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FIELD CONES AND POSITIVE OPERATORS ON REGULAR VECTOR BUNDLES

BY

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Introduction. This is a survey paper concerning the results obtained in the research (made between 1988–1995) of a structure formed by a regular vector bundle (E, p, M) endowed with a field K of cones. This structure is denoted by $[(E, p, M); K]$. The topics considered are:

— the geometry of an arbitrary structure $[(E, p, M); K]$ (the geometry of a locale fibre E_x, the global geometry of (E, p, M), the geometry of the set of global sections of (E, p, M)).

— the geometry of the set of operators from a structure $[(E, p, M); K_1]$ to a structure $[(E', p', M); K_2]$ (the set of positive operators and structure $(\Omega/\Omega_0; \boldsymbol{K})$; ordering, norms and distances for this last structure),

— the geometry of natural lifted structure of a tensor bundle associated to a structure $[(E, p, M); K]$,

— some applications of these results.

A regular bundle *is a vector bundle (E, p, M) for which E and M are real topological paracompact connected without boundary manifolds and* $\dim M = n, \dim E = n + m$.

A field of cones *is a map K,*

$$K : x \in M \to K(x) \subset E_x \subset E, \ (E_x = p^{-1}(x)).$$

for which are satisfied the following two axioms:

A_1. $K(x) \subset E_x$, $(\forall x \in M)$ *is a convex pointed closed with interior points cone (in the topological vector space* E_x).

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A_2. *The sets* $\cup \mathrm{Int}\, K(x)$ *and* $\cup(E_x - K(x))$ *are open subsets of the topological manifold* E.

The structure formed by a regular vector bundle (E, p, M) endowed with a field of cones K will be denoted by $[(E, p, M); K]$.

Remarks. 1. A regular vector bundle (E, p, M) has a field of cones if and only if there exists for (E, p, M) a continuous global section of non zero vectors ζ, that is,

$$\zeta : M \to E,\ p \circ \zeta = \mathrm{id}_M,\ \zeta(x) \neq 0,\ (\forall x \in M).$$

2. A good example of $[(E, p, M); K]$ is $[(TM, p, M); K]$, where M is a time-oriented Lorentzian manifold and $K(x)$ $(\forall x \in M)$ is the quadratic cone of non spacelike time oriented tangent vectors of M.

The study of a structure $[(E, p, M); K]$ is suggested by the existence of time-oriented Lorentzian manifolds and justified by a certain necessity of globalisation of the results obtained by Mark Krasnosel'skij and his group of mathematicians in their study of positive operators (see the monographs [6], and [7], 1989).

The first paper where a vector bundle endowed with a field of cones appeared is probably that of Dennis Sullivan ([15]). From 1988, the geometry of a differentiable manifold endowed with a field of tangent cones was studied by Dan I.Papuc, ([8]-[10]). The general case of structure $[(E, p, M)]$ was studied for the first time in [11] and [12]. The general lines of a new geometry of a time-oriented Lorentzian manifold were established by Dan I.Papuc [13]. Liana David studied operators on a vector bundle endowed with an homogeneous $n-$ *hedral* cone-field in [2]. Some spectral properties of positive linear operators on a vector bundle endowed with an arbitrary cone-field were established by Liana David in [3]. In [14] it was established a general theory of positive operators between two regular vector bundles endowed with fields of cones. We present here a survey of all these results.

1. The general theory of a structure $[(E, p, M); K]$.

1.1. The geometry of a local fibre of the structure $[(E, p, M); K]$.

The results of this section are taken from [5] and [11].

The pair $(E_x; K(X)), \forall x \in M$, is *a Krein space*. Hence it follows:

a). There is *an ordering relation* on E_x:

$$X \leq Y \Leftrightarrow Y - X \in K(x),\ (X, Y \in E_x).$$

The pair $(E, \leq)$ is *an ordered topological vector space*.

b). For every $Z \in \mathrm{Int}\, K(x)$, E_x is Z–*measurable* i.e.

$$\forall X \in E_x,\ \exists \lambda \in \boldsymbol{R}\, \lambda \geq 0, -\lambda Z \leq X \leq \lambda Z.$$

E_x has *a norm* determined by $K(x)$ and a fixed $Z \in \mathrm{Int}\, K(x)$:

$$|\ |_Z : X \in E_x \to |X|_Z \in \boldsymbol{R},$$

$$|X|_Z = \min\{\lambda | \lambda \in \boldsymbol{R}, \lambda \geq 0, -\lambda Z \leq X \leq \lambda Z\}.$$

This norm is *monotone* $(0 \leq X \leq Y \Rightarrow 0 \leq |X|_Z \leq |Y|_Z)$ and directed on both sides.

c). *The open-balls of* E_x *in the norm* $|\ |_Z$ and *the open ordered intervals in* $\leq$, both determined by the same $X_0 \in E_x, \varepsilon \in \boldsymbol{R}, \varepsilon > 0$ and $Z \in \mathrm{Int}\, K(x)$ coincide i.e.

$$B(X_0, Z, \varepsilon) = (X_0 - \varepsilon Z, X_0 + \varepsilon Z),$$

where

$$B(X_0, Z, \varepsilon) = \{X,\ |X - X_0|_Z < \varepsilon\}$$

and

$$(X_0 - \varepsilon Z, X_0 + \varepsilon Z) = \{X \mid X \in E_x, \exists \varepsilon_1 < \varepsilon, X_0 - \varepsilon_1 Z \leq X \leq X_0 + \varepsilon_1 Z\}.$$

If for an arbitrary $X \in E_x$ we put

$$K_{X+} = X + K(x) = \{Y \mid X \leq Y\};\ K_{X-} = X - K(x) = \{Y \mid Y \leq X\}$$

then

$$B(X, Z, \varepsilon) = (X - \varepsilon Z, X + \varepsilon Z) = (\mathrm{Int} K_{(X-\varepsilon Z)+}) \cap (\mathrm{Int} K_{(X+\varepsilon Z)-}).$$

The three topologies of E_x, the first determined by the structure (E, p, M), the second by the norm $|\ |_Z$ and the third by the open ordered intervals, coincide.

d). Other properties of the Krein space $(E_x, K(x))(E_x$ being finite dimensional) are:

— the cone $K(x)$ is a *generating set* for E_x i.e.

$$\forall X \in E, \exists Z_1, Z_2 \in \mathrm{Int}\, K(x) | X = Z_1 - Z_2.$$

— the cone $K(x)$ is a *normal cone* i.e.

$$\delta_x = \min\{|(X/|X|_Z + (Y/|Y|_Z)|_Z, X, Y \in K(X), X \neq 0, Y \neq 0\} > 0.$$

$$(\forall Z \in \operatorname{Int} K(x)).$$

— the cone $K(x)$ is *a complete regular cone*, i.e. every non-decreasing upper bounded sequence of E_x is convergent in the ordered topological vector space E_x.

1.2. Global properties of a structure $[(E,p,M):k]$. The result from this section are taken from [11], p.43–49. We remark that the relations (23) from [11], p.47 are not true (see [12]).

a) Two topological properties of a structure $[(E,p,M):k]$:
— $\bigcup(\operatorname{Int} K(x)) = \operatorname{Int}(\bigcup K(x))$.
— If $\{X_{x_n}\} \to X_{x_0}$ and $(\forall n \in N)$. $0 \leq X_{x_n}$ then $0 \leq X_{x_0}$.

b) A very important result is the following theorem ([11], p.44):

Theorem. *If it is considered an arbitrary structure* $[(E,p,M):k]$, *then there exists a continuous global section* ζ *of* (E,p,M) *such that the vector* $\zeta(x)$ *is interior to the cone* $K(x)(\forall x \in M)$ *i.e.*

$$\zeta : M \to E,\ p \circ \zeta = \mathrm{id}_M,\ \zeta(x) \in \operatorname{Int} K(x).\ (\forall x \in M).$$

c) A main tool in study of a structure $[(E,p,M);K]$ is a function

$$\nu : (Z,X) \in \bigcup \operatorname{Int} K(x) \times E_x \longrightarrow \nu(Z,X) \in \boldsymbol{R}^2,$$

where $\nu(Z,X) = (\alpha(Z,X), \beta(Z,X))$ and
— $\alpha(Z,X) = \min\{\lambda | \lambda \in \boldsymbol{R}, Y \leq \lambda Z\}$,
— $\beta(Z,X) = \max\{\lambda | \lambda \in \boldsymbol{R}, \lambda Z \leq X\}$.
The function ν has the following **I–IV fundamental properties**:

$$\forall (Z,X), (Z,Y) \in \cup(\operatorname{Int} K(x) \times E_x),$$

I. $\forall \lambda \in \boldsymbol{R}, \lambda > 0|\ \nu(Z,\lambda X) = \lambda \cdot \nu(Z,X)$, $\nu(\lambda Z, X) = (1/\lambda) \cdot \nu(Z,X)$.
II. $\beta(Z,X) + \beta(Z,Y) \leq \beta(Z,X+Y) \leq \alpha(Z,X+Y) \leq \alpha(Z,X) + \alpha(Z,Y)$.
III. $\nu(Z,Z) = (1,1)$, $\nu(Z,X) = (0,0) \Rightarrow X = 0$.
IV. $\forall \lambda \in \boldsymbol{R}|\alpha(Z,\lambda Z - X) = \lambda - \beta(Z,X)$, $\beta(Z,\lambda Z - X) = \lambda - \alpha(Z,X)$.
The map ν is continuous.

Fundamental theorem. *Let be given*
— *a regular vector bundle* (E,p,M),
— *a continuous global section* ζ *of* (E,p,M) *such that* $\zeta(x) \neq 0(\forall x \in M)$,
— *a continuous function* $v_\zeta : E \longrightarrow \boldsymbol{R}^2$ *satisfying the conditions obtained from the fundamental properties I–IV of* v *substituting there* $v(\zeta(p(X)), X)$ *by* $v_\zeta(X)$.

The three elements $(E,p,M), \zeta, v_\zeta$, *uniquely determine a structure* $[(E,p,M);K]$, *where* $\forall x \in M | K(x) = \{X \in E_x | \beta_\zeta(X) \geq 0\}$. *If in* E_x *we consider the ordering* $X \leq Y \Leftrightarrow Y - X \in K(x)$, *then* $\alpha_\zeta(X) = \min\{\lambda | \lambda \in \boldsymbol{R}, X \leq \lambda\zeta(x)\}$, $\beta_\zeta(X) = \max\{\lambda | \lambda \in \boldsymbol{R}, \lambda\zeta(x) \leq X\}$.

d) Some important relations: $\forall x \in M$, $\forall X, Y \in E_x$, $Z \in \operatorname{Int} K(x)|$
— $X \in K(x) \Leftrightarrow \beta(Z,Y) \geq 0$;
— $\beta(Z,X) \leq \alpha(Z,X)$, $\alpha(Z,X) = -\beta(Z,-X)$;
— $X - \beta(Z,X)Z \in Fr K(x)$, $\alpha(Z,X)Z - X \in Fr K(x)$;
— $Y \in B(X,Z,\varepsilon) = (X - \varepsilon Z, X + \varepsilon Z) \Leftrightarrow$

$$\exists \varepsilon_1 \in \boldsymbol{R}, 0 \leq \varepsilon_1 < \varepsilon |\ \alpha(Z,X) - \varepsilon_1 \leq \alpha(Z,Y) \leq \alpha(Z,X) + \varepsilon_1,$$

$$\beta(Z,X) - \varepsilon_1 \leq \beta(Z,Y) \leq \beta(Z,X) + \varepsilon_1;$$

— $\nu(Z,X) = (\lambda,\lambda) \Leftrightarrow X = \lambda Z$; $(\forall \lambda \in \boldsymbol{R})$;
— $X \leq Y \Leftrightarrow \alpha(Z,X) \leq \alpha(Z,Y), \beta(Z,X) \leq \beta(Z,Y)$;
— $|X|_Z = \max\{|\alpha(Z,X)|, |\beta(Z,X)|\}$.

e) If for the structure $[(E,p,M);K]$ it is given a continuous global section ζ of (E,p,M) such that $\forall x \in M | \zeta(x) \in \operatorname{Int} K(x)$, then we can define the function

$$|\,|_\zeta : X_x \in E_x \to |X_x|_\zeta \in \boldsymbol{R},$$

where

$$|X_x|_\zeta = |X_x|\zeta(x) = \max\{|\alpha(\zeta(x), X_x)|, |\beta(\zeta(x), X_x)|\}.$$

For every $x \in M$, the restriction $|\,|_\zeta |E_x$ of the function $|\,|_\zeta$ is the norm $|\,|_{\zeta(x)} : E_x \to \boldsymbol{R}$ above considered ((1.1.b)).

1.3. The space of global sections of a structure $[(E,p,M);K]$. The results of this section are taken from [11], p.49–53.

a) The set of all global sections of the vector bundle (E,p,M) is

$$\Sigma = \{\sigma | \sigma : M \to E, p_\circ\sigma = \mathrm{id}_M, \sigma \text{ is } \boldsymbol{C}^0\}.$$

Σ is a real vector space and a module over ring $\Phi = \{f | f : M \to \mathbf{R}, f$ is $C^0\}$.

For Σ it is used the WO^0-topology (the C^0-Whitney open topology). The set Σ with WO^0-topology is a topological additive abelian group.

The topological space $(\Sigma, WO^0\text{-}top.)$ is a real topological vector space if and only if M is a compact manifold.

b) A positive global section of the structure $[(E,p,M);K]$ is an element σ of Σ for which $\sigma(x) \in K(x), (\forall x \in M)$.

The set K_Σ of all positive global sections of $[(E,p,M);K]$ is a *convex pointed closed* with *interior points cone* of the real vector space Σ, endowed with the *WO^0-topology.*

We obtained this way a structure $(\Sigma; K_\Sigma)$. The fundamental 1. properties of this structure were studied in [11]. are essentially different if M is compact or not. $(\Sigma; K_\Sigma)$ is a Krein space if and only if M is a compact space. We remark:

$$\mathrm{Int}K_\Sigma = \{\sigma | \sigma \in \Sigma, \ \sigma(x) \in \mathrm{Int}K(x), \forall x \in M\}.$$

c) Some properties of the structure $(\Sigma; K_\Sigma)$:

1. K_Σ is a generating cone for Σ, i.e.

$$\forall \sigma \in \Sigma; \exists \zeta_1, \zeta_2 \in \mathrm{Int}K_\Sigma | \sigma - \zeta_1 - \zeta_2.$$

2. $\sigma_1 \leq \sigma_2 \Leftrightarrow \sigma_2 - \sigma_1 \in K_\Sigma; \ (\sigma_1, \sigma_1 \in \Sigma)$.

$$\sigma_1 \leq \sigma_2 \Leftrightarrow \sigma_1(x) \leq \sigma_2(x), (\forall x \in M).$$

$(\Sigma; \leq)$ is an ordered vector space.

3. For an element $\zeta \in \mathrm{Int}K_\Sigma, \Sigma_\zeta$ will be the set of all ζ-measurable elements of Σ, i.e.

$$\Sigma_\zeta = \{\sigma | \sigma \in \Sigma, \exists \lambda \in \mathbf{R}, \lambda \geq 0, -\lambda\zeta \leq \sigma \leq \lambda\zeta\}.$$

We shall have:

3_1. $\Sigma_\zeta = \Sigma$ if and only if M is compact,

3_2. Σ_ζ has a norm $|\ |_\zeta : \sigma \in \Sigma \to |\sigma|_\zeta \in \mathbf{R}$, where

$$|\sigma|_\zeta = \min\{\lambda | \lambda \in \mathbf{R}, \lambda \geq 0, -\lambda\zeta \leq \sigma \leq \lambda\zeta\}.$$

3_3. If Σ_c is the set of all sections with compact support then

$$\Sigma_c \subset \Sigma_\zeta. \ (\forall \zeta \in \mathrm{Int}K).$$

3_4. The set $\{\Sigma_\zeta | \zeta \in \mathrm{Int}K_\Sigma\}$ is a covering of Σ.

4. For $\zeta \in intK_\Sigma$ it is defined the equivalence relation on Σ:

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1 - \sigma_2 \in \Sigma_\zeta, \ (\sigma_1, \sigma_2 \in \Sigma).$$

5. The norm $|\ |_\zeta$ of Σ_ζ determines a distance d_ζ on every equivalence class $\zeta + \Sigma_\zeta$:

$$\forall \sigma_1, \sigma_2 \in \sigma + \Sigma_\zeta | d_\zeta(\sigma_1, \sigma_2) = |\sigma_1 - \sigma_2|_\zeta.$$

6. **Theorem:** *If ζ is an arbitrary fixed element of $\mathrm{Int}K_\Sigma$, then*

a) for every equivalence class $\sigma + \Sigma_\zeta$, the metric topology determined by d_ζ and the topology of open ordered in $\geq$ determined by ζ intervals coincide.

b) the WO^0-topology of Σ is the topology for which a base of neighborhoods is the set of all open balls $\{B(\sigma, \zeta\varepsilon)\}$ defined by means of the distance $d\zeta$,

c) the equivalencies classes $\sigma + \Sigma_\zeta$ are closed and open subsets of the topological space $(\Sigma, WO^0\text{-}top)$.

If we consider only algebraic structure of linear space on Σ and the cone K_Σ. then we can define *open ordered intervals* $(\sigma_0 - \varepsilon\zeta, \sigma_0 + \varepsilon\zeta)$ for $\forall \sigma \in \Sigma, \forall \varepsilon \in \mathbf{R}, \varepsilon > 0, \forall \zeta \in \Sigma, \zeta(x) \in \mathrm{Int}K(x)$. The topology defined by these open ordered intervals is exactly the WO^0-topology of Σ.

2. Operators for a structure $[(E,p,M);K]$. All results of the following three sections (2. 1-2-3), are taken from [4].

2.1. The cone of positive operators. The structure $(\Omega/\Omega_0; K)$.

a) An operator from a vector bundle (E,p,M) to a vector bundle $(E'.p',M)$ is a continuous map $A : E \to E'$ for which $A(p^{-1}(x)) \subset p'^{-1}(x)$. $(\forall x \in M)$. We shall denote the set of all these operators by Ω. So: $\Omega = \{A | A : E \to E', A \textit{ is continuous}, \ A(p^{-1}(x)) \subset p'^{-1}(x), \ \forall x \in M)\}$.

The set Ω of all operators between the regular vector bundles $(E,p,M).(E',p',M)$ is a real vector space and a module over the ring of real continuous functions defined on M or on E.

Some particular sets of operators from (E,p,M) to (E',p',M), i.e. subsets of Ω:

— the linear subspace of Ω, denoted by Ω_L, of **linear operators** (the morphisms from the vector bundle (E,p,M) to the vector bundle (E',p',M)),

— the linear subspace of Ω, denoted by $\Omega_{\Sigma'}$, of **section-operators**. Every such operator is defined by a global section of (E',p',M). If $\sigma' \in \Sigma', \sigma' : M \to E', p' \circ \sigma' = \mathrm{id}_M$, then the section-operator $A_{\sigma'}$ determined by σ' will be $A_{\sigma'} : E \to E'$, $A_{\sigma'}(X) = \sigma'(p(X))$ for $\forall X \in E$, i.e. $A_{\sigma'} = \sigma' \circ p$.

Obviously $\Omega_L \cap \Omega_{\Sigma'} = \{A_0\}$ (A_0 is the constant operator determined by the trivial global section of (E',p',M)).

b) Let be two structures $[(E,p,M);K_1]$ and $[(E',p',M);K_2]$.

A positive operator from the structure $[(E,p,M);K_1]$ to the structure $[(E',p',M);K_2]$ is an operator $A : E \to E'$ such that $A(K_1(x)) \subset K_2(x)$, $\forall x \in M$. The subset of all positive operators of Ω will be denoted by K_Ω. So:

$$K_\Omega = \{A | A \in \Omega, A(K_1(x)) \subset K_2(x), (\forall x \in M)\}.$$

The set $K - \Omega$ of all positive operators from $[(E,p,M);K_1]$ to $[(E,p,M);K_2]$ is a convex generating cone of Ω.

c) Among the positive operators from the structure $[(E,p,M);K_1]$ to the structure $[(E,p,M);K_2]$ there are considered the operators A which satisfy the relation $A(K_1(x)) = 0 \in K_2(x), \forall x \in M$. This subset of Ω is denoted by Ω_0. So:

$$\Omega_0 = \{A | A \in \Omega, A(K_1(x)) = 0 \in K_2(x), \forall x \in M\}.$$

Ω_0 is a linear subspace of Ω and $\Omega_0 \subset K_\Omega, \Omega_L \cap \Omega_0 = \Omega_{\Sigma'} \cap \Omega_0 = \{A_0\}$.

By means of structures Ω and Ω_0 it is considered **the quotient linear space**

$$\Omega/\Omega_0 = \{A + \Omega_0 | \forall A \in \Omega\}.$$

An arbitrary element of $\Omega/\Omega_0, A + \Omega_0$, will be denoted by $[A]$.

The maps $A \in \Omega_L \to [A] \in \Omega/\Omega_0$ and $A_{\sigma'} \in \Omega_{\Sigma'} \to [A_{\Omega'}] \in \Omega/\Omega_0$ are injective. Also, $[A_1] = [A_2]$ if and only if $A_1(X) = A_2(X), \forall X \in K_1(p(x))$.

The last remark permit to consider the following important subset of Ω/Ω_0:

$$\boldsymbol{K} = \{[A] | A \in K_\Omega\}.$$

Obviously, $[A] \in \boldsymbol{K}$ if and only if $A \in \Omega, A(K_1(x)) \subset K_2(x)$, $(\forall x \in M)$.

The set $\boldsymbol{K}$ is a convex pointed generating cone of the linear space Ω/Ω_0.

In virtue of a remark made above, it is given the following definition:

It is called the interior of the cone $\boldsymbol{K}$ *and it is denoted by* $\mathrm{Int}\boldsymbol{K}$ *the subset of* Ω/Ω_0 *defined by*

$$\mathrm{Int}\boldsymbol{K} = \{[B] | [B] \in \Omega/\Omega_0, \ B(\mathrm{Int}K_1(x)) \subset \mathrm{Int}K_2(x), \forall x \in M\}.$$

Obviously, $\mathrm{Int}\boldsymbol{K} \subset \boldsymbol{K}$.

2.2. Ordering relation for the structure $(\Omega/\Omega_0; \boldsymbol{K})$

a) For the linear space Ω/Ω_0, it is defined, by means of the cone $\boldsymbol{K}$, an ordering relation [5]:

$$[A_1], [A_2] \in \Omega/\Omega_0 | [A_1] \leq [A_2] \Leftrightarrow [A_2] - [A_1] \in \boldsymbol{K}.$$

Obviously $[A_2] - [A_1] \in \boldsymbol{K} \Leftrightarrow [A_2 - A_1] \in \boldsymbol{K} \Leftrightarrow \forall X \in K_1(x) | (A_2 - A_1)(X) \in K_2(x) \Leftrightarrow \forall X \in K_1(x) | A_1(X) \leq A_2(X)$ (in the ordering defined by K_2 for the structure $[(E',p',M);K_2]$).

The pair $(\Omega/\Omega_0; \leq)$ is an ordered vector space, directed on both sides.

If $[B_1], [B_2] \in \mathrm{Int}\boldsymbol{K}$ then there exists two others elements $[B_3], [B_4] \in \mathrm{Int}\boldsymbol{K}$ such that $[B_3] \leq [B_1], [B_2] \leq [B_4]$.

b) The above ordering relation defined on the linear space Ω/Ω_0 permits us to determine topological structures of Ω/Ω_0.

An ordered open interval centered in $[A*] \in \Omega/\Omega_0$, **determined by an element** $[B] \in \mathrm{Int}\boldsymbol{K}$, is a set

$$\ll [A*] - \varepsilon[B], [A*] + \varepsilon[B] \gg := \{[A] \in \Omega/\Omega_0 | \exists \varepsilon, \varepsilon_1 \in \boldsymbol{R},$$

$$0 < \varepsilon_1 < \varepsilon : [A*] - \varepsilon_1[B] \leq [A] \leq A*] + \varepsilon_1[B]\}.$$

The set of all ordered open intervals determined by the same element $[B] \in \mathrm{Int}\boldsymbol{K}$ is a base of a topology of Ω/Ω_0, denoted $\tau_{[B]}$.

2.3. Norms and distances for the structure $(\Omega/\Omega_0; \boldsymbol{K})$

a) An element $[A] \in \Omega/\Omega_0$ is called $[B]$-**measurable**, $[B] \in \mathrm{Int}\boldsymbol{K}$, if there is $\lambda \in \boldsymbol{R}, \lambda > 0$ such that $-\lambda[B] \leq [A] \leq \lambda[B]$.

There are elements $[A] \in \Omega/\Omega_0$ which are not $[B]$-**measurable**. For example, if $f : E_1 \to \boldsymbol{R}$ is a continuous positive function, unbounded on a certain $\mathrm{Int}K_1(x)$, then the element $[fB]$ is not $[B]$-**measurable.**

The set of all elements of Ω/Ω_0 which are $[B]$-**measurable** will be denoted by $\Delta_{[B]}$. This set is a real linear subspace of Ω/Ω_0.

The set of all linear spaces $\{\Delta_{[B]} | [B] \in \mathrm{Int}\,\boldsymbol{K}\}$ is a covering of Ω/Ω_0.

b) On the set $\Delta = \{\Delta_{[B]} \in \mathrm{Int}\,\boldsymbol{K}\}$ it is defined the relation

$$\Delta_{[B']}, \Delta_{[B'']} \in \Delta.\ \Delta_{[B']} \le \Delta_{[B'']} :\Rightarrow \Delta_{[B']} \subset \Delta_{[B'']}.$$

Obviously $\Delta_{[B']} \subset \Delta_{[B'']} \Leftrightarrow [B'] \in \Delta_{[B'']} \Leftrightarrow \exists \lambda \in \boldsymbol{R}, \lambda > 0, [B'] \le \lambda[B'']$.

The relation $\le$ defined above is an ordering relation directed on both sides.

c) On the set $\Delta_{[B]}$, ($[B] \in \mathrm{Int}\,\boldsymbol{K}$), it is considered the map $\|\ \|_{[B]} : \Delta_{[B]} \to \boldsymbol{R}$, where

$$(\forall [A] \in \Delta_{[B]}),\ \|[A]\|_{[B]} = \min\{\lambda | \lambda \in \boldsymbol{R}, \lambda \ge 0, -\lambda[B] \le [A] \le \lambda[B]\}.$$

This map is a monotone norm of the linear space $\Delta_{[B]}$.

d) By means of the linear subspace $\Delta_{[B]} \subset \Omega/\Omega_0$ it is considered the linear factor space $(\Omega/\Omega_0)/\Delta_{[B]}$. An element of this linear factor space determinate by an arbitrary $[A]$ ($[A] \in \Omega/\Omega_0$) will be the class of equivalence $[A] + \Delta_{[B]}$.

For every class $[A] + \Delta_{[B]}$ we consider the map

$$d_{[B]} : ([A] + \Delta_{[B]}) \times ([A] + \Delta_{[B]}) \to \boldsymbol{R},\ d_{[B]}([A] + [A'], [A] + [A'']) =$$

$$= \|[A'] - [A'']\|_{[B]} \in \boldsymbol{R}.$$

The function $d_{[B]}$ is a distance on $[A] + \Delta_{[B]}$.

3. Field of cones on the associated tensor bundles. All results of the following four sections (3. 1–2–3–4) are taken from [12].

3.1. The structure $[(E_{p^q}, \pi, M); K_{p^q}]$

a) For the vector bundle (E, p, M), it is considered the dual vector bundle (E^*, p^*, M) where $E^* = \{\xi_x | \xi_x : E_x \to \boldsymbol{R}, \forall x \in M\}$, ξ_x being a linear map.

If for (E, p, M) the field K of cones exists and it is given, then for (E^*, p^*, M) it is also defined a field of cones K^*. In this case K^* is the map $K^* : x \in M \to K*(x) \subset E*_x \subset E*$ where

$$K^*(x) = \{\xi_x | \xi_x \in E^*_x, \xi_x(X_x) \ge 0, \forall X_x \in K(x)\}.$$

The pair $[E^*, p^*, M); K^*]$ satisfies the axioms A_1 and A_2.

A linear functional $\zeta_x \in E^*_x$ belongs to $\mathrm{Int}K^*(x)$ if and only if $\zeta_x(X_x) > 0$ for every $X_x \in K(x), X_x \neq 0$.

Let (E_{p^q}, π, M) the tensor bundle of tensors of type (p^q). A (p^q) tensor t_x is a polylinear (with regard to $\boldsymbol{R}$) map $t_x : \underbrace{E_x \times E_x \times \ldots \times E_x}_{p \text{ times}} \times \underbrace{E^*_x \times E^*_x \times \ldots \times E^*_x}_{q \text{ times}} \to \boldsymbol{R}$. The space E_{p^q} is defined by $E_{p^q} = \{t_x | t_x \textit{ is a type } (p^q) \textit{ tensor}, \forall x \in M\}$. By means of the fields of cones K and K^* it is defined a field K_{p^q} of cones on the tensor bundle (E_{p^q}, π, M). K_{p^q} is the map $K_{p^q} : x \in M \to K_{p^q}(x) \subset E_{p^q x} \subset E_{p^q}$ where:

$$K_{p^q}(x) = \{t_x | t_x \in E_{p^q_x}, t_x(X_1, X_2, \ldots, X_p, \xi^1, \xi^2, \ldots, \xi^q) \ge 0,$$

$$\forall X_i \in K(x), \forall \xi^j \in K^*(x);\ i = 1, \ldots, p; j = 1, \ldots, q\}.$$

The $\mathrm{Int}K_{p^q}(x)$ in the topological space $E_{p^q_x}$ is the set

$$\mathrm{Int}K_{p^q}(x) = \{t_x | t_x \in E_{p^q_x}, t_x(X_1, X_2, \ldots, X_p, \xi^1, \xi^2, \ldots, \xi^q) \ge 0,$$

$$\forall X_i \in \boldsymbol{K}(x), \forall \xi^j \in \boldsymbol{K}^*(x);\ i = 1, \ldots, p; j = 1, \ldots, q\}.$$

The pair $[(E_{p^q}, \pi, M); K_{p^q}]$ satisfies the axioms A_1 and A_2.

In the following we shall put $[(E, p, M); K] = [(E^1_0, \pi, M); K^1_0]$ and $[(E^*, p^*, M); k] = [(E^0_1, \pi, M); K^0_1]$.

The geometry of a pair $[(E_{p^q}, \pi, M); K_{p^q}]$ is the geometry of a regular vector bundle endowed with a field of cones, (i.e. a structure $[(E, p, M); K]$), the geometry that was exposed in [12]. The most important properties of this geometry was presented in § 1 of this paper.

3.2. The heredity with respect to associated tensors bundles. If a structure $[(E, p, M); K]$ is given then there are associated to it the structures $[(E^*, p^*, M); K^*]$ and $[(E_{p^q}, \pi, M); K_{p^q}], (\forall p, q \in \boldsymbol{N})$, which are uniquely determined. If for the first two structures, $[(E, p, M); K]$ and $[(E^*, p^*, M); K^*]$, are fixed two vectors fields, Z and ζ, such that $Z(x) \in \mathrm{Int}K(x), \forall x \in M)$ and $\zeta(x) \in \mathrm{Int}K^*(x), (\forall x \in M$ then

$$\Xi = \underbrace{\zeta \otimes \zeta \otimes \ldots \otimes \zeta}_{p \text{ times}} \otimes \underbrace{Z \otimes Z \otimes \ldots \otimes Z}_{q \text{ times}},$$

(where $\otimes$ is the tensorial product) is a section of the tensor bundle (E_{p^q}, π, M) so that $\forall x \in M | \Xi(x) \in \mathrm{Int}K_{p^q}(x)$.

The last remark shows that **the structure** $[(E,p,M):K]$ **endowed with a continuous interior vector field** Z **and a continuous linear functional field** ξ, $(\forall x \in M|\xi(x): E_x \to \mathbf{R}, \xi(x)$ is a linear map and $\xi(x)(X_x) > 0$ for every $X_x \in K(x), X_x \neq 0)$ determines in a natural manner a unique structure $[(E_{pq}, \pi, M); K_{pq}]$ and a continuous interior tensor field Ξ. All these prove the "hereditary power" of the properties of the initial structure.

3.3. Global sections of a tensor bundle endowed with a field of cones. We consider the set of all global sections of the tensor bundle (E_{pq}, π, M):

$$\Sigma = \{\sigma | \sigma :: M \to E_{pq}, \pi_o\sigma = \mathrm{id}_M, \sigma \text{ is continuous}\}.$$

Σ is a real vector space and a module over the ring $\Phi = \{f|f: M \to \mathbf{R}, f \text{ is } C^0\}$.

For Σ we shall use the WO^0–topology (the C^0 – Whitney open topology). The set Σ with WO^0–topology is a topological additive abelian group. The topological space $(\Sigma, WO^0 - top.)$ is a real topological vector space if and only if M is a compact manifold.

A positive global section of the pair $[(E_{pq}, \pi, M); K_{pq}]$ is an element σ of Σ for which $\sigma(x) \in K_{pq}(x), (\forall x \in M)$. The set $K_{pq\Sigma}$ of all positive global sections of $[E_{pq}, \pi, M); K_{pq}]$ is a convex pointed closed with interior points cone of the real vector space Σ, endowed with the WO^0-*topology*. The fundamental properties of the structure $(\Sigma; K_{pq}\Sigma)$ were studied in [6]. They are essential different if M is compact or not.

3.4. Positive operators on tensor bundles endowed with fields of cones. A positive operator in this context was defined in [6]. For two regular vector bundles over the same base space M, (E,p,M) and (E',p',M), endowed with the fields of cones K, respectively K', a positive operator is a map (linear or not linear) $\Phi: E \to E'$ so that $p' \circ \Phi = p, \Phi(K(x)) \subset K'(x)$. To every positive operator Φ it is associated a positive operator for the vector spaces of global sections, $\Phi_\Sigma: (\Sigma, K_\Sigma) \to (\Sigma', K'_\Sigma)$, determined by the relations:

$$\Phi_\Sigma : \sigma \in \Sigma \to \Phi_\Sigma(\sigma) = \sigma' \in \Sigma', \text{ where } \sigma' = \Phi(\sigma) = \Phi_o\sigma.$$

The fundamental properties of arbitrary positive operators were established in [6].

For the case of tensors tangent to a differentiable manifold endowed with a field of tangent cones, some positive operators between associated tensor bundle were defined in [11] p.56–57, by means of the tensorial operations: tensorial multiplication and addition (determined by means of global positive sections of some appropriate tensor bundles).

4. Applications.

4.1. Field of tangent cones to smooth manifold. A structure $[(E,p,M);K]$, where M is a C^∞-diff. manifold and (E,p,M) is the tangent vector bundle (TM,p,M) of M, was considered in [8] and [9]. Obviously, the geometry of a structure $[(TM,p,M);K]$ contains more new facts than the geometry of a general structure $[(E,p,M);K]$.

a). In [8] it was defined a regular partial ordering on a smooth manifold $(M, \leq)$. (Definition 5, p.63) and the connections of this structure with a structure $[(TM,p,M);K]$ were established.

For a structure $[(TM,p,M);K]$, an oriental curve of M is a map $x: t \in [a,b] \subset \mathbf{R} \to x(t) \in M, (a < b)$ which is injective, continuous, piecewise C^1–diff. and every tangent vector $dx(t)/dt$ belongs to $K(x(t))$. The point $x(a)$ is the start point and the point $x(b)$ is the endpoint of this curve.

The field of tangent cones K is oriented if the following condition is satisfied: every map $x: t \in [a,b] \subset \mathbf{R} \to x(t) \in M$ which is continuous, piecewise C^1–diff., with non zero tangent vectors, every tangent vector $dx(t)/dt$ belongs to $K(x(t))$, **is an injective map**; (Definition 6, p.64,[8]).

If for a structure $[(TM,p,M),K]$ the field of cones K is an oriented field, then for M there exists an ordering ≤ 0 of its points i.e. $x \leq y$ if $x = y$ or if $x \neq y$ then there is an oriented curve for which x is the start point and y is the endpoint, (Theorem 4, p.6,[8]).

For the obtained structure $(M, \leq)$ an oriented curve of $[(TM,p,M),K]$ is an ordered curve (p.63,[8]).

b). If we suppose that for the structure $[(TM,p,M),K]$ a C^∞–diff. global section ζ of interior to $K(x)(\forall x \in M)$ vectors is given (see 1.2.b) then for every smooth curve c of M, we can consider **a canonical parameter**. If the curve c is determined in a local admissible chart of M by $x = x(t)$ then a canonical parameter of c is determined by the relation: s is a canonical parameter of c if $|dx/ds|\zeta_{(x(s))} = 1$ ([9], p.48).

The canonical parameters of a curve permit us to consider **the length** of this curve:

$$L_\zeta(c; s_1, s_2) = |s_2 - s_1|,$$

otherwise

$$L_\zeta(c; t_1, t_2) = \left|\int |dx/dt|\zeta_{(x(t))} dt\right|,$$

where $x = x(t), s_1 = x(t_1), s_2 = x(x_2)$.

Using the length of curves of a structure $[(TM,p,M),K]$ for which a global section ζ was fixed, in a natural way it is introduced **a distance on**

M. By means of this distance it is obtained a metric geometry of M. In this geometry there are established some properties analogous with some properties from classical Riemannian Geometry.

For the metric geometry of a structure $[(TM,p,M),K]$ with the global section ζ fixed, many problems are open: the study of geodesics, the study of metric connections, problems concerning the definition and the study of curvature a.s.o.

c). For a smooth manifold M for which there is and it is fixed a structure $[(TM,p,M),K]$, associated linear connections and (for the directions tangent to M) an intrinsic parallelism can be defined ([9], p.47–48).

A linear connection Γ is **an weak associated linear connection** of the structure $[(TM,p,M),K]$, if for every smooth curve c and for the parallel displacement τ_c determined by Γ and c we have $\tau_c(K(x(t))) \subset K(x(t))$.

A direction tangent to M is a 1-dimensional linear subspace of $T_x(M)$. Two distinct fields of directions $\{\rho X\}$ and $\{\rho Y\}$, defined on the same smooth curve, for which the vectors $X(x(t))$ and $Y(x(t))$ do not belong to the $FrK(x(t))$, are intrinsic parallel if the **anharmonic ratio** determined by these directions and by the directions obtained by the intersection of 2-dimensional linear subspace $[X(x(t)),Y(x(t))]$ of $T_x(M)$ with $FrK(x(t))$ is constant.

Many problems concerning associated linear connections and intrinsic parallelism (existence, characteristics properties, autoparallel curves a.s.o.) are open.

4.2. A new metric geometry of a time-oriented Lorentzian manifold. All results of this section are taken from [13]. A paper dedicated to memory of Professor Albert Crumeyrolle, Paul Sabatier University, Toulouse, France).

a). *A* **Lorentzian manifold** *is a structure, denoted by* $[M,g]$, *where:*

— M *is a real,* $n+1$ *dimensional,* C^∞ *-diff. paracompact connected without boundary manifod;*

— g *is a* C^∞ *-diff. global field of tensors tangent to* M, *these being two-times covariant symmetric of rank* $n+1$ *and signature* n.

This structure is fundamental in the General Relativity Theory (see [5]). The classical geometry of a structure $[M,g]$ is founded on the study of tensor field g. This study is guided by the analogies with the study of Riemannian metrics. So in every tangent to M vector space $T_x(M)$, a Minkovski metric (i.e. "a scalar product" and "a norm") is determined by means of the tensor $g(x)$. But this scalar product and this norm are not scalar product and norm in general admitted sense. The obtained results

are sometimes very different from those obtained in the case of Riemannian metrics (see [1]).

After some preparative, in [13] it was defined another norm in the tangent spaces to M and this way, a new metric geometry for the structure $[M,g]$ is obtained. We shall present this norm here.

b) For a Lorentzian manifold $[M,g]$ these are considered the cones of nonspacelike tangent vectors i.e. the following subsets of tangent linear spaces to M:

$$C(x) = \{X | X \in T_x(M), g(x)(X,X) \geq 0\} \qquad (\forall x \in M).$$

Every set $C(x)$ is determined by a degenerate hyperquadric and it is composed from two *convex pointed closed with interior vectors (in the topological space* $T_x(M)$*) cones* which are denoted by $C^+(x)$ and $C^-(x)$. These cones satisfy the relations : $C^+(x) \cup C^-(x) = C(x), C^+(x) \cap C^-(x) = \{0\}, X \in C^+(x) \Leftrightarrow -X \in C^-(x)$.

A Lorentzian manifold is **time-orientable** if there is a global C^∞ -diff. field of timelike vectors tangent to M i.e.

$$\forall x \in M | g(x)(\zeta x, \zeta x) > 0.$$

A time-orientable Lorentzian manifold is called **a space-time manifold.**

If for a space-time manifold $[M,g]$, a global C^∞ -diff. field of timelike vectors ζ it is fixed (determined), then this space-time manifold is called a **time-normalized space-time manifold.**

For a time-normalized space-time manifold the field ζ uniquely determines, for every point x of M, one of cones $C^+(x)$ or $C^-(x)$ which are subsets of the set of nonspacelike tangent vectors. This is the cone $C^+(x)$ or $C^-(x)$ which contains the vector $\zeta(x)$. This subset of $C(x)$ will be denoted by $K(x)$ and it will be determined by the relation:

$$K(x) = \{X | [X \in T_x(M)] \wedge [X \neq 0 \wedge g(X,X) > 0$$
$$\wedge g(\zeta(x), X) > 0) \vee (X = 0]\}.$$

This way it is defined a map

$$K : x \in M \to K(x) \subset T_x(M) \subset T(M)$$

which satisfies the axioms A_1, A_2, from [11], p.40. It is obtained a structure $[(T(M), p; M)K]$ from [11], p.40. Consequently: all results from § 1–4 regarding a structure $[(E,p,M);K]$ can be applied for a time-orientable Lorentzian manifold, for which a C^∞-diff. global field ζ of timelike vectors it is specified. This way, as in 4.1 it is obtained for a time-normalized space-time manifold a new metric geometry.

Convention: A time-normalized space-time manifold is denoted by $[M,g;\zeta,K]$. Obviously here K is the field of cones tangent to M uniquely determined by the pair g,ζ.

Remarks : 1. The field K can be determined also by another arbitrary tensor field globally conformal to g, i.e. by an arbitrary element of the set $G(M,g)=\{\rho g|\rho : M\to(0,\infty), \rho \text{ is } C^\infty - \text{diff.}\}$.

2. For an arbitrary Lorentzian manifold $[M,g]$ the condition to be *time-orientable* (the existence of a global C^∞-diff. field of tangent timelike vectors) it is a strong restrictive topological condition for M. But it is easy to see that **every Lorentzian manifold can be locally** endowed with a structure of time-normalized space-time manifold. This means that for every point $x\in M$ there is an open set of M, U_x, such that the Lorentzian manifold (restricted to U_x) $[U_x;g|U_x]$ can be endowed with a structure of time-normalized space-time: $[U_x,g|U_x;\zeta_{U_x},K_{U_x}]$.

c) Some elements of the new geometry of $[M,g;\zeta,H]$.

1) $\forall X\in T_x(m)|X\in K(x)\Leftrightarrow(X=0)\vee(X\neq 0\wedge g(X,X)\geq 0\wedge g(\zeta(x),X)>0)X\in \operatorname{Int}K(x)\Leftrightarrow g(X,X)>0\wedge g(\zeta(x),X)>0$.

2) $\forall X,Y\in T_x(M)|X\leq Y\Leftrightarrow Y-X\in K(x)\Leftrightarrow(X=Y)\vee((X\neq Y)\wedge$

$$(g(X,X)+g(Y,Y)\geq 2\cdot g(X,Y)\wedge(g(\zeta(x),Y)>g(\zeta(x),X))$$

3) $\forall X\in T_x(M);\forall Y\in \operatorname{Int}K(X)|$

$$X\notin\{\rho Y|\rho\in \boldsymbol{R}\}\Rightarrow(g(Y,X)))^2-g(Y,Y)\cdot g(X,X)>0.$$

4) If we set $D(X)=[(g(\zeta(x),X))^2-g(\zeta(x),\zeta(x))\cdot g(X,X)]^{1/2}\cdot(g(\zeta(x),\zeta(x)))^{-1}$, then the map

$$D():X\in T_x(M)\to D(X)\in\boldsymbol{R},$$

is a pseudo-norm of $T_x(M)$ i.e.

$$D(X)\geq 0,\ D(\rho X)=|\rho|\cdot D(X),\ D(X)+D(Y)\geq D(X+Y).$$

We remark that $D(X)=0\Leftrightarrow X\in\{\rho\zeta(x)|\rho\in\boldsymbol{R}\}$.

5) The function $v:(Z,X)\in\cup(\operatorname{Int}K(x)\times T(x))\to v(Z,X)\in\boldsymbol{R}^2$, (see 1.2.c)) where $v(Z,X)=(\alpha(Z,X),\beta(Z,X))\in\boldsymbol{R}^2$ has the real numbers $\alpha(Z,X)$ and $\beta(Z,X)$ determined by the relations

$$\alpha(Z,X)=g(Z,X)/g(Z,Z)+D(X),\beta(Z,X)=g(Z,X)/g(Z,Z)-D(X).$$

6) The map $|\ |_\zeta : X\in T(M)\to|X|_\zeta\in\boldsymbol{R}$, where

$$|X|_\zeta=\max\{|\alpha(\zeta(p(X)),X)|,|\beta(\zeta(p(X)),X)|\}=$$

$$=|g(\zeta p(X),X)|/g(\zeta(p(X)),\zeta p(X))+D(X)$$

restricted to the tangent space $T_x(M),(x=p(X))$, **is a norm on this vector space.**

A physical interpretation of this norm: if for $n=3$ it is considered the Minkowski 4-dimensional space tangent to $[M,g]$, i.e. $[T_{p(X)}(M),g(p(X))]$ and in a local chart of M, normal in x (i.e. in this local chart for the tensor g in $x, g_{ab}=-\delta_{ab}; g_{n+1n+1}=c^2\ a,b\in\{1,2,3\}$) we have $\zeta(p(X))=(0,0,0,1/c)$, where c is the speed of light, **then for an event** $X(X^1,X^2,X^3,t)$ (an element of $[T_{p(X)}(M),g(p(X))]$) **the norm** $|X|_\zeta$ **is the sum of the distances** $|ct|$ **and** $(\delta_{ab}X^aX^b)^{1/2}$.

Another geometrical properties of the structure $[M,g;\zeta,K]$ are exposed in [13].

4.3. Applications in the theory of monotone dynamical systems. The structure $[(E,p,M);K]$ can be used to obtain a generalization of the theory of monotone dynamical systems ([14]). In order to see how this generalization works there are presented here the definitions of some concepts similar to those which can be found in [14]:

A semi-flow on the vector bundle (E,p,M) is a continuous map $\Phi: E\times\boldsymbol{R}_+\to E$ which satisfies the following two conditions:

i) $\forall t\geq 0,\forall x\in M:\Phi_t(E_x)\subset E_x,\ \ (E_x=p^{-1}(x))$.

ii) $\forall t\geq 0,\forall s\geq 0:\Phi_{t+s}=\Phi_t\circ\Phi_s;\Phi_0=\mathrm{id}_E$.

The subset $A\subset E$ is called positively invariant if $\forall t\geq 0,\Phi_t(A)\subset A$. The subset A is called invariant if $\Phi_t(A)=A$.

If Φ is a semi-flow on the vector bundle (E,p,M) then the orbit of a section σ of (E,p,M) is the subset of E

$$G(\sigma):=\{\Phi_t(\sigma(x))|\forall t\geq 0,\forall x\in M\}.$$

The section σ is called an equilibrium section of Φ if $G(\sigma)=Im\sigma$.

It is clear that σ is an equilibrium section of Φ if and only if $\forall x \in M, \sigma(x)$ is an equilibrium point on the flow $\Phi_x : E_x \times \boldsymbol{R} \to E_x$, where $\Phi_x = \Phi|_{E_x \times \boldsymbol{R}_+}$.

For a structure $[(E,p,M);K]$ a semi-flow Φ on the regular vector bundle (E,p,M) is called:

— monotone if for $\forall t \geq 0, \forall x \in M, \forall X,Y \in E_x | X \leq Y \Leftrightarrow \Phi_t(X) \leq \Phi_t(Y)$;

— strongly order preserving if it is monotone ad for $\forall x \in M, \forall X,Y \in E_x | X < Y$ (i.e. $Y - X \in \mathrm{Int}K(x)) \Rightarrow \exists U$ a neighbourhood of X in E and V a neighborhood of Y in E and $t_0 \in \boldsymbol{R}$ such that

$$\forall X' \in U, \forall Y' \in V | \rho(X') = \rho(Y') \Rightarrow \Phi_{to}(Y') \leq \Phi_{to}(X');$$

— almost strongly monotone if it is monotone and for $\forall x \in M$, $\forall X,Y \in E_x | X < Y \Rightarrow \exists t_0 \in \boldsymbol{R}$ such that $\Phi_{to}(X) < \Phi_{to}(Y)$ (i.e. $(\Phi_{to}(Y) - \Phi_{to}(X)) \in \mathrm{Int}K(x)$).

Proposition. *If for a structure $[(E,p,M;K]$ a semi-flow Φ on the vector bundle (E,p,M) is almost strongly monotone then Φ is strongly order preserving.*

This *Proposition* shows a possible manner in which results obtained for monotone dynamical systems in Banach spaces, ([14]) can be generalized for a structure $[(E,p,M);K]$. For this last structure endowed with a monotone (strongly order preserving, almost strongly monotone) semi-flow different topics can be studied: the behavior of the orbits of the sections of (E,p,M), problems of stability, equilibrium sections and points, positively invariant sets etc.

4.4. Results concerning particular positive operators on particular structures $[(E,p,M);K]$.

a) In [2], Liana David considered a structure $[(E,p,M);K]$ for which the cones of the field K are $n-hedral$ cones and the topological manifold M is compact. In this case two theorems are proved:

Theorem 1. *There is a finite number of continuous global sections of $(E,p,M), \sigma_1, \sigma_2, \ldots, \sigma_s$ such that every positive local linear continuous operator $A : (\Sigma, K_\Sigma) \to (\Sigma, K_\Sigma)$ which satisfies the condition $A(\sigma_i) = \sigma_i, \forall i \in \{1,2,\ldots,s\}$ is the identity.*

Theorem 2. *There is a finite number of continuous global sections of $(E,p,M), \sigma_1, \sigma_2, \ldots, \sigma_s$ such that every sequence of positive local linear continuous operators $A_n : (\Sigma, K_\Sigma) \to (\Sigma, K_\Sigma)$ which satisfies the condition $A_n(\sigma_i) \to \sigma_i, \forall i \in \{1,2,\ldots,s\}$ also satisfies $A_n(\sigma) \to \sigma, \forall \sigma \in \Sigma$.*

b) In [3] Liana David proved that for every structure $[(E,p,M);K]$ with M compact there is a "plastering" structure $[(E,p,M);K_1]$ i.e. every cone of the field K admits as a plastering cone, a cone of the field K_1 (see [7], p.50).

Also in [3] are presented some results concerning spectral properties of linear positive operators of an arbitrary structure $[(E,p,M);K]$. For example, the spectral radius of an arbitrary positive operator of $[(E,p,M);K]$ belongs to the closure of the spectrum of this operator. Many others results are established.

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A NEW DEFINITION FOR CERTAIN LIFTS ON A VECTOR BUNDLE

BY

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Introduction. In the study of vector bundles one considers prolongation of certain geometrical objects on the base manifold or on the bundle, to the total space. Among these prolongation there are the vertical and horizontal lifts for tensor fields and linear connections. The definitions of these lifts, given up to now, have some deficiencies as follows.

a) For certain objects the vertical or horizontal lift is not a vertical or an horizontal object [4].

b) The definition given for certain lifts on a general vector bundle [1, 4 - 6] does not reduce to the definition before known [7, 8], in the particular case of a tangent bundle.*

c) The definition for the horizontal lift is given only for certain particular types of tensor fields [1, 4 - 6].

The aim of this work is to modify, to complete and to extend the definitions of certain lifts on a vector bundle according with a more natural, more consequent and more simple point of view and to give geometrical characterizations for these lifts.

1. d–Tensor fields and d–Lifts. Let $\xi = (E, \pi, M)$ be a vector bundle with total space E, projection π and base manifold M, connected and paracompact. Let be, in adapted charts on M, ξ and E, the local coordinates $(x^i), (y^a), (x^i, y^a)$ respectively and the pairs of corresponding dual basses $(\partial_i, d^i), (e_a, e^a), (\partial_i, \partial_a, d^i, d^a)$, where $\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y_a, d^i = dx^i, d^a = dy^a$ and $i, j, k = 1, 2, \ldots, m$, $a, b, c = 1, 2, \ldots, n$. Setting for each $z = (x, y) \in E$, $V_z E = \ker T_z \pi$, we obtain the *vertical distribution* and so

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