$$dC(p_0) = (1, 0, 0)$$

so $\omega_0 = \frac{1}{I_1}$.

Set $L_\omega = H - \omega C$. Choose $p_\omega = \left(\frac{1}{\sqrt{\omega I_1}}, 0, 0\right)$. Therefore

$$d(\omega) = H(p_\omega) - \omega C(p_\omega) = \frac{1}{4\omega I_1^2}$$

Differentiating this twice we get,

$$d''(\omega) = \frac{3}{2 I_1^2 \omega^3}$$

and so

$$d''(\omega_0) = \frac{3 I_1}{2} > 0$$

from the above. Now consider the Hessian $\delta^2 L_{\omega_0}(\cdot)$. By simple computations, we get :

$$\delta^2 L_{\omega_0}(p_0) = -\tfrac{2}{I_1}(\delta m_1)^2 + \left(\tfrac{1}{I_2} - \tfrac{1}{I_1}\right)(\delta m_2)^2 + \left(\tfrac{1}{I_3} - \tfrac{1}{I_1}\right)\tfrac{a_3}{a_3 - k}(\delta m_3)^2 .$$

Hence, for $k < a_3$ we can apply Theorem 1 and so we obtain the desired result. ■

Remark 4. For the particular case $k = 0$, we refined the Oh's result for the free rigid body.

Remark 5. The same result can be also obtained via the energy - Casimir method [2].

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DIFFUSIONS ON LAGRANGE MANIFOLDS

BY

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The Riemannian theory of diffusion processes and stochastic development was extended by P.L.Antonelli and T.Zastawniak to the case of Finsler manifolds, the extension being motivated by important problems in Biology ([1], [2], [3]). We extend here this theory to some special Lagrange manifolds.

1. Nonlinear connection on TM; parallelism. Let M be a smooth n dimensional manifold and $\pi : TM \to M$ its tangent bundle. If (x^i) is a local system of coordinates on a domain U of a chart on M then (x^i, y^i) is the induced local system of coordinates on $\pi^{-1}(U)$ in TM.

We set $\partial_i := \partial/\partial x^i$, $\dot{\partial}_i := \partial/\partial y^i$

Let (M, L) be a Lagrange manifold and $N = (N^i_j(x, y))$ a nonlinear connection on M, $(\delta_i, \dot{\partial}_i)$ the adapted frame and $(dx^i, \delta y^i)$ its dual. We have

$$\delta_i = \partial_i - N^j_i \dot{\partial}_j, \qquad \delta y^i = dy^i + N^i_j dx^j \tag{1.1}$$

We denote by

$$G(x, y) = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j$$

the metrical structure on TM induced by L and N where

$$g_{ij} := \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L \tag{1.2}$$

We can associate to N and L a metric d-connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with its coefficients expressed as follows:

$$L^i_{jk} = \frac{1}{2} g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}), \; C^i_{jk} = \frac{1}{2} g^{ih} \dot{\partial}_j g_{hk} \tag{1.3}$$

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A systematic presentation of the geometry of a Lagrange manifold is given in the monograph of Miron and Anastasiei [8].

Let $x : [0, T] \to M$ be a smooth curve on M and $0 \neq y_0 \in T_{x(0)}M$.

The vector field $y(t) \in TM_{x(t)}$ is called the nonlinear parallel transport of y_0 along $x(t)$ if

$$\frac{dy^i}{dt} + N^i_j(x(t), y(t))\frac{dx^j}{dt} = 0 \tag{1.4}$$

$$y(0) = y_0 \tag{1.5}$$

Generally this nonlinear differential equation has only a local solution. We are interested to find a nonlinear connection such that the solution of (1.4), (1.5) is defined on the whole interval $[0, T]$.

For a regular Lagrangian on M we have the semispray associated to L (the Euler - Lagrange field)

$$\xi = y^i\partial_i + \xi^i\dot{\partial}_i \ , \ \xi^i = g^{ij}(\partial_j L - y^k\dot{\partial}_j\partial_k L) \tag{1.6}$$

The solutions of the Euler - Lagrange equations

$$\frac{d}{dt}(\dot{\partial}_i L) - \partial_i L = 0 \ , \ y^i = \frac{dx^i}{dt} \tag{1.7}$$

are the integral curves of ξ, that is the solutions of the equations

$$\frac{d^2x^i}{dt^2} = \xi^i(x(t), \frac{dx}{dt}) \tag{1.8}$$

We can associate to ξ a canonical nonlinear connection with coefficients given by

$$\overset{c}{N}{}^i_j = -\frac{1}{2}\dot{\partial}_j\xi^i \tag{1.9}$$

Generally, the paths of $\overset{c}{N}{}^i_j$, that is the solutions of the differential equations

$$\frac{d^2x^i}{dt^2} + \overset{c}{N}{}^i_j(x(t))\frac{dx^j}{dt} = 0 \tag{1.10}$$

are different from the integral curves of ξ.

Let $E = y^i\dot{\partial}_i L - L$ be the energy function associated to L.

Definition 1.1. *We say that the nonlinear connection $N = (N^i_j)$ on M is conservative with respect to L if*

$$E(x(t), y(t)) = \text{const.} \tag{1.11}$$

Generally $\overset{c}{N} = (\overset{c}{N}{}^i_j)$ is a non conservative nonlinear connection and its paths are different from the integral curves of ξ. Let $N = (N^i_j)$ a nonlinear connection on M such that

$$N^i_j = \overset{c}{N}{}^i_j + T^i_j \tag{1.12}$$

where T^i_j are the components of a (1, 1) d-tensor field. We are interested to find T^i_j such that N is a conservative nonlinear connection.

Theorem 1.1. *If $T^i_j g_{ik} = T^i_k g_{ij}$, the following conditions are equivalent:*

(i) The paths of N coincide with the integral curves of ξ

(ii) The nonlinear connection N is conservative.

Proof: We use the following identities:

$$g_{ij}\dot{\partial}_k g^{jh} = g_{kj}\dot{\partial}_i g^{jk}$$

$$\overset{c}{N}{}^i_j g_{ik} - \overset{c}{N}{}^i_k g_{ij} = \dot{\partial}_k\partial_j L - \dot{\partial}_j\partial_k L$$

Remark. $N = (N^i_j)$ is a conservative nonlinear connection if T^i_j are solutions of the system of equations

$$T^i_j g_{ik} y^j = y^h\dot{\partial}_k\partial_h L - \partial_k L - \overset{c}{N}{}^i_h g_{ik} y^h \tag{1.13}$$

$$T^i_j g_{ik} = T^i_k g_{ij} \tag{1.14}$$

A solution of this system is given by:

$$T^i_j = \frac{B}{2A}y_j y^i - \frac{1}{2A}(\xi^{*i}y_j + \xi^*_j y^i) \tag{1.15}$$

where

$$A = y_i y^i = g_{ij}y^j y^i, \ B = \xi^*_i y^i, \ \xi^{*i} = y^k\dot{\partial}_k\xi^i - 2\xi^i$$

We suppose that the energy function has the following properties:

1) E is continuous on TM, smooth on $\overset{\circ}{T}M = TM \setminus \{0_x, x \in M\}$
2) The local expression of E fulfills the conditions

a) $\lim\limits_{\|y\| \to \infty} |E(x, y)| = \infty$

b) $E(x, y) = 0 \iff y = 0$, for every x.

Examples. For the Lagrangians

a) $L(x, y) = \frac{1}{2}F^2(x, y)$ (Finslerian case)

b) $L(x, y) = \frac{1}{2}F^2(x, y) + A_i(x)y^i$

c) L a m-homogeneous non degenerate Lagrangian E verifies 1) and 2) above.

Theorem 1.2. *If $N = (N_j^i)$ is a conservative nonlinear connection with respect to the regular Lagrangian L and the energy function has the properties 1) and 2) above then the equation (1.1) with the initial condition (1.5) has a unique solution defined on $[0, T]$.*

Throughout this paper we use Lagrangians for which the energy function has properties 1) and 2).

Let $x : [0, T] \to M$ be a smooth curve on M and $c(t) = (x(t), y(t))$ its horizontal lift, that is $\delta y^i = 0$.

If $u_0 \in T_{x(0)}M$, we say that the vector field $u(t) \in T_{x(t)}M$ is the parallel transport of u_0 along $x(t)$ if its horizontal lift is the parallel transport along $c(t)$, of the horizontal lift of u_0.

That is:

$$\frac{du^i}{dt} + L^i_{jk}(x(t), y(t))u^j\frac{dx^k}{dt} = 0, \quad u(0) = u_0 \tag{1.16}$$

If $u(t), v(t)$ are two solutions of (1.16) then

$$g_{ij}(x(t), y(t))u^i v^j = \text{const.} \quad \forall t \in [0, T] \tag{1.18}$$

2. Stochastic parallel transport. In this section we extend the concept of stochastic parallel transport along a trajectory of an horizontal diffusion on TM.

Let $x(t)$ be a diffusion on M defined for all $t \in [0, T]$, where T is a stopping time. (For an introduction in this field see [4], [5], [7]).

We shall approximate the diffusion by piecewise sample path on M.

Given a partition $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ we take a piecewise smooth curve $x_\pi(t)$, with first, second and third order derivative such that $x_\pi(t_\alpha) = x(t_\alpha), \alpha \in \{0, \cdots, n\}$. We can lift each sample function, and

thus we obtain a process $y_\pi(t) \in T_{x_\pi(t)}M$ whose sample paths satisfy the ordinary differential equation:

$$\frac{dy^i_\pi(t)}{dt} + N^i_j(x_\pi(t), y_\pi(t))\frac{dx^j}{dt} = 0 \tag{2.1}$$

of the nonlinear parallel transport, with the initial condition

$$y_\pi(0) = y_0 \tag{2.2}$$

Theorem 2.1. *The solution of the family of ordinary differential equations (2.1), (2.2) converges in probability as mesh $\pi \to 0$ to the solution of the Stratonovich stochastic differential equation:*

$$dy^i + N^i_j(x(t), y(t)) \circ dx^j = 0 \tag{2.3}$$

$$y(0) = y_0 \tag{2.4}$$

Definition 2.2. *The solution of (2.3), (2.4) is called the stochastic nonlinear parallel transport of y_0 along the diffusion $x(t)$ and the diffusion $(x(t), y(t))$ will be called the horizontal lift of the diffusion $x(t)$.*

Theorem 2.3. *If $y(t)$ is a solution of (2.3) then*

$$E(x(t), y(t)) = \text{const.} \quad a.s. \ on \ [0, T]. \tag{2.5}$$

In the same manner we can define the stochastic parallelism along a horizontal diffusion on $\overset{\circ}{T}M$.

Let $c(t) = (x(t), y(t))$ be an horizontal diffusion on $\overset{\circ}{T}M$ starting at (x_0, y_0) and $c_\pi(t) = (x_\pi(t), y_\pi(t))$ a piecewise smooth bounded first, second and third order derivative such that $c_\pi(t_\alpha) = c(t_\alpha)$, $\alpha \in \{0, \cdots, n\}$ where $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of [0, T].

Let $u_\pi(t) \in T_{x_\pi(t)}M$ be a process whose sample paths satisfy the ordinary differential equations of parallel transport:

$$\frac{du^i_\pi}{dt} + L^i_{jk}u^j\frac{dx^k_\pi}{dt} = 0, \quad u_\pi(0) = u_0 \tag{2.6}$$

where $u_0 \in T_{x_0}M$ ia a $\mathcal{F}_0$ measurable random vector and π is an arbitrary partition.

Theorem 2.4. *The solution of the family of differential equations (2.6) converges in probability as mesh $\pi \to 0$ to the solution $u(t)$ of the stochastic differential equation*

$$du^i + L^i_{jk} u^j \circ dx^k = 0, \quad u(0) = u_0 \tag{2.8}$$

Theorem 2.5. *If $u(t), v(t)$ are two solutions of (2.8) then*

$$g_{ij}(x(t), y(t)) u^i v^j = \text{const.} \qquad a.s. \qquad on \quad [0, T].$$

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CONNECTION MAP IN THE HIGHER ORDER GEOMETRY

BY

IOAN BUCATARU

Recently, R. Miron and Gh. Atanasiu ([4], [5], [6]) have developed a geometrization of the higher order Lagrangians which parallels that of the first order Lagrangians expounded in a monograph by R.Miron and M.Anastasiei ([3]).

In this geometrization a central role is played by certain nonlinear connections in the k-oscalator bundle.

Our aim is to associate to every nonlinear connection in the k-osculator bundle a special map called connection map, which to be similar to that considered by P.Dombrovski ([1]) for linear connections and extended in [3], ch. 2. This map will be used for providing alternative expressions for some facts from the higher order geometry.

In §1 we recall the notion of nonlinear connection in the k-osculator bundle. The connection map associated to a nonlinear connection in the k-osculator bundle is introduced in §2. It appears as a special morphism of vector bundles in such a way that for k=1 it reduces to the usual connection map associated to a nonlinear connection ([3],pag.28). Then it is shown that its kernel is just the horizontal subbundle.

1. Nonlinear connections in the k-osculator bundle. Let M be a real, smooth manifold of dimension n and $(Osc^k M, \pi, M)$ its k-osculator bundle. Then $Osc^k M$ is a real smooth manifold of dimension (k+1)n. We set $E = Osc^k M$.

Let (x^i) be the local coordinates in a local chart $U \subset M$. The local coordinates on $\pi^{-1}(U) \subset E$ will be denoted by $(x^i, y^{(1)i}, \ldots, y^{(k)i})$.

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