$\tau^{(k)}, M)$ *defined on fibres by* $K_u : T_uE \longrightarrow \underbrace{T_{\pi(u)}M \times \ldots \times T_{\pi(u)}M}_{k \text{ times}}$ *by*

$K_u X_u = (\overset{(1)}{K_u} X_u, \overset{(2)}{K_u} X_u, \ldots, \overset{(k)}{K_u} X_u)$ *is a π-morphism of vector bundles.*

Definition 2.1. *The $\pi-$morphism K defined in the above Proposition will be called the connection map associated to the nonlinear connection N.*

Proposition 2.4. *The kernel of the morphism K is the horizontal subbundle NE.*

Proof. Since K is a $\pi-$epimorphism of vector bundles, Ker K is subbundle of the bundle (TE, τ_E, E)

Let be $u \in E$ and $X_u \in T_uE$. Then $X_u \in N_0(u)$ if and only if $\overset{(\alpha)}{K_u} X_u = 0, \forall \alpha = \overline{1,k}$, that is, if and only if $X_u \in \mathrm{Ker}\, K_u$. This means that for every $u \in E$, $\mathrm{Ker}\, K_u = N_0(u)$ hence $\mathrm{Ker}\, K = NE$.

Other properties of the map K will be given in a forthcoming paper.

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A CLASS OF GENERALIZED LAGRANGE SPACES

BY

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The geometry of a general metric i.e. depending on point and on direction in that point is now well-known. A manifold endowed with such a general metric is called a generalized Lagrange space, $GL-$space for brevity and the metric itself is called a $GL-$metric (see R.Miron and M.Anastasiei [3]). Any Finsler space is a particular $GL-$space.

In this paper we associate to a given Finsler metric a very general $GL-$metric which provides as particular cases all the $GL-$metrics studied in [3,Ch.X-XI]. The applications to Physics and Biology of those particular GL - metrics justify the study of the present $GL-$metric. Although we shall not go very far with its geometry it will become clear how this can be done using the Cartan nonlinear connection of the given Finsler metric.

1. Finsler spaces. $GL-$metrics. Let M be a real smooth manifold of finite dimension n and TM its tangent manifold. Let $x = (x^i)$ and $(x, y) = (x^i, y^i)$ be the coordinates on M and TM, respectively. The indices will run from 1 to n and the Einstein convention on summation is implied. Let $F^n = (M, F)$ be a Finsler space . Then $F : TM \longrightarrow R$ is a smooth function on $T_0M = TM \setminus \{0\}$ positively 1-homogeneous with respect to y. The Finsler metric of F^n is

$$\gamma_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j F^2/2, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i} \tag{1.1}$$

and it is assumed that the quadratic form $\gamma_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in \mathbf{R}^n$ is positive defined. Notice that $\gamma_{ij}(x, y)$ are positively 0-homogeneous functions

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and $F^2 = \gamma_{ij}(x, y)y^i y^j$. When a change of coordinates $(x, y) \longrightarrow (\tilde{x}, \tilde{y})$ on TM is performed, from $F(\tilde{x}(x), \tilde{y}(x, y)) = F(x, y)$ it follows

$$\gamma_{ij}(x, y) = \partial_i \tilde{x}^k \partial_j \tilde{x}^h \tilde{\gamma}_{kh}(\tilde{x}(x), \tilde{y}(x, y)), \quad \partial_i := \frac{\partial}{\partial x^i}. \tag{1.2}$$

If the provenance of γ_{ij} from F is forgotten but (1.2) is preserved one gets the notion of $GL-$ metric. More precisely we set

Definition 1.1. *A matrix $(g_{ij}(x, y))$ is called a $GL-$ metric if*

1) $g_{ij}(x, y) = g_{ji}(x, y)$,
2) $g_{ij}(x, y) = \partial_i \tilde{x}^k \partial_j \tilde{x}^h \tilde{g}_{kh}(\tilde{x}(x), \tilde{y}(x, y))$.
3) $\det(g_{ij}(x, y) \neq 0$.
4) The signature of the quadratic form $g_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in \boldsymbol{R}^n$ is constant.

2. A class of $GL-$ metrics associated to a Finsier metric. Let a and b two smooth real functions on T_0M so that $a(x, y) > 0$ and $b(x, y) \geq 0$ for every $(x, y) \in T_0M$. We set $y_i = \gamma_{ij}y^j$ and consider the matrix

$$g_{ij}(x, y) = a(x, y)\gamma_{ij}(x, y) + b(x, y)y_i y_j. \tag{2.1}$$

Proposition 2. *The matrix $(g_{ij}(x, y))$ in (2.1) is a positive definite $GL-$ metric.*

Proof. The symmetry is obvious. The tensorial character of $g_{ij}(x, y)$ follows from (1.2) noticing first that it implies $y_i = \partial_i \tilde{x}^k y_k$. Now if we put $g^{jk}(x, y) = \frac{1}{a}(\gamma^{jk} - \frac{b}{a+bF^2} y^j y^k)$, where $\gamma^{jk}\gamma_{ki} = \delta^j_i$, one verifies that $g_{ij}g^{jk} = \delta^k_i$. Hence $\det(g_{ij}(x, y)) \neq 0$. Note that $a + bF^2 > 0$. For $(\xi^i) \in \boldsymbol{R}^n$, from $g_{ij}(x, y)\xi^i\xi^j = a\gamma_{ij}\xi^i\xi^j + b(\gamma_{ik}\xi^i y^k)(\gamma_{jh}\xi^j y^h)$ it follows that $(g_{ij}(x, y)$ is positive definite. ■

Remark 2.1. When $\gamma_{ij}(x, y)$ has only a constant signature then $(g_{ij}(x, y))$ has a constant signature only on some subsets of T_0M.

Remark 2.2. The following particular forms of the $GL-$ metric (2.1) were studied in [3, Ch. X-XI]:

1) $a = \exp(2\Phi(x, y))$, $b = 0$: $g_{ij}(x, y) = \exp(2\Phi(x, y))\gamma_{ij}(x, y)$
2) $a = \exp(2\alpha(x))$, $b = 0$, $\gamma_{ij}(x, y) = \gamma_{ij}(y)$: $a_{ij}(x, y) =$ $= \exp(2\alpha(x))\gamma_{ij}(y)$ is the Antonelli metric.
3) $a \equiv 1$, $b = 1 - \frac{1}{n^2(x,y)}$: $g_{ij}(x, y) = \gamma_{ij}(x, y) + (1 - \frac{1}{n^2(x,y)})y_i y_j$.

This metric was applied to Relativistic Optics. There n means the refractive index of the medium. The cases n constant or n depending on x only are remarkable ones.

The function $\varepsilon = g_{ij}(x, y)y^i y^j$ is called the absolute energy of the $GL-$ metric $(g_{ij}(x, y))$.

Proposition 2.2. *The absolute energy of the $GL-$ metric (2.1) is* $\varepsilon = (a + bF^2)F^2$.

A $GL-$ metric is said to be regular if its absolute energy is a regular Lagrangian, that is, the matrix with the entries $\dot{\partial}_i\dot{\partial}_j\varepsilon/2$ is nonsingular. In general, the $GL-$ metric (2.1) is not regular. However, when a and b do not depend on y we get $\dot{\partial}_i\dot{\partial}_j\varepsilon/2 = (a + 2bF^2)\gamma_{ij} + by_i y_j$ which is a matrix of the form (2.1). Thus in this case the $GL-$ metric (2.1) is regular.

One says that a $GL-$ metric $(g_{ij}(x, y))$ is provided by a regular Lagrangian or it is an $L-$ metric if there exists a function $L : TM \longrightarrow R$ so that $g_{ij}(x, y) = \dot{\partial}_i\dot{\partial}_j L/2$. It is clear that, when exists, L is not unique. One proves that a $GL-$ metric $(g_{ij}(x, y)$ is an $L-$ metric if and only if $\dot{\partial}_k g_{ij} = \dot{\partial}_j g_{ik}$. A direct calculation gives

Proposition 2.3. *The $GL-$ metric (2.1) is provided by a Lagrangian if and only if*

$$(\dot{\partial}_k a)\gamma_{ij} - (\dot{\partial}_i a)\gamma_{kj} + (\dot{\partial}_k b)y_i - (\dot{\partial}_i b)y_k + b(y_i\gamma_{kj} - y_k\gamma_{ij}) = 0. \tag{2.2}$$

The equation (2.2) holds in very few circumstances. In other words the $GL-$ metric (2.1) with arbitrary a and b provides a large class of $GL-$ metrics which are not reducible to $L-$ metrics. For instance, we have

Proposition 2.4. *Even a and b do not depend on y, the $GL-$ metric (2.1) is not reducible to a $L-$ metric.*

Proof. In our hypothesis, (2.2) reduces to $\gamma_{kj}y_i - \gamma_{ij}y_k = 0$ which multiplied by γ^{ij} gives $(1 - n)y_k = 0$. This is false for $n > 1$. Hence (2.2) does not hold. ■

Proposition 2.5. *The $GL-$ metric $g_{ij}(x, y) = \exp(2\Phi(x, y))\cdot$ $\gamma_{ij}(x, y)$ is not reducible to an $L-$ metric.*

Proof. For $b = 0$ and $a = \exp(2\Phi(x, y))$, (2.2) reduces to $(\dot{\partial}_k\Phi)\gamma_{ij} - (\dot{\partial}_i\Phi)\gamma_{kj} = 0$. Multiplying this by γ^{ij} one obtains $(n - 1)\dot{\partial}_k\Phi = 0$ which is false when Φ depends on y. Thus (2.2) is false. ■
We notice that the Antonelli metric (a_{ij}) is an $L-$ metric. The corresponding Lagrangian is just its absolute energy $\varepsilon = \exp(2\alpha(x))F^2(x, y)$.

The non reducibility of some $GL-$ metrics to $L-$ metrics shows that the means provided by the symplectic geometry are not sufficient for the study of the $GL-$ metrics. The metrical approach systematically used in [3] is more powerful in this respect.

Now let be $a(x,y) = \alpha(F^2(x,y))$, $b = \beta(F^2(x,y))$ with $\alpha, \beta : R_+ \longrightarrow R_+^*$. Then (2.1) becomes

$$g_{ij}(x,y) = \alpha(F^2)\gamma_{ij} + \beta(F^2)y_i y_j. \tag{2.3}$$

Proposition 2.6. *The $GL-$ metric (2.3) is reducible to an $L-$ metric if and only if $\beta = 2\alpha'$.*

Proof. For the $GL-$ metric (2.3) the eq. (2.2) is equivalent to $(2\alpha' - \beta)(y_k\gamma_{ij} - y_i\gamma_{kj}) = 0$ or $\beta = 2\alpha'$ since $y_k\gamma_{ij} - y_i\gamma_{kj} \neq 0$. ∎

By the Proposition 2.6 the $GL-$ metric $g_{ij} = \alpha(F^2)\gamma_{ij} + 2\alpha'(F^2)y_i y_j$ is an $L-$ metric but we do not know L. Looking for L in the form $L = \phi(F^2)$, $\phi : R_+ \longrightarrow R_+^*$ we find $\phi' = \alpha$. Thus the $GL-$ metric $g_{ij}(x,y) = \phi'(F^2)\gamma_{ij} + 2\phi''(F^2)y_i y_j$ with $\phi'(t) + t\phi''(t) > 0$ is an $L-$ metric provided by the Lagrangian $L = \phi(F^2)$.
This fact can be also checked by a direct calculation noticing that $2y_i = \dot\partial_i F^2$. The Lagrange spaces with $L = \phi(F^2)$ (called $\phi-$ Lagrange spaces) were studied by P.L. Antonelli and D. Hrimiuc in [2] as Lagrangians generating sprays. The problem to find all Lagrangians generating sprays was posed and a first class of such Lagrangians was studied by M. Anastasiei and P.L. Antonelli in [1].

3. Canonical $N-$linear connection of the $GL-$ metric (2.1).

Let N be the Cartan nonlinear connection of F. Then $N_j^i = \gamma_{jk}^i y^k$, where γ_{jk}^i are the "generalized Christoffel symbols". If one sets $\delta_i = \partial_i - N_i^k\dot\partial_k$ the canonical $N-$linear connection of $g_{ij}(x,y)$ has the local coefficients as follows, [3, Ch. X–XI] :

$$\begin{aligned} F_{jk}^i &= \tfrac{1}{2}g^{ih}(\delta_j g_{hk} + \delta_k g_{hj} - \delta_h g_{jk}), \\ C_{jk}^i &= \tfrac{1}{2}g^{ih}(\dot\partial_j g_{hk} + \dot\partial_k g_{hj} - \dot\partial_h g_{jk}). \end{aligned} \tag{3.1}$$

Let $(\overset{\circ}{F}{}_{jk}^i, \overset{\circ}{C}{}_{jk}^i)$ be the local coefficients of the Cartan connection of F^n. These are obtained from by (3.1) by replacing g_{ij} with γ_{ij}. We note that $\overset{\circ}{C}_{hjk} = \gamma_{hi}\overset{\circ}{C}{}_{jk}^i = \dot\partial_h\dot\partial_j\dot\partial_k F^2/4$ is totally symmetric and $\overset{\circ}{C}_{hjk}y^k = 0$.

We set

$$\begin{aligned} &a = \exp(2\Phi),\ \Phi_k = \delta_k\Phi,\ \Phi^k = \gamma^{ki}\Phi_i,\ \dot\Phi_k = \dot\partial_k\Phi,\ \dot\Phi^k = \gamma^{kj}\dot\Phi_j \\ &b_k = \delta_k b,\ b^k = \gamma^{ki}b_i,\ \dot b_k = \dot\partial_k b,\ \dot b^k = \gamma^{kj}\dot b_j, \\ &\{h, jk\} = (\partial_k\gamma_{hj} + \partial_j\gamma_{hk} - \partial_h\gamma_{jk})/2 \end{aligned} \tag{3.2}$$

By a long calculation one gets

$$\begin{aligned} F_{jk}^i &= \overset{\circ}{F}{}_{jk}^i + \Phi_k\delta_j^i + \Phi_j\delta_k^i - \Phi^i\gamma_{jk} + bg^{ih}A_{hjk} - \tfrac{b}{a+bF^2}E_{jk}^i, \\ C_{jk}^i &= \overset{\circ}{C}{}_{jk}^i + \dot\Phi_k\delta_j^i + \dot\Phi_j\delta_k^i - \dot\Phi^i\gamma_{jk} + \tfrac{y^i}{a+bF^2}[P_j y_k + P_k y_j + \\ &\quad + \gamma_{jk}(1 + \dot\Phi_0) + \tfrac{b\dot b_0}{2a}y_j y_k] \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &E_{jk}^i = y^i(\{0, jk\} + \gamma_{00}^s\overset{\circ}{C}_{sjk} + B_k y_j + B_j y_k - \Phi_0\gamma_{jk} - \tfrac{b}{2a}y_j y_k), \\ &A_{hjk} = [\delta_k(y_h y_j) + \delta_j(y_h y_k) - \delta(y_j y_k)]/2. \\ &B_k = \Phi_k + \tfrac{b_k}{2a}F^2, \\ &P_j = \tfrac{\dot b_j}{2} - b\dot\Phi_j \end{aligned}$$

and the upper script "0" means the contraction by (y). The formulae (3.2) simplify when a and b are $h-$ covariant constant, that is, $\delta_k a = 0$ and $\delta_k b = 0$ or a and b do not depend on y or when the both situation hold, that is, a and b are constant. In the latter case we get

$$\begin{aligned} F_{jk}^i &= \overset{\circ}{F}{}_{jk}^i + bg^{ih}A_{hjk} - \tfrac{b}{a+bF^2}y^i(\{0, jk\} + \gamma_{00}^s\overset{\circ}{C}_{sjk}), \\ C_{jk}^i &= \overset{\circ}{C}{}_{jk}^i + \tfrac{1}{a+bF^2}y^i\gamma_{jk}. \end{aligned} \tag{3.4}$$

Further on, the torsions and the curvatures of the canonical connection (3.2) could be determined. The calculation simplifies if the idea of transformation of Finsler connections is used, cf. [3, Ch. X–XI]

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