Theorem 2.4. *The solution of the family of differential equations (2.6) converges in probability as mesh $\pi \to 0$ to the solution $u(t)$ of the stochastic differential equation*

$$du^i + L^i_{jk} u^j \circ dx^k = 0, \quad u(0) = u_0 \tag{2.8}$$

Theorem 2.5. *If $u(t), v(t)$ are two solutions of (2.8) then*

$$g_{ij}(x(t), y(t)) u^i v^j = \text{const.} \qquad a.s. \qquad on \quad [0, T].$$

REFERENCES

1. P.L.Antonelli, T.J.Zastawniak, *Diffusions on Finsler manifolds*, Proc. of the XXV^{th} Symposium on Mathematical Physics, Toruń, Poland (1992).
2. P.L.Antonelli, T.J.Zastawniak, *Stochastic calculus on Finsler manifolds and an application in Biology*, Nonlin.World, 1 (1994), pp 149–171.
3. P.L.Antonelli, T.J.Zastawniak, *Introduction to Diffusion on Finsler Manifolds*, Math.Comput.Modelling, 20 (1994), pp 109–116.
4. Elworthy, *Stochastic Differential Equations on Manifolds*, Cambridge University Press, Cambridge, 1982.
5. M.Emery, *Stochastic Calculus in Manifolds*, Springer-Verlag, Berlin, Heidelberg, 1989.
6. D.Hrimiuc, *Diffusion on the total space of a vector bundle*, (to appear).
7. H. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam, Kodansher, Tokyo, 1989.
8. R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces, Theory and Applications*, Kluwer Academic Press, FTPH, No.59, 1994.

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CONNECTION MAP IN THE HIGHER ORDER GEOMETRY

BY

IOAN BUCATARU

Recently, R. Miron and Gh. Atanasiu ([4], [5], [6]) have developed a geometrization of the higher order Lagrangians which parallels that of the first order Lagrangians expounded in a monograph by R.Miron and M.Anastasiei ([3]).

In this geometrization a central role is played by certain nonlinear connections in the k-oscalator bundle.

Our aim is to associate to every nonlinear connection in the k-osculator bundle a special map called connection map, which to be similar to that considered by P.Dombrovski ([1]) for linear connections and extended in [3], ch. 2. This map will be used for providing alternative expressions for some facts from the higher order geometry.

In §1 we recall the notion of nonlinear connection in the k-osculator bundle. The connection map associated to a nonlinear connection in the k-osculator bundle is introduced in §2. It appears as a special morphism of vector bundles in such a way that for k=1 it reduces to the usual connection map associated to a nonlinear connection ([3],pag.28). Then it is shown that its kernel is just the horizontal subbundle.

1. Nonlinear connections in the k-osculator bundle. Let M be a real, smooth manifold of dimension n and $(Osc^k M, \pi, M)$ its k-osculator bundle. Then $Osc^k M$ is a real smooth manifold of dimension (k+1)n. We set $E = Osc^k M$.

Let (x^i) be the local coordinates in a local chart $U \subset M$. The local coordinates on $\pi^{-1}(U) \subset E$ will be denoted by $(x^i, y^{(1)i}, \ldots, y^{(k)i})$.

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A change of coordinates $(x, y^{(1)}, \ldots, y^{(k)}) \to (\tilde{x}, \tilde{y}^{(1)}, \ldots, \tilde{y}^{(k)})$ on E is given by:

$$
(1.1)\qquad \begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, x^2, \ldots, x^n); \qquad \operatorname{rank}\left\|\dfrac{\partial \tilde{x}^i}{\partial x^j}\right\| = n \\ \tilde{y}^{(1)i} = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\ 2\tilde{y}^{(2)i} = \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \\ \ldots \\ k\tilde{y}^{(k)i} = \dfrac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + \ldots + k\dfrac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j} \end{cases}
$$

As $d\pi : (TE, \tau_E, E) \to (TM, \tau, M)$ is an epimorphism of vector bundles, it results that its kernel is a vector subbundle of the bundle (TE, τ_E, E). This will be denoted by VE and will be called the vertical subbundle of the TE. The fibres of VE determine an integrable distribution $V : u \in E \to V_u \subset T_uE$ which has the dimension kn, called vertical distribution.

A nonlinear connection on E is a vector subbundle NE of the tangent bundle TE such that the Whitney sum:

$$
(1.2)\qquad TE = NE \oplus VE, \text{ holds.}
$$

The $\mathcal{F}$(E)-linear mapping $J : \chi(E) \to \chi(E)$ defined by:

$$
(1.3)\qquad J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, \ldots, J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}}, J\left(\frac{\partial}{\partial y^{(k)i}}\right) = 0
$$

is a k-tangent structure, that is $J^{k+1} = 0$.

Let us consider $N_0 = N, N_1 = J(N_0), \ldots, N_{k-1} = J(N_{k-2})$ and V_k the distribution locally generated by $\{\frac{\partial}{\partial y^{(k)i}}\}_{i=\overline{1,n}}$. We have $k+1$ distributions $(N_0, \ldots, N_{k-1}, V_k)$, each of the dimension n, such that:

$$
(1.4)\qquad T_uE = N_0(u) \oplus N_1(u) \oplus \ldots N_{k-1}(u) \oplus V_k(u), \forall u \in E
$$

A local basis for the $\mathcal{F}$(E)-module $\chi(E)$, adapted to the distributions $N_0, N_1, \ldots, N_{k-1}$ and V_k, is:

$$
(1.5)\qquad \left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\partial}{\partial y^{(k)i}}\right\}, \quad \text{where :}
$$

$$
(1.6)\qquad \begin{cases} \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{(1)}{N}{}_i^j \dfrac{\partial}{\partial y^{(1)j}} - \ldots - \underset{(k)}{N}{}_i^j \dfrac{\partial}{\partial y^{(k)j}} \\ \dfrac{\delta}{\delta y^{(1)i}} = J\left(\dfrac{\delta}{\delta x^i}\right), \ldots, \dfrac{\delta}{\delta y^{(k-1)i}} = J\left(\dfrac{\delta}{\delta y^{(k-2)i}}\right). \end{cases}
$$

The functions $\underset{(1)}{N}{}_i^j, \ldots, \underset{(k)}{N}{}_i^j$ are called the coefficients of the nonlinear connection N. Under a change of coordinates (1.1) on E, the basis (1.5) changes as follows:

$$
(1.7)\qquad \frac{\delta}{\delta \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\delta}{\delta x^j}, \quad \frac{\delta}{\delta \tilde{y}^{(1)i}} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\delta}{\delta y^{(1)j}} \ldots \quad \frac{\partial}{\partial \tilde{y}^{(k)i}} = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{\partial}{\partial y^{(k)j}}.
$$

The dual basis of the basis (1.5) is:

$$
(1.8)\qquad \{dx^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i}\}, \quad \text{where :}
$$

$$
(1.9)\qquad \begin{cases} \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M}{}_j^i dx^j \\ \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M}{}_j^i dy^{(1)j} + \underset{(2)}{M}{}_j^i dx^j \\ \ldots \\ \delta y^{(k)i} = dy^{(k)i} + \underset{(1)}{M}{}_j^i dy^{(k-1)j} + \ldots + \underset{(k)}{M}{}_j^i dx^j \end{cases}
$$

The functions $\underset{(1)}{M}{}_j^i, \ldots, \underset{(k)}{M}{}_j^i$ are called the dual coefficients of the nonlinear connection N.

The coefficients $\underset{(1)}{N}{}_j^i, \ldots, \underset{(k)}{N}{}_j^i$ and the dual coefficients $\underset{(1)}{M}{}_j^i, \ldots, \underset{(k)}{M}{}_j^i$ are related by:

$$
(1.10)\qquad \begin{cases} \underset{(1)}{M}{}_j^i = \underset{(1)}{N}{}_j^i, \quad \underset{(2)}{M}{}_j^i = \underset{(2)}{N}{}_j^i + \underset{(1)}{N}{}_m^i \underset{(1)}{M}{}_j^m \quad \ldots \\ \underset{(k)}{M}{}_j^i = \underset{(k)}{N}{}_j^i + \underset{(k-1)}{N}{}_m^i \underset{(1)}{M}{}_j^m + \ldots + \underset{(1)}{N}{}_m^i \underset{(k-1)}{M}{}_j^m \end{cases}
$$

A nonlinear connection N is completely determined by a system of function $\underset{(1)}{M}{}_j^i, \ldots, \underset{(k)}{M}{}_j^i$, given on every domain of local chart and which, under a change of coordinates (1.1) verify (6.6) from [4].

2. The connection map associated to a nonlinear connection N on Osc$^k M$. Let be $(E = \mathrm{Osc}^k M, \pi, M)$, the k-osculator bundle, $(TM^{(k)}, \tau^{(k)}, M)$ the Whitney sum of the tangent bundle (TM, τ, M) whith itself of k-times and N a nonlinear connection on E.

Proposition 2.1. *Let be $u \in E$, $X_u = \overset{(0)}{X}{}^i \frac{\partial}{\partial x^i}\big|_u + \ldots + \overset{(k)}{X}{}^i \frac{\partial}{\partial y^{(k)i}}\big|_u \in T_u E$. Then:*

$$(\overset{(1)}{X}{}^i + \underset{(1)}{M}{}^i_j \overset{(0)}{X}{}^j)\frac{\partial}{\partial x^i}\Big|_{\pi(u)} \in T_{\pi(u)}M$$

$$(\overset{(2)}{X}{}^i + \underset{(1)}{M}{}^i_j \overset{(1)}{X}{}^j + \underset{(2)}{M}{}^i_j \overset{(0)}{X}{}^j)\frac{\partial}{\partial x^i}\Big|_{\pi(u)} \in T_{\pi(u)}M \quad \ldots$$

$$(\overset{(k)}{X}{}^i + \underset{(1)}{M}{}^i_j \overset{(k-1)}{X}{}^j + \ldots + \underset{(k)}{M}{}^i_j \overset{(0)}{X}{}^j)\frac{\partial}{\partial x^i}\Big|_{\pi(u)} \in T_{\pi(u)}M.$$

Proof. Let us set

$$(2.1)\quad \begin{cases} (\overset{(1)}{K}_u X_u)^i := \overset{(1)}{X}{}^i + \underset{(1)}{M}{}^i_j \overset{(0)}{X}{}^j; \\ (\overset{(2)}{K}_u X_u)^i := \overset{(2)}{X}{}^i + \underset{(1)}{M}{}^i_j \overset{(1)}{X}{}^j + \underset{(2)}{M}{}^i_j \overset{(0)}{X}{}^j; \quad \ldots \\ (\overset{(k)}{K}_u X_u)^i := \overset{(k)}{X}{}^i + \underset{(1)}{M}{}^i_j \overset{(k-1)}{X}{}^j + \ldots + \underset{(k)}{M}{}^i_j \overset{(0)}{X}{}^j. \end{cases}$$

By a straightforward calculation one obtains:

$$X_u = \overset{(0)}{X}{}^i \frac{\delta}{\delta x^i}\Big|_u + (\overset{(1)}{K}_u X_u)^i \frac{\delta}{\delta y^{(1)i}}\Big|_u + \ldots + (\overset{(k)}{K}_u X_u)^i \frac{\partial}{\partial y^{(k)i}}\Big|_u$$

Acording to (1.7), $\{\frac{\delta}{\delta x^i}\big|_u, \frac{\delta}{\delta y^{(1)i}}\big|_u, \ldots, \frac{\partial}{\partial y^{(k)i}}\big|_u\}$ are d-vectors on E. Thus a change of coordinates (1.1) on E implies

$$(2.2)\quad \begin{cases} \overset{(0)}{X}{}^j = \frac{\partial x^j}{\partial \tilde{x}^i}\widetilde{\overset{(0)}{X}}{}^i \\ (\overset{(1)}{K}_u X_u)^j = \frac{\partial x^j}{\partial \tilde{x}^i}(\widetilde{\overset{(1)}{K}_u X_u})^i \quad \ldots \quad (\overset{(k)}{K}_u X_u)^j = \frac{\partial x^j}{\partial \tilde{x}^i}(\widetilde{\overset{(k)}{K}_u X_u})^i \end{cases}$$

Consequently $\overset{(0)}{X}{}^i, (\overset{(1)}{K}_u X_u)^i, \ldots, (\overset{(k)}{K}_u X_u)^i$ are the components of some vectors, that is, $\overset{(0)}{X}{}^i \frac{\partial}{\partial x^i}\big|_{\pi(u)}, (\overset{(1)}{K}_u X_u)^i \frac{\partial}{\partial x^i}\big|_{\pi(u)}, \ldots, (\overset{(k)}{K}_u X_u)^i \frac{\partial}{\partial x^i}\big|_{\pi(u)}$ belong to $T_{\pi(u)}M$, q.e.d.

We shall put $\overset{(1)}{K}_u X_u = (\overset{(1)}{K}_u X_u)^i \frac{\partial}{\partial x^i}\big|_{\pi(u)}, \ldots, \overset{(k)}{K}_u X_u = (\overset{(k)}{K}_u X_u)^i \frac{\partial}{\partial x^i}\big|_{\pi(u)}$ and we note that if the vector X is given in the adapted basis (1.5) then the expressions $(\overset{(\alpha)}{K} X)_{(\alpha=\overline{1,k})}$ are very simple. Indeed we have:

Proposition 2.2. *Let $u \in E$, $X_u \in T_u E$, $X_u = \overset{(0)}{X}{}^i \frac{\delta}{\delta x^i}\big|_u + \overset{(1)}{X}{}^i \frac{\delta}{\delta y^{(1)i}}\big|_u + \ldots + \overset{(k)}{X}{}^i \frac{\partial}{\partial y^{(k)i}}\big|_u$.*

Then $\overset{(\alpha)}{K}_u X_u = \overset{(\alpha)}{X}{}^i \frac{\partial}{\partial x^i}\big|_{\pi(u)} \quad (\alpha = \overline{1,k})$.

Proof. Taking into consideration (1.6) we have:

$$X_u = \overset{(0)}{X}{}^i \frac{\partial}{\partial x^i}\Big|_u + (\overset{(1)}{X}{}^i - \overset{(0)}{X}{}^j \underset{(1)}{N}{}^i_j)\frac{\partial}{\partial y^{(1)i}}\Big|_u + (\overset{(2)}{X}{}^i - \overset{(1)}{X}{}^j \underset{(1)}{N}{}^i_j - \overset{(0)}{X}{}^j \underset{(2)}{N}{}^i_j)\frac{\partial}{\partial y^{(2)i}}\Big|_u + \ldots + (\overset{(k)}{X}{}^i - \overset{(k-1)}{X}{}^j \underset{(1)}{N}{}^i_j - \ldots - \overset{(0)}{X}{}^j \underset{(k)}{N}{}^i_j)\frac{\partial}{\partial y^{(k)i}}\Big|_u$$

and further we obtain:

$$(\overset{(1)}{K}_u X_u)^i = (\overset{(1)}{X}{}^i - \overset{(0)}{X}{}^j \underset{(1)}{N}{}^i_j) + \overset{(0)}{X}{}^j \underset{(1)}{M}{}^i_j = \overset{(1)}{X}{}^i + \overset{(0)}{X}{}^j(\underset{(1)}{M}{}^i_j - \underset{(1)}{N}{}^i_j) = \overset{(1)}{X}{}^i,$$

$$(\overset{(2)}{K}_u X_u)^i = (\overset{(2)}{X}{}^i - \overset{(1)}{X}{}^j \underset{(1)}{N}{}^i_j - \overset{(0)}{X}{}^j \underset{(2)}{N}{}^i_j) + \underset{(1)}{M}{}^i_j(\overset{(1)}{X}{}^j - \overset{(0)}{X}{}^m \underset{(1)}{N}{}^j_m) + \overset{(0)}{X}{}^j \underset{(2)}{M}{}^i_j =$$

$$= \overset{(2)}{X}{}^i + \overset{(1)}{X}{}^j(\underset{(1)}{M}{}^i_j - \underset{(1)}{N}{}^i_j) + \overset{(0)}{X}{}^j(\underset{(2)}{M}{}^i_j - \underset{(2)}{N}{}^i_j - \underset{(1)}{M}{}^i_m \underset{(1)}{N}{}^m_j) = \overset{(2)}{X}{}^i.$$

Similarly, using (1.10) we get:

$(\overset{(\alpha)}{K}_u X_u)^i = \overset{(\alpha)}{X}{}^i, \forall(\alpha = \overline{3,k}), (i = \overline{1,n})$.

These consideration make abvious

Proposition 2.3. *The map $K : (TE, \tau_E, E) \longrightarrow (TM^{(k)},$*

$\tau^{(k)}, M)$ *defined on fibres by* $K_u : T_uE \longrightarrow \underbrace{T_{\pi(u)}M \times \ldots \times T_{\pi(u)}M}_{k \text{ times}}$ *by*

$K_u X_u = (\overset{(1)}{K_u} X_u, \overset{(2)}{K_u} X_u, \ldots, \overset{(k)}{K_u} X_u)$ *is a* π*-morphism of vector bundles.*

Definition 2.1. *The* $\pi-$*morphism* K *defined in the above Proposition will be called the connection map associated to the nonlinear connection* N.

Proposition 2.4. *The kernel of the morphism* K *is the horizontal subbundle* NE.

Proof. Since K is a $\pi-$epimorphism of vector bundles, Ker K is subbundle of the bundle (TE, τ_E, E)

Let be $u \in E$ and $X_u \in T_uE$. Then $X_u \in N_0(u)$ if and only if $\overset{(\alpha)}{K_u} X_u = 0, \forall \alpha = \overline{1,k}$, that is, if and only if $X_u \in \operatorname{Ker} K_u$. This means that for every $u \in E$, $\operatorname{Ker} K_u = N_0(u)$ hence $\operatorname{Ker} K = NE$.

Other properties of the map K will be given in a forthcoming paper.

REFERENCES

1. D o m b r o v s k i, P. *On the geometry of the tangent bundles*. Jour. reine und angew. Math., 210(1962) 73-88.
2. K o v a l s k i, O. - *Curvature of the Riemannian metric on the tangent bundles of a Riemannian manifold*. Jour. reine und angew. Math., 250(1971), 124-129.
3. M i r o n, R. and A n a s t a s i e i, M. *Vector bundles. Lagrange spaces. Applications in relativity*. Acad.Română, Bucureşti, 1987, (In Romanian)
4. M i r o n, R. and A t a n a s i u, Gh. *Differential geometry of the k-osculator bundle*. Rev Roum. de Math. Pures et Appl. 1996 (to appear)
5. M i r o n, R. and A t a n a s i u, Gh. *Prolongation of the Riemannian, Finslerian and Lagrangian structures*. Rev Roum. de Math. Pures et Appl. 1996 (to appear)
6. M i r o n, R. and A t a n a s i u, Gh. *Higher order Lagrange spaces*. Rev Roum. de Math. Pures et Appl. 1996 (to appear)

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A CLASS OF GENERALIZED LAGRANGE SPACES

BY

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The geometry of a general metric i.e. depending on point and on direction in that point is now well-known. A manifold endowed with such a general metric is called a generalized Lagrange space, $GL-$space for brevity and the metric itself is called a $GL-$metric (see R.Miron and M.Anastasiei [3]). Any Finsler space is a particular $GL-$space.

In this paper we associate to a given Finsler metric a very general $GL-$metric which provides as particular cases all the $GL-$ metrics studied in [3,Ch.X-XI]. The applications to Physics and Biology of those particular GL - metrics justify the study of the present $GL-$ metric. Although we shall not go very far with its geometry it will become clear how this can be done using the Cartan nonlinear connection of the given Finsler metric.

1. Finsler spaces. $GL-$metrics. Let M be a real smooth manifold of finite dimension n and TM its tangent manifold. Let $x = (x^i)$ and $(x, y) = (x^i, y^i)$ be the coordinates on M and TM, respectively. The indices will run from 1 to n and the Einstein convention on summation is implied. Let $F^n = (M, F)$ be a Finsler space. Then $F : TM \longrightarrow R$ is a smooth function on $T_0M = TM \setminus \{0\}$ positively 1-homogeneous with respect to y. The Finsler metric of F^n is

$$\gamma_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j F^2/2, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i} \tag{1.1}$$

and it is assumed that the quadratic form $\gamma_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in R^n$ is positive defined. Notice that $\gamma_{ij}(x, y)$ are positively 0-homogeneous functions

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