$$
\begin{aligned}
dP_2 &= -\omega_1^2 P_1 + \omega_2^3 P_3 - \omega_0^3 \overline{P}_1 + \omega_0^1 \overline{P}_3 = \\
&= \tfrac{1}{1-K^2}\{(\gamma - Kq)\omega_0^1 + (q - K\gamma)\omega_3^2\} P_1 + \omega_2^3 P_3 + \\
&+ \tfrac{1}{1-K^2}\{(K\gamma - q)\omega_0^1 + (Kq - \gamma)\omega_3^2\} \overline{P}_1 + \omega_0^1 \overline{P}_3 \\
dP_3 &= -\omega_1^3 P_1 - \omega_2^3 P_2 + \omega_0^2 \overline{P}_1 - \omega_0^1 \overline{P}_2 = \\
&= \omega_0^2(\overline{P}_1 + KP_1) + \omega_3^2 P_2 - \omega_0^1 \overline{P}_2 \\
d\overline{P}_1 &= \omega_0^3 P_2 - \omega_0^2 P_3 + \omega_1^2 \overline{P}_2 + \omega_1^3 \overline{P}_3 = \\
&= \tfrac{1}{1-K^2}\{(q - K\gamma)\omega_0^1 + (\gamma - Kq)\omega_3^2\} P_2 - \omega_0^2(P_3 + K\overline{P}_3) + \\
&+ \tfrac{1}{1-K^2}\{Kq - \gamma)O_0^1 + (K\gamma - q)\omega_3^2\} \overline{P}_2 \\
d\overline{P}_2 &= -\omega_0^3 P_1 + \omega_0^1 P_3 - \omega_1^2 \overline{P}_1 + \omega_2^3 \overline{P}_3 = \\
&= \tfrac{1}{1-K^2}\{(K\gamma - q)\omega_0^1 + (Kq - \gamma)\omega_3^2\} P_1 + \omega_0^1 P_3 + \\
&+ \tfrac{1}{1-K^2}\{(\gamma - Kq)\omega_0^1 + (q - K\gamma)\omega_3^2\} \overline{P}_1 + \omega_2^3 \overline{P}_3 \\
d\overline{P}_3 &= \omega_0^2 P_1 - \omega_0^1 P_2 - \omega_1^3 \overline{P}_1 - \omega_2^3 \overline{P}_2 = \\
&= \omega_0^2(P_1 + K\overline{P}_1) - \omega_0^1 P_2 + \omega_3^2 \overline{P}_2
\end{aligned}
$$

which represent a 3-dimensional surface.

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A NEW APPLICATION OF THE OH'S STABILITY CRITERION

BY

MIRCEA PUTA AND GHEORGHE IVAN

Recently Oh [1] has given a simple criterion for the orbital stability and instability of the relative equilibrium of an Hamilton - Poisson system.

More precisely, let P be a Poisson manifold with codimension one symplectic leaves at regular points, H be a Hamiltonian function and p_0 be an equilibrium. Assume that p_0 is regular point. Let C be a given Casimir function whose level sets are symplectic leaves near p_0. Then by Lagrange multiplier theorem

$$dH(p_0) - \omega_0 dC(p_0) = 0$$

for some $\omega_0 \in \mathbf{R}$.

Assume that we have a smooth family of equilibrium p_ω near $p_{\omega_0} = p_0$, and define L_ω by:

$$L_\omega(p_\omega) \stackrel{def}{=} H(p_\omega) - \omega C(p_\omega)$$

Assume that $d^2 L_\omega(p_0)$ has one negative eigenvalue and all the others are positive. If we define:

$$d(\omega) \stackrel{def}{=} H(p_\omega) - \omega C(p_\omega),$$

then we have:

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Theorem 1. *(Oh) If $d''(\omega_0) > 0$ (resp., < 0) then $p_{\omega_0} = p_0$ is orbitally stable (resp., unstable).*

Remark 1. (see [1]) One interesting feature of this theorem is that it does not involve the eigenvalue analysis of the linearized part of our initial system at p_0 which is usually necessary to check the linear or non-linear stability, but it only involves the analysis of the Hessian and an auxiliary function of one variable.

Remark 2.(see [1]) We can also apply this criterion to the case where the Hessian has only one positive eigenvalue by considering $-L_\omega$ and $-d''(\omega)$.

Now we shall apply this criterion to the rigid body motion with a single control applied to its minor axis.

The rigid body equations with a single control about the minor axis can be written in the following form:

$$(1)\qquad \begin{cases} \dot m_1 = a_1 m_2 m_3 \\ \dot m_2 = a_2 m_1 m_3 \\ \dot m_3 = a_3 m_1 m_2 + u_3 \end{cases}$$

where $m = (m_1, m_2, m_3)$ is the angular momentum viewed in the body coordinate system, $I = (I_1, I_2, I_3)$ is the diagonalized moment of inertia tensor, $I_1 > I_2 > I_3$ and as usually:

$$a_1 = \frac{I_2 - I_3}{I_2 I_3};\quad a_2 = \frac{I_3 - I_1}{I_3 I_1};\quad a_3 = \frac{I_1 - I_2}{I_1 I_2}.$$

Now we employ the feedback :

$$(2)\qquad u_3 = -k m_1 m_2$$

where k is the feedback gain parameter. We refer to the system (1) with the feedback (2) as the **controlled system**.

Theorem 2. *The controlled system (1), (2) is a Hamilton-Poisson one with respect to the phase space $P = \mathbf{R}^3$ the Poisson structure :*

$$\{f, g\} \stackrel{def}{=} -\nabla\left(\frac{a_3 - k}{a_3} M^2\right) \cdot (\nabla f \times \nabla g),$$

where

$$M^2 = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2 \frac{a_3}{a_3 - k}\right)$$

and the Hamiltonian H given by :

$$H = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3} \cdot \frac{a_3}{a_3 - k}\right)$$

Proof. Indeed we have :

$$\dot m_1 = \{m_1, H\} = -\det\begin{bmatrix} \frac{a_3-k}{a_3} m_1 & \frac{a_3-k}{a_3} m_2 & m_3 \\ 1 & 0 & 0 \\ \frac{m_1}{I_1} & \frac{m_2}{I_2} & \frac{a_3}{a_3-k}\frac{m_3}{I_3} \end{bmatrix} =$$

$$= m_2 m_3\left(\frac{1}{I_3} - \frac{1}{I_2}\right) = a_1 m_2 m_3$$

$$\dot m_2 = \{m_2, H\} = -\det\begin{bmatrix} \frac{a_3-k}{a_3} m_1 & \frac{a_3-k}{a_3} m_2 & m_3 \\ 0 & 1 & 0 \\ \frac{m_1}{I_1} & \frac{m_2}{I_2} & \frac{a_3}{a_3-k}\frac{m_3}{I_3} \end{bmatrix} =$$

$$= -m_1 m_3\left(\frac{1}{I_3} - \frac{1}{I_1}\right) = a_2 m_1 m_3$$

$$\dot m_3 = \{m_3, H\} = -\det\begin{bmatrix} \frac{a_3-k}{a_3} m_1 & \frac{a_3-k}{a_3} m_2 & m_3 \\ 0 & 0 & 1 \\ \frac{m_1}{I_1} & \frac{m_2}{I_2} & \frac{a_3}{a_3-k}\frac{m_3}{I_3} \end{bmatrix} =$$

$$= m_1 m_2\left(\frac{1}{I_2} - \frac{1}{I_1}\right)\frac{a_3 - k}{a_3} = a_3 m_1 m_2 - k m_1 m_2,$$

as required. ∎

Remark 3. Let us note that the level surfaces $M^2 =$ constant are ellipsoids of codimension one when $a_3 > k$ and spheres when $k = 0$, so they are regular leaves of $so(3) \cong \mathbf{R}^3$ of codimension one and we can apply Oh's theorem.

Theorem 3. *The controlled system (1), (2) may be stabilized about the relative equilibrium $p_0 = (1, 0, 0)$ for $k < a_3$.*

Proof. Let us choose $C = \left(\frac{1}{2}M^2\right)^2$. Then

$$dH(p_0) = \left(\frac{1}{I_1}, 0, 0\right)$$

$$dC(p_0) = (1, 0, 0)$$

so $\omega_0 = \frac{1}{I_1}$.

Set $L_\omega = H - \omega C$. Choose $p_\omega = \left(\frac{1}{\sqrt{\omega I_1}}, 0, 0\right)$. Therefore

$$d(\omega) = H(p_\omega) - \omega C(p_\omega) = \frac{1}{4\omega I_1^2}$$

Differentiating this twice we get,

$$d''(\omega) = \frac{3}{2I_1^2\omega^3}$$

and so

$$d''(\omega_0) = \frac{3I_1}{2} > 0$$

from the above. Now consider the Hessian $\delta^2 L_{\omega_0}(c)$. By simple computations, we get :

$$\delta^2 L_{\omega_0}(p_0) = -\frac{2}{I_1}(\delta m_1)^2 + \left(\frac{1}{I_2} - \frac{1}{I_1}\right)(\delta m_2)^2 + \left(\frac{1}{I_3} - \frac{1}{I_1}\right)\frac{a_3}{a_3 - k}(\delta m_3)^2 .$$

Hence, for $k < a_3$ we can apply Theorem 1 and so we obtain the desired result. ■

Remark 4. For the particular case $k = 0$, we refined the Oh's result for the free rigid body.

Remark 5. The same result can be also obtained via the energy - Casimir method [2].

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DIFFUSIONS ON LAGRANGE MANIFOLDS

BY

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The Riemannian theory of diffusion processes and stochastic development was extended by P.L.Antonelli and T.Zastawniak to the case of Finsler manifolds, the extension being motivated by important problems in Biology ([1], [2], [3]). We extend here this theory to some special Lagrange manifolds.

1. Nonlinear connection on TM; parallelism. Let M be a smooth n dimensional manifold and $\pi : TM \to M$ its tangent bundle. If (x^i) is a local system of coordinates on a domain U of a chart on M then (x^i, y^i) is the induced local system of coordinates on $\pi^{-1}(U)$ in TM.

We set $\partial_i := \partial/\partial x^i$, $\dot{\partial}_i := \partial/\partial y^i$

Let (M, L) be a Lagrange manifold and $N = (N^i_j(x, y))$ a nonlinear connection on M, $(\delta_i, \dot{\partial}_i)$ the adapted frame and $(dx^i, \delta y^i)$ its dual. We have

(1.1) $$\delta_i = \partial_i - N^j_i \dot{\partial}_j, \qquad \delta y^i = dy^i + N^i_j dx^j$$

We denote by

$$G(x, y) = g_{ij} dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$$

the metrical structure on TM induced by L and N where

(1.2) $$g_{ij} := \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L$$

We can associate to N and L a metric d-connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with its coefficients expressed as follows:

(1.3) $$L^i_{jk} = \frac{1}{2}g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}), C^i_{jk} = \frac{1}{2}g^{ih}\dot{\partial}_j g_{hk}$$

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