ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLII, Supliment, s.I a, Matematică, 1996

THE KLEIN IMAGES OF A CLASS OF LINE COMPLEXES FOR WHICH TWO FAMILIES OF PRINCIPAL SURFACES CONSIST OF CYLINDERS

BY

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Introduction. We consider a 3-dimensional elliptic space S_3 referred to a moving frame as an orthonormal polar-tetrahedron $T(A_0, A_1, A_2, A_3)$ of four linearly independent analytic points $A_K, k = 0,1,2,3$. An infinitesimal displacement of such a frame is determined by the matrix equation:

$$dA = \omega A, \tag{1}$$

where $A = (A_0, A_1, A_2, A_3)$ is column matrix and $\omega = (\omega_k^j)$, $(k, j = 0,1,2,3)$ is a skew-symmetric square matrix of rank four, which is called Cartan's matrix corresponding to the elliptic space S_3 [3]. The elements ω_k^j are the invariant one forms (Pfaff's differential forms) of the elliptic group of transformation, whose structural equations have the form:

$$D\omega_i^j = \omega_i^\alpha \wedge \omega_\alpha^j - \omega_0^i \wedge \omega_0^j, \qquad (\alpha, i, j = 0,1,2,3). \tag{2}$$

where D denotes the exterior differential operator and $\wedge$ denotes the exterior product between the differential forms.

The Klein mapping $K: Gr(1,3) \longrightarrow p^4 \subset S_5$, is from the set of lines $Gr(1,3)$ (Grassmann manifold) of the space S_3 onto the points of the four dimensional Klein hyperquadric p^4 (K-quadric) as the absolutum of the five-dimensional elliptic space S_5. It is well-known that the Grassmann manifold $Gr(1,3)$ has dimension four [4]. The mapping K which maps the set of all lines $Gr(1,3)$ bijectively onto the points of the K-quadric $p^4 \subset S_5$ (the Klein

Communicated at the 25-th National Conference on Geometry and Topology, September 18-23, 1995, Iaşi, România

space) is called the Klein map. In virtue of this definition the line A_iA_j corresponds a point $\ell = [A_i, A_j]$ which is called the Klein point (K-point) belongs to $p^4 \subset S_5$, where $[A_i, A_j]$ denotes the Grassmann product of the two points A_i, A_j. In this case ℓ is determined by six-tuples $(\ell_1, \ell_2, \dots, \ell_6)$ such that ℓ satisfies the Plucker condition $\Omega(\ell, \ell) = \ell_1.\ell_4 + \ell_2\ell_5 + \ell_3\ell_6 = 0$. From this definition, it follows that the line manifolds immersed in S_3 can be studied as point submanifolds on $p^4 \subset S_5$ [4].

In order to study the geometrical properties of the K-images of line manifolds immersed in S_3 on the K-quadric p^4, it is necessary to construct the moving frame attached to any point on $p^4 \subset S_5$ [5]. We take the rectangle hexahedron formed by the six edges of the frame of the frame $T(A_0, A_1, A_2, A_3)$ as a moving frame conjugate to any K-point on p^4. We denote this frame by $H(P_1, P_2, P_3; \overline{P}_1, \overline{P}_2, \overline{P}_3)$ where the K-points $P_i, \overline{P}_i, i = 1, 2, 3$ satisfy the following conditions:

$$\Omega(P_i, \overline{P}_j) = \delta_{ij}, \quad < P_i, \overline{P}_j >= 0, \quad j = 1, 2, 3,$$
$$< P_i, P_j >= \delta_{ij}, \quad < \overline{P}_i, \overline{P}_j >= \delta_{ij}$$

and

$$P_1 = [A_0, A_1], \quad P_2 = [A_0, A_2], \quad P_3 = [A_0, A_3],$$
$$\overline{P}_1 = [A_2, A_3], \quad \overline{P}_2 = [A_3, A_1], \quad \overline{P}_3 = [A_1, A_2].$$

Such a frame is called a normalized polar hexahedron and is denoted by $H(P_i, \overline{P}_i)$, $i = 1, 2, 3$. The infinitesimal displacements of the frame $H(P_i, \overline{P}_i)$ are given by

$$(3) \qquad \begin{bmatrix} dP_1 \\ dP_2 \\ dP_3 \\ d\overline{P}_1 \\ d\overline{P}_2 \\ d\overline{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_1^2 & \omega_1^3 & 0 & \omega_0^3 & -\omega_0^2 \\ -\omega_1^2 & 0 & \omega_2^3 & -\omega_0^3 & 0 & \omega_0^1 \\ -\omega_1^3 & -\omega_2^3 & 0 & \omega_0^2 & -\omega_0^1 & 0 \\ 0 & \omega_0^3 & -\omega_0^2 & 0 & \omega_1^2 & \omega_1^3 \\ -\omega_0^3 & 0 & \omega_0^1 & -\omega_1^2 & 0 & \omega_2^3 \\ \omega_0^2 & -\omega_0^1 & 0 & -\omega_1^3 & -\omega_2^3 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \overline{P}_1 \\ \overline{P}_2 \\ \overline{P}_3 \end{bmatrix}$$

1. Line complexes immersed in the space S_3. We take the generator ℓ of the Grassmann manifold $Gr(1,3)$ as the edge A_0A_3 of the frame $T(A_0, A_1, A_2, A_3)$. The invariance conditions of the line $\ell = A_0A_3$ under the transformation group of S_3 are $\omega_\gamma^\nu = 0$, $\gamma = 0, 3, \nu = 1, 2$. The line complex defined as a 3-dimensional submanifold of $Gr(1,3)$ is denoted by $Gr(1,3{:}3)$ i.e., any line of $Gr(1,3{:}3)$ depends only upon three principal forms say $\omega_0^2, \omega_0^1, \omega_3^2$. The differential equations which characterize this line complex related to a canonical moving frame (the planes $A_0A_3A_1$, $A_0A_3A_2$

are coincident with the planes which correspond to the points A_0, A_3 in the normal correlation respectively) are given by

$$(4) \qquad \begin{bmatrix} \omega_3^1 \\ dK \\ \omega_0^3 - K\omega_1^2 \\ K\omega_0^3 - \omega_1^2 \end{bmatrix} = \begin{bmatrix} K & 0 & 0 \\ P & \alpha & \beta \\ \alpha & q & \gamma \\ \beta & \gamma & r \end{bmatrix} \begin{bmatrix} \omega_0^2 \\ \omega_0^1 \\ \omega_3^2 \end{bmatrix}, \qquad 1 - K^2 \neq 0$$

where K determine the curvature of the line complex and $\alpha, \beta, \gamma, \mathbf{p}, \mathbf{q}, \mathbf{r}$ are invariants in the first and second order contact elements of the ray ℓ of the line complex respectively [6].

Now, we consider the line complex (4) for which two families of principal surfaces consist of cylinders. This complex has been studied in [7], and characterized by the following system of Pfaff's equations

$$(5) \qquad \begin{bmatrix} \omega_3^1 \\ dK \\ \omega_0^3 - K\omega_1^2 \\ K\omega_0^3 - \omega_1^2 \end{bmatrix} = \begin{bmatrix} K & 0 & 0 \\ P & 0 & 0 \\ 0 & q & \gamma \\ 0 & \gamma & q \end{bmatrix} \begin{bmatrix} \omega_0^2 \\ \omega_0^1 \\ \omega_3^2 \end{bmatrix}, \qquad 1 - K^2 \neq 0$$

where $\alpha = \beta = 0$, $\mathbf{q} = \mathbf{r}$

Ahmed S.A. in [7], studied the existence of this complex. He proved the following theorem

Theorem 1. *The complex of lines in S_3, whose two families of principal surfaces coincide with two families of cylinders in the sense of Clifford exists within three arbitrary functions of one variable.*

The following theorems study the geometrical properties of the complex under discussion.

Theorem 2. *One of the principal surfaces, which corresponds the characteristic value $S = \frac{-p}{2K}$, is the coordinate surface $\omega_0^1 = 0$, $\omega_3^2 = 0$ and the others coincide with the surfaces*

$$\omega_0^2 = 0, \quad \omega_0^1 - \omega_3^2 = 0, \qquad \omega_0^2 = 0, \quad \omega_0^1 + \omega_3^2 = 0.$$

Theorem 3. *If the ray A_0A_3 describes a congruence of lines $\omega_0^1 + \omega_3^2 = 0$ then the ray A_0A_1 describes a congruence with imaginary focal points harmonic devise the vertices A_0, A_1.*

Theorem 4. *If the ray A_0A_3 describes a congruence of lines $\omega_0^1 - \omega_3^2 = 0$ then the ray A_0A_1 describes a congruence with real focal points harmonic devise the vertices A_0, A_1.*

In what follows we state the following theorem which study the geometrical construction.

Theorem 5. *Take one parametric family of surfaces with null Gaussian curvature cut orthogonally rays of normal congruence. For every surface of the family construct its polar conjugate surface. Take these two surfaces as focal surfaces of some congruence, one parametric family of these congruences construct the complex characterized by $\alpha = \beta = 0$,* $\mathbf{q} = \mathbf{r}$.

We refer to [7] for the proofs and more detailed discussion on the previous theorems.

2. The K-images of line complexes on the absolutum of the K-space S_3. It is well known that the K-image of a line complex is a 3-dimensional point manifold (3-dimensional surface) on the absolutum of the K-space S_5 [4]. We denote the K-image of the line complex (5) by $\alpha_3(P_3)$ which described by the vertex P_3 of the frame $H(P_i, \overline{P}_i)$ of S_5 where

$$P_3 = [A_0A_3] \qquad dP_3 = \omega_3^1 P_1 + \omega_3^2 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2$$

But $\omega_3^1 = K\omega_0^2$. Then

$$dP_3 = \omega_0^2(\overline{P}_1 + KP_1)\omega_3^2 P_2 - \omega_0^1 \overline{P}_2 \tag{6}$$

The second-osculating K-space denoted by $T^2_{P_3}(\alpha_3)$ of the 3-dimensional K-manifold α_3 is obtained from

$$\begin{aligned}
d^2P_3 &= \omega_0^2(d\overline{P}_1 + KdP_1 + dKP_1) + (D\omega_0^2)(\overline{P}_1 + KP_1) + \\
&\quad + \omega_3^2 dP_2 + (D\omega_3^2)P_2 - \omega_0^1 d\overline{P}_2 - (D\omega_0^1)\overline{P}_2 \\
&= \omega_0^2[(\omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \overline{P}_3) + K(\omega_1^2 P_2 + \omega_1^3 P_3 + \\
&\quad + \omega_0^3 \overline{P}_2 - \omega_0^2 \overline{P}_3) + (p\omega_0^2)P_1] + (\overline{P}_1 + KP_1)(D\omega_0^2) + \\
&\quad + \omega_3^2(-\omega_1^2 P_1 + \omega_2^3 P_3 - \omega_0^3 \overline{P}_1 + \omega_0^1 \overline{P}_3) + (D\omega_3^2)P_2 - \\
&\quad - \omega_0^1(-\omega_0^3 P_1 + \omega_0^1 P_3 - \omega_1^2 \overline{P}_1 - \omega_2^3 \overline{P}_3) - (D\omega_0^1)\overline{P}_2 \\
&= (p(\omega_0^2)^2 - \omega_1^2\omega_3^2 + \omega_0^1\omega_0^3)P_1 + (-\omega_3^2\omega_0^3 + \omega_0^1\omega_1^2)\overline{P}_1 + \\
&\quad + (\omega_0^2\omega_0^3 + K\omega_1^2\omega_0^2 + D\omega_3^2)P_2 + ((\omega_0^2\omega_1^2 + K\omega_0^2\omega_0^3 - \\
&\quad - D\omega_0^1)\overline{P}_2 + (-(\omega_0^2)^2 + K\omega_0^2\omega_1^3 - (\omega_3^2)^2 - (\omega_0^1)^2)P_3 + \\
&\quad + (\omega_0^2\omega_1^3 - K(\omega_0^2)^2 + \omega_3^2\omega_0^1 - \omega_0^1\omega_2^3)\overline{P}_3 + (D\omega_0^2)(\overline{P}_1 + KP_1)
\end{aligned}$$

Then

$$\begin{aligned}
d^2P_3 &= (p(\omega_0^2)^2 - \omega_1^2\omega_3^2 + \omega_0^1\omega_0^3 + K\omega_3^2\omega_0^3 - K\omega_0^1\omega_1^2)P_1 + \\
&\quad + (-2K(\omega_0^2)^2 + 2\omega_3^2\omega_0^1)\overline{P}_3 \bmod P_2, P_3, \overline{P}_2, \overline{P}_1 + KP_1)
\end{aligned} \tag{7}$$

From (7) it follows that the normal space $N_{P_3}(\alpha_3)$ contains the K-points $P_1, \overline{P}_3$. Therefore, the first and the 2nd fundamental forms of α_3 denoted by I and II_1, II_2:

$$I = \langle dP_3, dP_3 \rangle, \quad II_1 = \langle d^2P_3, P_1 \rangle, \quad II_2 = \langle d^2P_3, \overline{P}_3 \rangle \tag{8}$$

are as follows:

$$\begin{aligned}
I &= \langle dP_3, dP_3 \rangle = (\omega_0^1)^2 + (1 + K^2)(\omega_0^2)^2 + (\omega_3^2)^2. \\
II_1 &= p(\omega_0^2)^2 + q(\omega_0^1)^2 + 2\gamma\omega_0^1\omega_3^2 + q(\omega_3^2)^2. \\
II_2 &= -2K(\omega_0^2)^2 + 2\omega_0^1\omega_3^2.
\end{aligned}$$

The Gaussian curvature K_3 and the mean curvature vector $\underline{H}_3$ are defined by [4]

$$\begin{cases} K_3 = 1 + \dfrac{\text{Det } II_1 + \text{Det } II_2}{\text{Det } I} \\ \underline{H}_3 = (\text{trace } II_1)P_1 + (\text{trace } II_2)\overline{P}_3 \end{cases} \tag{9}$$

Thus we have the following results:

$$\begin{cases} \text{Gaussian curvature } K_3 = [p(q^2 - \gamma^2) + (1 + K)^2]/(1 + K^2), \\ \text{vector mean curvature } \underline{H}_3 = (p + 2q)P_1 + (-2K)\overline{P}_3. \end{cases} \tag{10}$$

Thus we have proved the following theorem:

Theorem 6. *The Klein image of the line complex (5) is a 3-dimensional surface immersed in $P^4 \subset S_5$ which is determined by its Gaussian curvature and mean curvature vector given by (10).*

We now study the geometrical properties of the line complex (5) on the quadric.

3. The geometrical properties of the line complex (5) on the quadric. The equation $\omega_0^1 - \omega_3^2 = 0$ determines a line congruence. For this line congruence we have

$$dP_3 = \omega_0^2(\overline{P}_1 + KP_1) + (\omega_0^1(P_2 - \overline{P}_2). \tag{11}$$

We note that the K-point P_3 describes a 2-dimensional surface on P^4 whose 2-dimensional tangent K-space $T_{P_3}(\alpha_2)$ has the tangential coordinates $[P_3, \overline{P}_1 + kP_1, P_2 - \overline{P}_2]$. We have

$$\begin{aligned} d[P_3, \overline{P}_1 + kP_1, P_2 - \overline{P}_2] = \\ &= [dP_3, \overline{P}_1 + kP_1, P_2 - \overline{P}_2] + [P_3, d(\overline{P}_1 + kP_1), P_2 - \overline{P}_2] + \\ &+ [P_3, \overline{P}_1 + kP_1, d(P_2 - \overline{P}_2)] = [\,\omega_0^2(\overline{P}_1 + kP_1) + \omega_0^1(P_2 - \overline{P}_2), \\ &\overline{P}_1 + kP_1, P_2 - \overline{P}_2] + [P_3, (\omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \overline{P}_3) + \\ &+ K(\omega_1^2 P_2 + \omega_1^3 P_3 + \omega_0^3 \overline{P}_2 - \omega_0^2 \overline{P}_3) + (p\omega_0^2)P_1, P_2 - \overline{P}_2] + \\ &+ [P_3, \overline{P}_1 + kP_1, (-\omega_1^2 P_1 + \omega_2^3 P_3 - \omega_0^3 \overline{P}_1 + \omega_0^1 \overline{P}_3) - (-\omega_0^3 P_1 + \omega_0^1 P_3 - \\ &- \omega_1^2 \overline{P}_1 + \omega_2^3 \overline{P}_3)] = [P_3, (\omega_0^3 + K\omega_1^2)P_2 + (\omega_1^2 + K\omega_0^3)\overline{P}_2 + \\ &+ (-\omega_0^2 + K\omega_1^3)P_3 + (\omega_1^3 - K\omega_0^2)\overline{P}_3 + (p\omega_0^2)P_1, P_2 - \overline{P}_2] + \\ &+ [P_3, \overline{P}_1 + KP_1, (\omega_0^3 - \omega_1^2)P_1 + (\omega_1^2 - \omega_0^3) + \\ &+ (\omega_2^3 - \omega_0^1)P_3 + (\omega_0^1 - \omega_2^3)\overline{P}_3] \end{aligned}$$

But $\omega_1^2 = -\omega_0^3$

Then

$$\begin{aligned} d[P_3, \overline{P}_1 + kP_1, P_2, \overline{P}_2] = \\ &= [P_3, \omega_1^2(K-1)(P_2 - \overline{P}_2) - \omega_0^2((1+K^2)P_3 - \\ &- 2K\omega_0^2 \overline{P}_3 + (p\omega_0^2)P_1, P_2 - \overline{P}_2] + \\ &+ [P_3, \omega P_1 + KP_1, 2\omega_1^2(\overline{P}_1 - P_1) + \\ &+ 2\omega_0^1(\overline{P}_3 - P_3)] \neq 0 \end{aligned}$$

This means that $[P_3, \overline{P}_1 + kP_1, P_2 - \overline{P}_2]$ is not stable.

The second-osculating K-space $T^2_{P_3}(\alpha_2)$ of the 2-dimensional K-surface

α_2 is obtained from

$$\begin{aligned} d^2P_3 = \; & \omega_0^2(d\overline{P}_1 + KdP_1 + dKP_1) + (D\omega_0^2)(\overline{P}_1 + KP_1) + \\ & + \omega_0^1(dP_2 - d\overline{P}_2) + (D\omega_0^1)(P_2 - \overline{P}_2) = \\ & = \omega_0^2[(\omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \omega P_3) + K(\omega_1^2 P_2 + \omega_1^3 P_3 + \\ & + \omega_0^3 \overline{P}_2 - \omega_0^2 \overline{P}_3) + (p\omega_0^2)P_1] + (D\omega_0^2)9\overline{P}_1 + KP_1) + \\ & + \omega_0^1[(-\omega_1^2 P_1 + \omega_2^3 P_3 - \omega_0^3 \overline{P}_1 + \omega_0^1 \overline{P}_3) - \\ & - (-\omega_0^3 P_1 + \omega_0^1 P_3 - \omega_1^2 \overline{P}_1 + \omega_2^3 \overline{P}_3)] + (D\omega_0^1)(P_2 - \overline{P}_2) = \\ & = (p(\omega_0^2)^2 - \omega_0^1\omega_1^2 + \omega_0^1\omega_0^3)P_1 + (-\omega_0^1\omega_0^3 + \omega_0^1\omega_1^2)\omega P_1 + (\omega_0^2\omega_0^3 + \\ & + K\omega_0^2\omega_1^2)P - 2 + (\omega_0^2\omega_1^2 + K\omega_0^2\omega_0^3)\overline{P}_2 + (-(\omega_0^2)^2 + k\omega_0^2\omega_1^3 + \\ & + \omega_0^1\omega_2^3 - (\omega_0^1)^2 P_3 + (\omega_0^2\omega_1^3 - K(\omega_0^2)^2 + 9\omega_0^1)^2 - \omega_0^1\omega_2^3)\overline{P}_3 + \\ & + (D\omega_0^2)(\overline{P}_1 + KP_1) + (D\omega_0^1)(P_2\overline{P}_2) \end{aligned}$$

Then

$$\begin{aligned} d^2P_3 = \; & (p(\omega_0^2)^2 - (1+K)\omega_0^1\omega_1^2 + (1+K)\omega_0^1\omega_0^3)P_1 + \\ & + (-2K(\omega_0^2)^2 + 2(\omega_0^1)^2)\overline{P}_3 \\ & (\mathrm{mod}\ P-3, \overline{P}_1 + KP_1, P_2 - \overline{P}_2). \end{aligned} \tag{12}$$

$$I = \langle dP_3, dP_3 \rangle = 2(\omega_0^1)^2 + (1+K^2)(\omega_0^2)^2.$$

$$\begin{aligned} II = \; & \langle d^2P_3, P_1 \rangle = p(\omega_0^2)^2 - (1+K)\omega_0^1\omega_1^2 + (1+K)\omega_0^1\omega_0^3 = \\ & = p(\omega_0^2)^2 + (1+K)(\tfrac{q+\gamma}{1+K})(\omega_0^1)^2 + (1+K)(\tfrac{q+\gamma}{1+K})(\omega_0^1)^2 = \\ & = p(\omega_0^2)^2 + 2(q+\gamma)(\omega_0^1)^2. \end{aligned}$$

$$II_2 = \langle d^2P_3, \overline{P}_3 \rangle = -2K(\omega_0^2)^2 + 2(\omega_0^1)^2.$$

$$\begin{cases} \text{Gaussian curvature } K_2 = [p(q+\gamma) + (1-K)^2]/(1+K^2). \\ \text{vector mean curvature } \underline{H}_2 = [p + 2(q+\gamma)]P_1 + [2(1-K)]\overline{P}_3. \end{cases} \tag{13}$$

Thus w have the following theorem:

T eorem 7. *The Klein image of the line congruence $\omega_0^1 - \omega_3^2 = 0$ is a 2- di ensional surface which is determined by its Gaussian curvature and vector ean curvature given by (13).*

e equation $\omega_0^1 = 0$, $\omega_3^2 = 0$ determines a principal surface. It is well k wn that the K-image of a ruled surface is a curve on the absolutum of the -space S_5.

For this surface we have

$$dP_3 = \omega_3^1 P_1 + \omega_3^2 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2 = \tag{14}$$
$$= (K\omega_0^2)P_1 + \omega_0^2 \overline{P}_1 = \omega_0^2(\overline{P}_1 + KP_1).$$

Note that dP_3 depends only on one principal form, then P_3 describe a curve on the absolute.

$$\begin{aligned} d[P_3, \quad \overline{P}_1 + KP_1] &= [dP_3, \overline{P}_1 + KP_1] + [P_3, d(\overline{P}_1 + KP_1)] = \\ &= [\omega_0^2(\overline{P}_1 + KP_1), \overline{P}_1 + KP_1] + [P_3, d\overline{P}_1 + KdP_1 + dKP_1] = \\ &= [P_3, (\omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \overline{P}_3) + \\ &+ K(\omega_1^2 P_2 + \omega_1^3 P_3 + \omega_0^3 \overline{P}_2 - \omega_0^2 \overline{P}_3) + (p\omega_0^2)P_1] \end{aligned}$$

Then

$$\begin{aligned} d[P_3, \overline{P}_1 + KP_1] = \; & [P_3, (p\omega_0^2)P_1 + (\omega_0^3 + K\omega_1^2)P_2 + (-\omega_0^2 + K\omega_1^3)P_3 + \\ & +(\omega_1^2 + K\omega_0^3)\overline{P}_2 + (\omega_1^3 - K\omega_0^2)\overline{P}_3] = \\ & = p\omega_0^2[P_3, P_1] + (\omega_0^3 + K\omega_1^2)[P_3, P_2] + \\ & +(-\omega_0^2 + K\omega_1^3)[P_3, P_3] + (\omega_1^2 + K\omega_0^3)[P_3, \overline{P}_2] + \\ & +(\omega_1^3 - K\omega_0^2)[P_3, \overline{P}_3] \end{aligned}$$

But $\omega_0^1 = 0, \;\; \omega_3^2 = 0 \longrightarrow \omega_1^2 = \omega_0^3 = 0$
Then $d[P_3, \overline{P}_1 + KP_1] = p\omega_0^2[P_3, P_1] - 2K\omega_0^2[P_3, \overline{P}_3] \neq 0$
Then the curve which describe the Klein image of the ruled surface $\omega_0^1 = 0, \;\; \omega_3^2 = 0$ on the absolutum of S_5 is not a straight line. The tangent plane $[P_3, \overline{P}_1 + kP_1]$ for the ruled surface $\omega_0^1 = 0, \;\; \omega_3^2 = 0$ is not constant. Then this ruled surface is not a plane line bundle. This means that its Klein image is not a generator of the absolutum $P_2^4 \subset S_5^K$.

$$\begin{aligned} d^2P_3 = \; & \omega_0^2(d\overline{P}_1 + KdP_1 + dKP_1) + (D\omega_0^2)(\overline{P}_1 + KP_1) = \\ & = \omega_0^2[(\omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \overline{P}_3) + \\ & +K(\omega_1^2 P_2 + \omega_1^3 P_3 + \omega_0^3 \overline{P}_2 - \omega_0^2 \overline{P}_3) + \\ & +(p\omega_0^2)P_1] + (D\omega_0^2)(\overline{P}_1 + KP_1) = (p(\omega_0^2)^2)P_1 + \\ & +(\omega_0^2\omega_0^3 + K\omega_0^2\omega_1^2)P_2 + (\omega_0^2\omega_1^2 + K\omega_0^2\omega_0^3)\overline{P}_2 + \\ & +(-(\omega_0^2)^2 + K\omega_0^2\omega_1^3)P_3 + (\omega_0^2\omega_1^3 - K(\omega_0^2)^2)\overline{P}_3 + \\ & +(D\omega_0^2)(\overline{P}_1 KP_1) \end{aligned}$$

But $\omega_1^2 = 0, \;\; \omega_0^3 = 0, \;\; \omega_3^1 = K\omega_0^2$
Then

$$d^2P_3 = (p(\omega_0^2)^2)P_1 + (-2K(\omega_0^2)^2)\overline{P}_3 \quad \mathrm{mod}\ (P_3, \overline{P}_1 + KP_1) \tag{15}$$

The geodesic curvature of the curve $(\omega_3^1 = K\omega_0^2, \;\; \omega_0^3 = 0, \;\; \omega_1^2 = 0)$ is denoted by K_9, where

$$\begin{aligned} K_9 = \; & [P_3, dP_3, d^2P_3, (\overline{P}_1 + KP_1), \overline{P}_3] = \\ & = [P_3, \omega_0^2(\overline{P}_1 + KP_1), (p(\omega_0^2)^2)P_1 + \\ & +(-2K(\omega_0^2)^2)\overline{P}_3, (\overline{P}_1 + KP_1), \overline{P}_3] = 0 \end{aligned}$$

Thus the curve is geodesic.

Theorem 8. *The Klein image of a ruled surface $\omega_0^1 = 0, \;\; \omega_3^2 = 0$ is a geodesic curve on the absolutum P_2^4 of the K-space S_5^k.*

This means that the ruled surface $\omega_0^1 = 0, \;\; \omega_3^2 = 0$ is of type helicoid.

The equation $\omega_0^2 = 0, \;\; \omega_0^1 - \omega_3^2 = 0$ determine a principal surface. For this surface we have:

$$\begin{aligned} dP_3 \; & = \omega_3^1 P_1 + \omega_3^2 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2 = \\ & = K\omega_0^2 P_1 + \omega_0^1 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2 = \\ & = \omega_0^1(P_2 - \overline{P}_2) \end{aligned} \tag{16}$$

Then

$$\begin{aligned} d[P_3, P_2 - \overline{P}_2] = \; & \tfrac{2(\gamma+q)}{1+K}\omega_0^1[P_3, P_1] - \\ & -\tfrac{2(\gamma+q)}{1+K}\omega_0^1[P_3, \overline{P}_1] + 2\omega_0^1[P_3, \overline{P}_3] \neq 0 \end{aligned}$$

Thus the curve which describe the Klein image of the cylinder $\omega_0^2 = 0, \;\; \omega_0^1 - \omega_3^2 = 0$ on the absolutum of S_5 is not a straight line.
For the cylinder $\omega_0^2 = 0, \;\; \omega_0^1 - \omega_3^2 = 0$ we have

$$dA_0 = \omega_0^1 A_1 + \omega_0^2 A_2 + \omega_0^3 A_3 = \omega_0^1 A_1 + \omega_0^3 A_3$$

$$dA_3 = \omega_3^0 A_0 + \omega_3^1 A_1 + \omega_3^2 A_2 = \omega_3^0 + \omega_3^2 A_2$$

Then

$$\begin{aligned} d(A_0 + iA_3) \; & = \omega_0^1(A_1 + iA_2) - i\omega_0^3(A_0 + iA_3) = \\ & = \omega_0^1(A_1 + iA_2) \quad (\mathrm{mod}\ (A_0 + iA_3)). \end{aligned} \tag{17}$$

$$d(A_0 - iA_3) = (\omega_0^1 A_1 + \omega_0^3 A_3) - i(\omega_3^0 A_0 + \omega_3^2 A_2) = \\ = \omega_0^1(A_1 - iA_2) \quad (\text{mod } (A_0 - iA_3)). \tag{18}$$

Therefore from (17), (18) we obtain the following $dL_1 = 0$ (mod L_1), $dL_2 = 0$ (mod L_2), where $L_1 = [A_0 + iA_3, A_1 + iA_2]$, $L_2 = [A_0 - iA_3, A_1 - iA_2]$, are two generators of the absolutum. Therefore L_1 and L_2 are stable and $(A_0 + iA_3)$, $(A_0 - iA_3)$ move along them, respectively.

The Klein image of the moving line A_0A_3 is a moving point on the absolutum while the Klein image of the generator L_1 is a fixed point. Since $\Omega(A_0A_3, L_1) = 0$ then the fixed point is a pole of the curve which is described by the point $A_0 + iA_3$ on the absolutum. The curve together with the pole are in polar conjugate.

Similarly the Klein image of the generator L_2 is a fixed point on the absolutum, this fixed point together with a curve described by a moving point on the absolutum are in polar conjugate. We have

$$\begin{aligned} L_1 &= [A_0 + iA_3, A_1 + iA_2] \\ dL_1 &= [d(A_0 + iA_3), A_1 + iA_2] + [A_0 + iA_3, d(A_1 + iA_2)] = \\ &= [\omega_0^1 A_1 + \omega_0^2 A_2 + \omega_0^3 A_3 + i\omega_3^0 A_0 + i\omega_3^1 A_1 + i\omega_3^2 A_2, \\ &\quad A_1 + iA_2] + [A_0 + iA_3, \omega_1^0 A_0 + \omega_1^2 A_2 + \omega_1^3 A_3 + i\omega_2^0 A_0 + \\ &\quad + i\omega_2^1 A_1 + i\omega_2^3 A_3] \end{aligned}$$

If $\omega_0^2 \neq 0$, $\omega_0^1 - \omega_3^2 = 0$ implies $\omega_0^3 = -\omega_1^2$

Then

$$\begin{aligned} dL_1 &= i\omega_3^0[A_0, A_1] - \omega_3^0[A_0, A_2] - \omega_3^1[A_1, A_2] + \omega_0^1[A_2, A_1] + \\ &\quad + \omega_0^3[A_3, A_1] + i\omega_0^3[A_3, A_2] + i\omega_0^3[A_0, A_1] - \omega_0^3[A_0, A_2] + \\ &\quad + \omega_1^3[A_0, A_3] + \omega_0^2[A_3, A_0] - \omega_0^3[A_3, A_1] - i\omega_0^3[A_3, A_2] = \\ &= \omega_0^2\{-K[A_1, A_2] + [A_2, A_1] - K[A_0, A_3] - [A_0, A_3]\} \end{aligned}$$

and if L_1 describes a ruled surface its Klein image is a curve on the quadric, i.e., the image of L_1 which is a pole moves along a curve on the absolute. We call this curve pole curve (polled).

Now we can state the theorem which gives the geometrical construction of the Klein image of the complex (5) on the absolute.

Theorem 9. *We take a curve on the absolutum considered as a pole curve (polled), each point of this curve determines a curve which is in polar conjugate to this point, all such curves represent the Klein image of the congruence and the Klein image of the complex under consideration represents as a one parametric family of the Klein image of this congruence.*

Proof. We take the rectangle hexahedron formed by the six edges of the frame $T(A_0, A_1, A_2, A_3)$ as a moving frame conjugate to any K-point on p^1. We denote this frame by $H(P_1, P_2, P_3, \overline{P}_1, \overline{P}_2, \overline{P}_3)$, where

$$\begin{aligned} P_1 &= [A_0, A_1], \quad P_2 = [A_0, A_2], \quad P_3 = [A_0, A_3], \\ \overline{P}_1 &= [A_2, A_3], \quad \overline{P}_2 = [A_3, A_1], \quad \overline{P}_3 = [A_1, A_2]. \end{aligned}$$

Consider a pole curve on the absolute. On this curve we take a moving point say P_3. Since P_3 describes a curve:

$$dP_3 = \omega_3^1 P_1 + \omega_3^2 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2 \tag{19}$$

must be depends on one principal form only say ω_0^1 then we have

$$\omega_0^2 = \lambda_1 \omega_0^1, \quad \omega_3^2 = \lambda_2 \omega_0^1, \quad \omega_3^1 = \lambda_3 \omega_0^1.$$

But dP_3 describe the Klein image of a cylinder. This image is given by

$$dP_3 = \omega_0^1(P_2 - \overline{P}_2). \tag{20}$$

Comparing the two equation (19), (20) we have $\omega_0^2 = \omega_3^1 = 0$, $\omega_3^2 = \omega_0^1$ and since ω_0^2, ω_3^1 are vanishing at the same time then they are linearly dependent, so we can take $\omega_3^1 = K\omega_0^2$.

The equation: $\omega_3^1 = K\omega_0^2$, $\omega_3^2 = \omega_0^1$

determines a line congruence its Klein image by the equation:

$$dP_3 = \omega_0^2(\overline{P}_1 + KP_1) + \omega_0^1(P_2 - \overline{P}_2)$$

which represent a two dimensional surface, one parametric family of these surfaces gives the Klein image of the complex $\omega_3^1 = K\omega_0^2$ (where $\alpha = \beta = 0$, $q = r$) which are determined by the equations

$$\begin{aligned} dP_1 &= \omega_1^2 P_2 + \omega_1^3 P_3 + \omega_0^3 \omega P_2 - \omega_0^2 \overline{P}_3 = \\ &= \tfrac{1}{1-K^2}\{(Kq - \gamma)\omega_0^1 + (K\gamma - q)\omega_3^2\} P_2 + \\ &+ \tfrac{-1}{1-K^2}\{(q - K\gamma)\omega_0^1 + (\gamma - Kq)\omega_3^2\} \overline{P}_2 - \\ &- \omega_0^2(\overline{P}_3 + kP_3) \end{aligned}$$

$$
\begin{aligned}
dP_2 &= -\omega_1^2 P_1 + \omega_2^3 P_3 - \omega_0^3 \overline{P}_1 + \omega_0^1 \overline{P}_3 = \\
&= \tfrac{1}{1-K^2}\{(\gamma - Kq)\omega_0^1 + (q - K\gamma)\omega_3^2\} P_1 + \omega_2^3 P_3 + \\
&+ \tfrac{1}{1-K^2}\{(K\gamma - q)\omega_0^1 + (Kq - \gamma)\omega_3^2\} \overline{P}_1 + \omega_0^1 \overline{P}_3 \\
dP_3 &= -\omega_1^3 P_1 - \omega_2^3 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2 = \\
&= \omega_0^2(\overline{P}_1 + KP_1) + \omega_3^2 P_2 - \omega_0^1 \overline{P}_2 \\
d\overline{P}_1 &= \omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \overline{P}_3 = \\
&= \tfrac{1}{1-K^2}\{(q - K\gamma)\omega_0^1 + (\gamma - Kq)\omega_3^2\} P_2 - \omega_0^2(P_3 + K\overline{P}_3) + \\
&+ \tfrac{1}{1-K^2}\{Kq - \gamma)O_0^1 + (K\gamma - q)\omega_3^2\} \overline{P}_2 \\
d\overline{P}_2 &= -\omega_0^3 P_1 + \omega_0^1 P_3 - \omega_1^2 \overline{P}_1 + \omega_2^3 \overline{P}_3 = \\
&= \tfrac{1}{1-K^2}\{(K\gamma - q)\omega_0^1 + (Kq - \gamma)\omega_3^2\} P_1 + \omega_0^1 P_3 + \\
&+ \tfrac{1}{1-K^2}\{(\gamma - Kq)\omega_0^1 + (q - K\gamma)\omega_3^2\} \overline{P}_1 + \omega_2^3 \overline{P}_3 \\
d\overline{P}_3 &= \omega_0^2 P_1 - \omega_0^1 P_2 - \omega_1^3 \overline{P}_1 - \omega_2^3 \overline{P}_2 = \\
&= \omega_0^2(P_1 + K\overline{P}_1) - \omega_0^1 P_2 + \omega_3^2 \overline{P}_2
\end{aligned}
$$

which represent a 3-dimensional surface.

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Received :22.IX.1995

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ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
Tomul XLII, Supliment, s.I.a, Matematică, 1996

A NEW APPLICATION OF THE OH'S STABILITY CRITERION

BY

MIRCEA PUTA AND GHEORGHE IVAN

Recently Oh [1] has given a simple criterion for the orbital stability and instability of the relative equilibrium of an Hamilton - Poisson system.

More precisely, let P be a Poisson manifold with codimension one symplectic leaves at regular points, H be a Hamiltonian function and p_0 be an equilibrium. Assume that p_0 is regular point. Let C be a given Casimir function whose level sets are symplectic leaves near p_0. Then by Lagrange multiplier theorem

$$dH(p_0) - \omega_0 dC(p_0) = 0$$

for some $\omega_0 \in \mathbf{R}$.

Assume that we have a smooth family of equilibrium p_ω near $p_{\omega_0} = p_0$, and define L_ω by:

$$L_\omega(p_\omega) \stackrel{def}{=} H(p_\omega) - \omega C(p_\omega)$$

Assume that $d^2 L_\omega(p_0)$ has one negative eigenvalue and all the others are positive. If we define:

$$d(\omega) \stackrel{def}{=} H(p_\omega) - \omega C(p_\omega),$$

then we have:

Communicated at the 25-th National Conference of Geometry and Topology, September 18-23, 1995, Iaşi, România