $\hat{g}$ is reduced, *not* to the Minkowski metric η, but rather to its isotopic image. i.e., $\hat{g} \to \hat{\eta} = \hat{T}\eta$

and, as such, they are the *conventional* normal coordinates (*principle of isoequivalence* [16b]). In different terms, the correct tangent space is not the conventional space $M(x, \eta, R)$, but the *isominkowskian space* $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ characterized by the same isounit and isotopic element of $\hat{\Re}(\hat{x}, \hat{g}, \hat{R})$, that is, with the *isominkowskian metric* $\hat{\eta} = \hat{T}\eta$, where $\hat{T}$ is the same as in the isoriemannian metric $\hat{g} = \hat{T}g$. Under these conditions, the isonormal coordinates reduce the g–component in $\hat{g} = \hat{T}g$ to the η–component of $\hat{\eta} = \hat{T}\eta$ and this illustrates that isonormal coordinates coincide with the conventional normal coordinates.

To have an idea of the possible applications, we mention that the isounits of Class I can always be diagonalized, thus expressed in the form

$$\hat{T} = \text{diag.}(n_1{}^{-2}, n_2{}^{-2}, n_3{}^{-2}, n_4{}^{-2}) \; ; \; n_\mu > 0 \; , \; \mu = 1, 2, 3, 4. \tag{3.2}$$

As such, the above isounits permit the achievement of the central geometric objectiv of Sect.1 : *the geometrization of a locally varying speed of light* $c = c_0/n_4$ within physical media, which occurs via the fourth component of the isoline element

$$\hat{x}^4 \hat{g}_{44} \hat{x}^4 = I\, c(\hat{x}, \mu, \tau, \ldots) g_{44}(x) I \; , \; c = c_0/n_4(x, \mu, \tau, \ldots), \tag{3.3}$$

where g_{44} is the ordinary metric element and n_4 is the familiar index of refraction.

The general isotopic rule $\hat{g} = \hat{T}g$ then permits the lifting into interior conditions of any given exterior metric, such as Schwarzschild's metric [20]. Note that the latter metric can only represent the *constant* speed c_0 and this illustrates the effectiveness of the isotopies for the geometric study of interior problems. The universality of the isotopies has been indicated in Sect.2.3. The preservation of Einstein's axioms occurs as follows. The representation of locally varying speeds of light technically occurs via the *isolight cone* [16a,b] $\hat{d}\hat{s}^2 = \hat{d}\hat{x}^\mu \hat{g}_{\mu\nu} \hat{d}\hat{x}^\nu = 0$ which is the image in isospace of the *deformation* of the light cone in the tangent Minkowski space caused by variable speeds. It has been proved that *the isolight cone is a perfect cone in isospace* and, moreover, that *the characteristic angles of the conventional and isolight cones coincides* (see ref. [16b], Ch.8. for details). In other words, *the isotopies reconstruct the speed of light in vacuum as the maximal causal speed in isospaces over isofields, thus preserving all Einsteinian axioms in their entirety*. The latter result is not a mere mathematical curiosity because it is important for specific applications, such as the correct calculations of gravitational horizons. In fact, the region outside these horizons is not empty, but filled up instead by very large and dense chromospheres.

It is well known that within these chromospheres the speed of light is locally varying with the density μ, temperature τ, etc., thus preventing the exact validity of the conventional light cone. The use of the isolight cone then permits more realistic studies of gravitational horizons because, while in isospace the maximal speed is c_0, the speed of light in its projection in the conventional space time is the actual speed $c = c_0/n(x, \mu, \tau, \ldots)$.

Note that the exterior geometrization in vacuum with constant speed of light c_0 is a particular case of the isoriemannian geometry occuring for $\hat{I} = I$. Thus illustrates the *covering* character of the isoriemannian geometry over the conventional form.

3.2. Isocovariant isodifferential and isoderivatives. In the first formulation of the isoriemannian geometry [13], differentials of contravariant isovector fields $\hat{X}^\beta$ on $\hat{\Re}$ were defined by $d\hat{X} = (\partial\hat{X}) \hat{\times} d\hat{x} = (\partial_\mu \hat{X}) \hat{T}_\nu{}^\mu d\hat{x}^\nu \neq dX = (\partial_\mu X) dx^\mu \; , \; \partial_\mu = \partial/\partial x^\mu$. The isodifferential calculus allows us to introduce the following alternative definition

$$\begin{aligned} \hat{d}\hat{X}^\beta &= (\hat{\partial}_\mu \hat{X}^\beta) \hat{d}\hat{x}^\mu = \hat{T}_\mu{}^\rho (\partial_\rho \hat{X}^\beta) \hat{I}^\mu{}_\sigma \hat{d}\hat{x}^\sigma \equiv \\ &\equiv (\partial_\mu \hat{X}^\beta) d\hat{x}^\mu = (\partial^\alpha \hat{X}^\beta) \hat{g}_{\alpha\varepsilon} d\hat{x}^\varepsilon, \end{aligned} \tag{3.4}$$

where the last expression is introduced to recall that the contractions are in isospace and not in conventional space. The preceding expression then shows that *isodifferentials of isovector fields coincide at the abstract level with conventional differential of vector fields for all Class I isotopies.*

The *isocovariant differential* can be defined by

$$\hat{D}\hat{X}^\beta = \hat{d}\hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\gamma \, \hat{X}^\alpha \hat{d}\hat{x}^\gamma, \tag{3.5}$$

with corresponding *isocovariant derivative*

$$\hat{X}^\beta{}_{\hat{|}\mu} = \hat{\partial}_\mu \hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\mu \, \hat{X}^\alpha, \tag{3.6}$$

where the *isochristoffel's symbols* are given by

$$\begin{aligned} \hat{\Gamma}_{\alpha\beta\gamma} &= \frac{1}{2}(\hat{\partial}_\alpha \hat{g}_{\beta\gamma} + \hat{\partial}_\gamma \hat{g}_{\alpha\beta} - \hat{\partial}_\beta \hat{g}_{\alpha\gamma}) = \hat{\Gamma}_{\gamma\beta\alpha} \, ; \\ \hat{\Gamma}_\alpha{}^\beta{}_\gamma &= \hat{g}^{\beta\rho} \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}_\gamma{}^\beta{}_\alpha \; , \; \hat{g}^{\beta\rho} = [(\hat{g}_{\mu\nu})^{-1}]^{\beta\rho}. \end{aligned} \tag{3.7}$$

One should note the abstract identity of the conventional and Class I isotopic connections. The extension to covariant isovector fields and covariant or contravariant isotensor fields is consequential (see [23]).

The isotopy of the proof of [3], pp. 80–81,yields to the following:

Lemma 1. *(Isoricci Lemma): Under the assumed conditions, the isocovariant derivatives of all isometrics on isoriemannian spaces are identically null,*

$$\hat{g}_{\alpha\beta|\gamma} \equiv 0\,, \quad \alpha, \beta, \gamma = 1, 2, 3, 4. \tag{3.8}$$

The novelty of the isoriemannian geometry is then illustrated by the fact that *the Ricci property persists under an arbitrary dependence of the metric.*

Despite the similarities with the conventional case, the lack of equivalence of the Riemannian and isoriemannian geometries can be ilustred via the *isotorsion* [13]

$$\hat{\tau}_{\alpha}{}^{\beta}{}_{\gamma} = \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} - \hat{\Gamma}_{\gamma}{}^{\beta}{}_{\alpha}, \tag{3.9}$$

which is identically null for the isoriemannian geometry here considered, but its projection in the space $\Re$ is not necessarily null. *Interior gravitational models treaded with the isoriemannian geometry are therefore theories with null isotorsion but generally non–null torsion.* This property is needed for a realistic treatment of *interior* problems to avoid excessive approximation such as the *tacit* assumption of the existence of the "perpetual motion" within a physical medium which is implied by interior theories with null torsion.

The occurrence also illustrates the property, verified at subsequent levels later on, that departures from conventional geometric properties must be studied in the *projection* of isoriemannian spaces in the original spaces because, when treated in their respective spaces, the two geometries coincide. Stated in different terms, when using the conventional geometry, exterior gravitation can only be studied in the spaces $\Re$. On the contrary, when using the isoriemannian geometry, interior gravitation can be studied in *two* different spaces, the isoriemannian spaces $\hat{\Re}$ and their projection into $\Re$.

Another way of identifying the differences between the Riemannian and isoriemannian geometries is by considering the following *isogeodesic equations*

$$\frac{\hat{D}\hat{x}_{\beta}}{\hat{D}\hat{s}} = \frac{\hat{d}\hat{v}_{\beta}}{\hat{d}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma}(\hat{x}, \hat{v}, \hat{a}, \ldots)\frac{\hat{d}\hat{x}^{\alpha}}{\hat{d}\hat{s}}\frac{\hat{d}\hat{x}^{\gamma}}{\hat{d}\hat{s}} = 0, \tag{3.10}$$

where $\hat{v} = \hat{d}\hat{x}/\hat{d}\hat{s} = \hat{I}_s \times dx/ds$, $\hat{s}$ is the proper isotime and $\hat{I}_s$ the related (one–dimensional) isounit. The preceeding equations must then be compared with the conventional equations

$$\frac{Dx_{\beta}}{Ds} = \frac{dv_{\beta}}{ds} + \hat{\Gamma}_{\alpha\beta\gamma}(x)\frac{dx^{\alpha}}{ds}\frac{dx^{\gamma}}{ds} = 0. \tag{3.11}$$

It is evident that the latter equations are most quadatric in the velocities while the isotopic equations are arbitrarily nonlinear in the velocities, as it occurs already in a flat space. Also, the later equations are local–differential and variationally selfadjoint while the former admit nonlocal–integral terms and are variationally nonselfadjoint in $\Re$ although selfadjoint in $\hat{\Re}$ [19].

We now introduce: the *isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} = \hat{\partial}_{\delta}\hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} - \hat{\partial}_{\gamma}\hat{\Gamma}_{\alpha}{}^{\beta}{}_{\delta} + \hat{\Gamma}_{\rho}{}^{\beta}{}_{\delta}\hat{\Gamma}_{\alpha}{}^{\rho}{}_{\gamma} - \hat{\Gamma}_{\rho}{}^{\beta}{}_{\gamma}\hat{\Gamma}_{\alpha}{}^{\rho}{}_{\delta}; \tag{3.12}$$

the *isoricci tensor* $\hat{R}_{\mu\nu} = \hat{R}_{\mu}{}^{\beta}{}_{\nu\beta}$; the *isocurvature isoscalar* $\hat{R} = \hat{g}^{\alpha\beta}\hat{R}_{\alpha\beta}$: the *isoeinstein tensor* $\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}$; and the *isotopic isoscalar*

$$\hat{\Theta} = \hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}(\hat{\Gamma}_{\rho\alpha\delta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\beta} - \Gamma_{\rho\alpha\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta}) = \hat{\Gamma}_{\rho\alpha\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta}(\hat{g}^{\alpha\delta}\hat{g}^{\gamma\beta} - \hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}). \tag{3.13}$$

the latter one being new for the isoriemannian geometry (see below).

3.3. The "five" basic identities of the isoriemannian geometry. Tedious but simple calculations then yields the following basic identies of the isoriemannian geometry:

Identity 1. *Antisymmetry of the last two indices of the isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} = -\hat{R}_{\alpha}{}^{\beta}{}_{\delta\gamma}; \tag{3.14}$$

Identity 2. *Symmetry of the first two indices of the isocurvature tensor*

$$\hat{R}_{\alpha\beta\gamma\delta} \equiv \hat{R}_{\beta\alpha\gamma\delta}; \tag{3.15}$$

Identity 3. *Vanishing of the totally antisymmetric part of the isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} + \hat{R}_{\gamma}{}^{\beta}{}_{\delta\alpha} + \hat{R}_{\delta}{}^{\beta}{}_{\alpha\gamma} \equiv 0; \tag{3.16}$$

Identity 4. *Isobianchi identity*

$$\underset{\alpha}{\hat{R}_{\alpha}}{}^{\beta}{}_{\gamma\delta\uparrow\rho} + \hat{R}_{\alpha}{}^{\beta}{}_{\rho\gamma\uparrow\delta} + \hat{R}_{\alpha}{}^{\beta}{}_{\delta\rho\uparrow\gamma} \equiv 0; \tag{3.17}$$

Identity 5. *Isofreud identity*

$$\hat{R}^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}\hat{R} - \frac{1}{2}\delta^{\alpha}{}_{\beta}\hat{\Theta} = \hat{U}^{\alpha}{}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}{}_{\beta}, \tag{3.18}$$

where $\hat{\Theta}$ is the isotopic isoscalar and

$$\hat{U}^{\alpha}{}_{\beta} = -\frac{1}{2}\frac{\hat{\partial}\hat{\Theta}}{\hat{\partial}\hat{g}^{\alpha\beta}{}_{\uparrow\alpha}}\hat{g}^{\alpha\beta}{}_{\uparrow\beta},$$

$$\hat{V}^{\alpha\rho}{}_{\beta} = \frac{1}{2}[\hat{g}^{\gamma\delta}(\delta^{\alpha}_{\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta} - \delta^{\rho}{}_{\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta})+$$

$$+(\delta^{\rho}{}_{\beta}\hat{g}^{\alpha\gamma} - \delta^{\alpha}{}_{\beta}\hat{g}^{\rho\gamma})\hat{\Gamma}_{\gamma}{}^{\delta}{}_{\delta} + \hat{g}^{\rho\gamma}\hat{\Gamma}_{\beta}{}^{\alpha}{}_{\gamma} - \hat{g}^{\alpha\gamma}\hat{\Gamma}_{\beta}{}^{\rho}{}_{\gamma}]. \tag{3.19}$$

A curiosity is that the conventional Riemannian geometry is generally thought to possess only *four* identities. In fact, the *fifth* identity, given in the above list by the Freud identity, is generally unknown in the contemporary technical literature in the field.

The latter identity was introduced by Freud [1] in 1939, treated in detail by Pauli [6] and then generally forgotten for a half a century, apparently, because of a conflict between the lack of source of Einstein's field equations in vacuum, $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R = 0$, and the evident need of a source in vacuum for the Freud identity $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R - \frac{1}{2}\delta^{\alpha}{}_{\beta}\Theta = U^{\alpha}{}_{\beta} + \partial_{\rho}V^{\alpha\rho}{}_{\beta}$ (there written in a conventional space, see [16b] for details).

Following a suggestion by the author, Rund [8] studied again the identity and proved that *the Freud identity is a bona fide identity for all Riemannian spaces irrespective of dimension and signature*, thus confirming the general need of a sourse also in vacuum (see below). In this note we have presented the *isotopies* of the Freud identity, that is, its formulation in isoriemannian spaces, as characterized by the isodifferential calculus.

Note that *all conventional and isotopic identities coincide at the abstract level*. This confirms that the conventional and isotopic geometries (exterior and interior problems) can be treated at the abstract, realization-free level via one single set of axioms, as desired (Sect.1), in both the exterior and interior formulations are *realizations* of the same axioms.

3.4. Interior isogravitation of matter and antimatter. The isotopy of the proof of the Theorem in [3], p.321, leads to the following property first identified in 1988 [13] (see also [16]) and which is here recovered via the isodifferential calculus.

Theorem 1. *(Fundamental Theorem for Interior Gravitation of Matter): Under the assumed regularity and continuity conditions, the most general possible isolagrange equations $\hat{E}^{\alpha\beta} = 0$ of Class I along an actual path P_o on a (3+1) dimensional isoriemannian space for the characterization of the interior gravitational problem of matter satisfying the properties: 1) Symmetry condition, $\hat{E}^{\alpha\beta} = \hat{E}^{\beta\alpha}$, 2) Contracted isobianchi identity, $\hat{E}^{\alpha\beta}{}_{\uparrow\beta} = 0$; and 3) The isofreud identity are given by*

$$\hat{E}^{\alpha\beta} = \alpha\hat{g}^{\frac{1}{2}}(\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{R} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{\Theta}) + \beta\hat{g}^{\alpha\beta} - \hat{g}^{\frac{1}{2}}\hat{D}^{\alpha\beta} = 0. \tag{3.20}$$

where $\hat{g}^{\frac{1}{2}} = (\det \hat{g})^{1/2}$; α and β are constants; and $\hat{D}^{\alpha\beta}$ is a source tensor. For $\alpha = 1$ and $\beta = 0$ the interior isogravitation field equations can be written

$$\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{\Theta} = \hat{t}^{\alpha\beta} - \hat{\tau}^{\alpha\beta} = \hat{U}^{\alpha}{}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}{}_{\beta}. \tag{3.21}$$

where $\hat{t}^{\alpha\beta}$ is a source tensor and $\hat{\tau}^{\alpha\beta}$ is a stress-energy tensor.

Note the appearance in Eqs. (3.21) of the isotopic isoscalar $\hat{\Theta}$ in the l.h.s and of source terms in the r.h.s., both originating from the isofreud identity. Additional studies not reported here for brevity (see [16b], Ch.9) have shown that the tensor $\hat{t}^{\alpha\beta}$ is nowhere null, of first order in magnitude and given by the electromagnetic tensor originating the mass of the elementary particles constituting the body considered. Therefore, *the isotopies permit the "identification" of the gravitational and electromagnetic field in the exterior problem* by eliminating the need of their "unification" (in the interior problem there are additional contributions from short range interactions).

Also, *the isotopic formulation of the interior problem permit a theory on the "origin" of the gravitational field*, rather than its "description", in which all mass terms are replaced by the fields which originate the same at the particle level. This establishes the need of a source also for the field equations in vacuum, exactly as requested by the exterior Freud identity $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R - \frac{1}{2}\delta^{\alpha}{}_{\beta}\Theta = U^{\alpha}{}_{\beta} + \partial_{\rho}V^{\alpha\rho}{}_{\beta} = t^{\alpha\beta} - \tau^{\alpha\beta}$, which therefore emerges as the exterior limit of interior equations (3.21).

Equivalently, it has been proved that the equations $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R = 0$ are incompatible with the primary, well established, electromagnetic origin of the mass of elementary particles, and are afflicted by other problematic aspects, such as: lack of weight in the tangent Minkowski space; problematic aspects in the relativistic limit of conservation laws; lack of uniqueness in the PPN approximation with consequential serious ambiguities in the interpretation of experimental data; and others. These problematic aspects are apparently resolved by the above isotopic theory on the "origin" of the gravitational field based on the Freud identity and its exterior limit in vacuum [16b].

We should finally note that the isotopic scalar is written in the l.h.s., rather than in the r.h.s. of the field equations because of geometric problems with Einstein's tensor $G^\alpha{}_\beta = R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R$ (lack of preservation of the Ricci Lemma under isotopies), which are resolved by the tensor $S^\alpha{}_\beta = R^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta R - \frac{1}{2}\delta^\alpha{}_\beta \Theta$ (see [16a], Ch.5 for details).

Theorem 1 applies, specifically and solely, for the characterization of matter. The corresponding representation of *antimatter* can be obtained via the *antiautomorphic map called "isoduality"* $\hat I > 0 \rightarrow \hat I^d = -\hat I < 0$ *of the entire formalism*, including the basic unit, numbers, spaces etc. This results in the *isodual isoriemannian geometry*, that of Class II, which is characterized by *isodual isoreal isofields* $\hat R^d(\hat n^d, +, \hat\times^d)$, *isodual isoriemannian spaces* $\hat{\Re}^d(\hat x^d, \hat g^d, \hat R^d)$, *isodual curvature tensor*, *isodual curvature scalar*, *isodual isotopic scalar* $\hat\Theta^d$ etc., whose study is omitted for brevity (see [16b] for all details).

The above isodual formulations essentially implies that *all quantities change sign under isoduality, including the energy momentum tensor, curvature etc., although these negative quantities are now referred to a negative unit*. In this way, the *gravitational* treatment of antimatter is brought in line with the *particle* treatment, that is, the characterization of antiparticle as historically discovered, that via *negative energy*, which now holds at all levels of study.

It should be noted that just the change of the sign of the charge is basically insufficient for a classical representation of antimatter in a mathematically and physically consistent way. In fact, the representation of antimatter requires an antiautomorphic map of the corresponding representation of matter *beginning with classical formulations*. This is established by *charge conjugation* which is precisely antiautomorphic, although only representable in Hilbert spaces. Isoduality is the only antiautomorphic map known to this author which first applies at the classical level and persists at the operator level where it becomes equivalent to change conjugation (see [16b] for details).

A first main result is that *antimatter-antimatter have a gravitational "attraction" equivalent to that of matter-matter systems* because a *negative* curvature referred to a *negative* unit $\hat I^d$ is fully equivalent to a *positive* curvature referred to the *positive* unit $\hat I$.

However the isodual theory predicts that *matter-antimatter and antimatter-matter systems experience a gravitational "repulsion"*, because in this case we have the projection of one system in the corresponding isodual space, resulting in *negative curvature referred to a positive unit, or vice versa, which represents repulsion* (see [16b] for details and proposed experimental verifications).

3.5. Isoparallel displacement and isogeodesics. An isovector field $\hat X^\beta$ on $\hat\Re$ is said to be transported by *isoparallel displacement* from a point $\hat m(\hat x)$ on a curve $\hat C$ on $\hat\Re$ to a neighboring point $\hat m'(\hat x + \hat d\hat x)$ on $\hat C$ if

$$\hat D\hat X^\beta = \hat d\hat X^\beta + \hat\Gamma_\alpha{}^\beta{}_\gamma \hat X^\alpha \hat d\hat x^\gamma \equiv 0. \tag{3.22}$$

or in integrated form

$$\hat X^\beta(\hat m') - \hat X^\beta(\hat m) = \int_{\hat m}^{\hat m'} \frac{\hat\partial \hat X^\beta}{\hat\partial \hat x^\alpha}\frac{\hat d\hat x^\alpha}{\hat d s}\hat d s, \tag{3.23}$$

where one should note the isotopic character of the integration. The isotopy of the conventional case [3] then yield the following:

Lemma 2. *Necessary and sufficient conditions for the existence of an isoparallel transport along a curve $\hat C$ on a (3+1)-dimensional isoriemannian space are that all the following conditions are identically verified along $\hat C$*

$$\hat R_\alpha{}^\beta{}_{\gamma\delta}\hat X^\alpha = 0, \qquad \beta, \gamma, \delta = 1, 2, 3, 4. \tag{3.24}$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. However, it is easy to see that the projection of the isoparallel transport in the conventional space $\Re$ is structurally different of the conventional parallel transport. In particular, if the latter is represented by an arrow, one would note a twisting action as occurring in the reality of motion within physical media, which is evidently absent in the exterior case.

Along similar lines, we say that a smooth path $\hat x_\alpha$ on $\hat\Re$ with isotangent $\hat v_\alpha = \hat d\hat x_\alpha/\hat d\hat s$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat D\hat x_\beta}{\hat D\hat s} = \frac{\hat d\hat v_\beta}{\hat d\hat s} + \hat\Gamma_{\alpha\beta\gamma}\frac{\hat d\hat x^\alpha}{\hat d\hat s}\frac{\hat d\hat x^\gamma}{\hat d\hat s} = 0. \tag{3.25}$$

It is easy to prove the following:

Lemma 3. *The isogeodesics of an isoriemannian space $\hat{\Re}$ are the curves verifying the isovariational principle*

$$\hat{\delta}\int [\hat{g}_{\alpha\beta}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \dots) d\hat{x}^{\alpha} d\hat{x}^{\beta}]^{\frac{1}{2}} = 0. \tag{3.26}$$

where again isointegration is understood.

Finally, we point out the property which is inherent in the notion of isotopies as realized in this paper according to which *geodesic trajectories in ordinary space coincide with the corresponding isogeodesic trajectories in isospace.* For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the *isocircle* [16b], even though its projection in original is an ellipse. This is due to the fact the semiaxes of the circle $1_k = +1, k = x, y$, are deformed under isotopies into expressions $1_k = +1 \to n_k^{-2}$ characterizing an ellipse *in conventional space.* But jointly the corresponding units along the x and y are deformed by the *inverse* amounts, $1_k \to n_k^2$, thus preserving the perfect circle *in isospace.* The same preservation in isopace occurs for all other curves.

The differences between a geodesic and an isogeodesic therefore emerge only when projecting the latter in the space of the former.

An empirical but conceptually effective rule is that *interior physical media "disappear" under their isoriemannian geometrization,* in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Riemannian space) are turned into isogeodesics in isospace with the shape of the geodesics in the absence of resistive forces.

In summary, a basic question raised in this paper is: *why use in interior problems the Riemannian geometry with metric $g(x)$ when the same axioms permit metrics $\hat{g}(\hat{x}, \hat{v}, \hat{a}, \dots)$ with an unrestricted functional dependence in the velocities and other variables?* Equivalently, we can ask the question: *why use the simplest possible realization of the Riemannian axioms when a structurally more general realization exists for a more adequate representation of interior problems?* Still equivalently, another question raised in this paper is: *why use geometries implying the sole constancy of the speed of light when the geometrization of the locally varying speed of light as occurring in the physical reality is available?*

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FIELD CONES AND POSITIVE OPERATORS ON REGULAR VECTOR BUNDLES

BY

DAN I. PAPUC

Introduction. This is a survey paper concerning the results obtained in the research (made between 1988–1995) of a structure formed by a regular vector bundle (E, p, M) endowed with a field K of cones. This structure is denoted by $[(E,p,M);K]$. The topics considered are:

— the geometry of an arbitrary structure $[(E,p,M);K]$ (the geometry of a locale fibre E_x, the global geometry of (E,p,M), the geometry of the set of global sections of (E,p,M)).

— the geometry of the set of operators from a structure $[(E,p,M);K_1]$ to a structure $[(E',p',M);K_2]$ (the set of positive operators and structure $(\Omega/\Omega_0; \boldsymbol{K})$; ordering, norms and distances for this last structure),

— the geometry of natural lifted structure of a tensor bundle associated to a structure $[(E,p,M):K]$,

— some applications of these results.

A regular bundle *is a vector bundle* (E,p,M) *for which* E *and* M *are real topological paracompact connected without boundary manifolds and* $\dim M = n, \dim E = n+m$.

A field of cones *is a map* K,

$$K : x \in M \to K(x) \subset E_x \subset E, \ (E_x = p^{-1}(x)).$$

for which are satisfied the following two axioms:

A_1. $K(x) \subset E_x$, $(\forall x \in M)$ *is a convex pointed closed with interior points cone (in the topological vector space* E_x *).*

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