13. P a p u c, D.I. — *About a new metric geometry of a time oriented Lorentzian manifold*. Seminarul de Mecanică, Univ. de Vest din Timişoara, nr.44, 1995.
14. S m i t h, H.L. — *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems.* AMS Mathematical Surveys and Monographs, vol.41, 1995.
15. S u l l i v a n, D. — *Cycles for the dynamical study of foliated manifolds and complex manifolds*. Inventiones Math. 36, 225–255, 1976.

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A NEW DEFINITION FOR CERTAIN LIFTS ON A VECTOR BUNDLE

BY

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Introduction. In the study of vector bundles one considers prolongation of certain geometrical objects on the base manifold or on the bundle, to the total space. Among these prolongation there are the vertical and horizontal lifts for tensor fields and linear connections. The definitions of these lifts, given up to now, have some deficiencies as follows.

a) For certain objects the vertical or horizontal lift is not a vertical or an horizontal object [4].

b) The definition given for certain lifts on a general vector bundle [1, 4 – 6] does not reduce to the definition before known [7, 8], in the particular case of a tangent bundle.*

c) The definition for the horizontal lift is given only for certain particular types of tensor fields [1, 4 – 6].

The aim of this work is to modify, to complete and to extend the definitions of certain lifts on a vector bundle according with a more natural, more consequent and more simple point of view and to give geometrical characterizations for these lifts.

1. d–Tensor fields and d–Lifts. Let $\xi = (E, \pi, M)$ be a vector bundle with total space E, projection π and base manifold M, connected and paracompact. Let be, in adapted charts on M, ξ and E, the local coordinates $(x^i), (y^a), (x^i, y^a)$ respectively and the pairs of corresponding dual basses $(\partial_i, d^i), (e_a, e^a), (\partial_i, \partial_a, d^i, d^a)$, where $\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y_a, d^i = dx^i, d^a = dy^a$ and $i, j, k = 1, 2, \ldots, m$, $a, b, c = 1, 2, \ldots, n$. Setting for each $z = (x, y) \in E$, $V_z E = \ker T_z\pi$, we obtain the *vertical distribution* and so

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*Conference presented at the 25-th National Conference on Geometry and Topology, September 18-23, 1995, Iaşi, România the *vertical subbundle* of TE, denoted by VE. This is the distribution tangent to the *vertical foliation* on E. Considering the quotient bundle $WE = TE/VE$, we obtain the following short exact sequence over E,

$$1) \qquad 0 \longrightarrow VE \xrightarrow{i} TE \xrightarrow{p} WE \longrightarrow 0,$$

where i and p are the canonical injection and projection, respectively. For VE we have the local basis (∂_a) and for WE, the basis $(\overline{\partial}_i = p(\partial_i))$. Then putting for each $z \in E, V_z^{\perp}E = \{\alpha \in T_z^*E | \alpha(A_z) = 0, \forall A_z \in V_zE\}$, we obtain a subbundle $V^{\perp}E$ of T^*E, called the *orthogonal dual* of VE. Setting then $W^{\perp}E = T^*E/V^{\perp}E$, we get the short exact sequence over E.

$$2) \qquad 0 \longrightarrow V^{\perp}E \xrightarrow{j} T^*E \xrightarrow{q} W^{\perp}E \longrightarrow 0,$$

with j and q the canonical injection and projection. For $V^{\perp}E$ we have the local basis (d^i) and for $W^{\perp}E$, the basis $(\overline{d}^a = q(d^a))$.

As base manifold for the vector bundle ξ, M has a supplementary structure and we can consider the following class of tensor fields.

Definition 1.1. *A distinguished, or shortly* d*-tensor field of type* (p,q,r,s) *on the base manifold* M *of the vector bundle* ξ, *is a section* t *of the vector bundle* $\otimes^p TM \otimes^r \xi \otimes^q T^*M \otimes^s \xi^*$ *over* M.

The local expression for such a tensor field is

$$3) \qquad t = t^{i_1 \cdots i_p a_1 \cdots a_r}_{j_1 \cdots j_q b_1 \cdots b_s}(x) \partial_{i_1} \otimes \cdots e_{a_1} \otimes \cdots d^{j_1} \otimes \cdots e^{b_1} \otimes \cdots \otimes e^{b_s},$$

Also, E being the total space of the vector bundle ξ, we can consider the following class of tensor fields.

Definition 1.2. *A* d*-tensor field of type* (p,q,r,s), *on the total space* E *of the vector bundle* ξ, *is a section* T *of the vector bundle* $\otimes^p WE \otimes^r VE \otimes^q V^{\perp}E \otimes^s W^{\perp}E$ *over* E.

$$4) \qquad T = T^{i_1 \cdots i_p a_1 \cdots a_r}_{j_1 \cdots j_q b_1 \cdots b_s}(x,y) \overline{\partial}_{i_1} \otimes \cdots \partial_{a_1} \otimes \cdots d^{j_1} \otimes \cdots \overline{d}^{b_1} \otimes \cdots \otimes \overline{d}^{b_s}.$$

These tensor fields were considered in [5] from another point of view.

The coordinates of the tensor fields t and T, have the same law of transformation under the change of the adapted charts and so we can give

Definition 1.3. *The* d *lift for a* d*-tensor field* t *of type* (p,q,r,s) *on* M, *given by 3), is the* d*-tensor field* $\mathrm{T} = \mathrm{t}^d$ *of the same type on* E, *given by 4) where*

$$5) \qquad T^{i_1 \cdots i_p a_1 \cdots a_r}_{j_1 \cdots j_q b_1 \cdots b_s}(x,y) = t^{i_1 \cdots i_p a_1 \cdots a_r}_{j_1 \cdots j_q b_1 \cdots b_s}(x).$$

Let $\mathcal{F}(M)$ be the ring of C^∞-real functions on M and $f^d = f \circ \pi$, for each $f \in \mathcal{F}(M)$. Putting $\mathcal{F}(M)^d = \{f^d | f \in \mathcal{F}(M)\}$, we can see that $\mathcal{F}(M)^d$ is a subring of $\mathcal{F}(E)$, isomorphic with $\mathcal{F}(M)$. Let $\mathcal{T}_q^p(M)$ and $\mathcal{T}(M)$ be the $\mathcal{F}(M)$-module of tensor fields of type (p,q) and the bigraded $\mathcal{F}(M)$-tensor algebra of M. Let then, $\mathcal{T}_s^r(\xi)$ and $\mathcal{T}(\xi)$ be the $\mathcal{F}(M)$-module of tensor field of type (r,s) and the bigraded $\mathcal{F}(M)$-tensor algebra of ξ. We denote by $\mathcal{T}^{p,r}_{q,s}(M,\xi), \mathcal{T}(M,\xi)$ and $\mathcal{T}^{p,r}_{q,s}(\xi,E), \mathcal{T}(\xi,E)$ the module of d-tensor fields and the corresponding fourgraded algebras on M and E, respectively.

Remark. The d-lift is a $\mathcal{F}(M)$-monomorphism of fourgraded algebras.

Setting for each 1-form $\mu \in \mathcal{T}_1(\xi)$, given by $\mu(x) = \mu_a(x)e^a$,

$$6) \qquad \gamma(\mu)(z) = \mu_a(x) y^a,$$

where $z = (x,y) \in E$, we obtain a class of functions on E with the following important property. For two vector fields A and B on E, we have $A = B$ if and only if $A(\gamma\mu) = B(\gamma\mu), \forall \mu \in \mathcal{T}_1(\xi)$. The operator γ can be extended to tensor fields $T \in \mathcal{T}_1^r(\xi)$ by

$$7) \qquad \gamma(T_b^{a_1 \cdots a_r}(x) e_{a_1} \otimes \cdots \otimes e_{a_r} \otimes e^b)(z) = y^b T_b^{a_1 \cdots a_r}(x) \partial_{a_1} \otimes \cdots \otimes \partial_{a_r}.$$

In particular, for $T = I_\xi$, the identical automorphism of ξ, we get the *canonical* vector field $K = \gamma(I_\xi)$ on E, given by

$$8) \qquad K(\gamma\mu) = \gamma\mu, \qquad \forall \mu \in \mathcal{T}_1(\xi),$$

with the local expression $K = y^a \partial_a$.

Definition 1.4. *A vertical vector field on E, is a section of the vertical subbundle VE.*

Hence, a vertical vector field is a d-tensor field of type $(0,0,1,0)$. It has the local expression $A(z) = A^a(x,y)\partial_a$.

Definition 1.5. *The vertical lift, for a section u of the vector bundle ξ, is the vertical vector field u^v on the total space E, given by*

$$u^v(\gamma\mu) = \mu(u)^v, \qquad \forall \mu \in \mathcal{T}_1(\xi). \tag{9}$$

where for $f \in \mathcal{F}(M)$, $f^v = f \circ \pi = f^d$.

Locally, if $u = u^a e_a$, then $u^v = u^a(x)\partial_a$. For $u = e_a$, we get

$$(e_a)^v = \partial_a, \qquad a = 1,2,\ldots,n. \tag{10}$$

For the vertical lift we note the properties

$$[u^v, v^v] = 0, \qquad \mathcal{L}_K u^v = -u^v, \tag{11}$$

where $\mathcal{L}_K$ is the Lie derivation with respect to the canonical vector field.

2. Normalization of the Vertical Foliation. The total space E of the vector bundle ξ being a manifold endowed with the vertical foliation, for its study it is convenient to consider a normalization of this foliation, that is, a distribution on E, supplementary to the vertical one. Such a distribution will be called *horizontal* distribution and denoted by HE. With HE we denote too the corresponding subbundle of TE and we shall call it the *horizontal* subbundle. M being paracompact, we can consider a normalization N defined by a linear connection D in the vector bundle ξ [6]. Setting in local charts

$$D_{\partial_i} e_b = \Gamma^a_{ib}(x) e_a, \qquad N^a_i(z) = \Gamma^a_{ib}(x) y^b, \qquad \delta_i = \partial_i - N^a_i \partial_a, \tag{12}$$

we can see that (δ_i), $i = 1,2,\ldots,m$, are m linear independent local vector fields and they generate locally the horizontal subbundle HE, associated to the linear connection D.

Definition 2.1. *A horizontal vector field on the total space E of ξ, with respect to the linear connection D (or the normalization N), is a section of the horizontal subbundle HE.*

Locally, for such a section we have $A(z) = A^i(x,y)\delta_i$.

Definition 2.2. *The horizontal lift for a vector field X on the base manifold M of ξ, with respect to the connection D on ξ, is the vector field X^h on the total space E, given by*

$$X^h(\gamma\mu) = \gamma(D_X\mu), \qquad \forall \mu \in \mathcal{T}_1(\xi). \tag{13}$$

Locally, if $X = X^i\partial_i$, then $X^h = X^i\delta_i$. For $X = \partial_i$, we obtain

$$(\partial_i)^h = \delta_i, \qquad i = 1,2,\ldots,m. \tag{14}$$

For the horizontal lift, the following properties hold

$$\begin{aligned} &(fX)^h = f^v X^h,\ [X^h, Y^h] = [X,Y]^h - \gamma R^D_{XY}, \\ &[X^h, u^v] = (D_X u)^v,\ \mathcal{L}_K X^h = 0. \end{aligned} \tag{15}$$

where $f \in \mathcal{F}(M)$, $X, Y \in \mathcal{T}^1(M)$, $u \in \mathcal{T}^1(\xi)$ and R^D is the curvature of D.

Definition 2.3. *A vertical 1-form on E is a 1-form which vanishes on each horizontal vector field.*

So, a vertical 1-form on E is a section on the subbundle $H^\perp E \subset T^*E$, the orthogonal dual of HE. Locally, for such a 1-form one has $\alpha(z) = \alpha_a(x,y)(d^a + N^a_i d^i)$.

Definition 2.4. *The vertical lift for a 1-form $\mu \in \mathcal{T}(\xi)$ is the 1-form $\mu^v \in \mathcal{T}_1(E)$ given by the relations*

$$\mu^v(X^h) = 0, \quad \mu^v(u^v) = \mu(u)^v, \quad \forall X \in \mathcal{T}^1(M),\ u \in \mathcal{T}^1(\xi). \tag{16}$$

If $\mu = \mu_a(x)e^a$, then $\mu^v(z) = \mu_a(x)(d^a + N^a_i(x,y)d^i)$. For $\mu = e^a$, we get

$$(e^a)^v = \delta^a = d^a + N^a_i d^i, \quad a = 1,2,\ldots,n. \tag{17}$$

The 1-forms (δ^a), $a = 1,2,\ldots n$, generate locally the subbundle $H^\perp E$.

Remark. The vertical 1-form on E and the vertical lift for a 1-form on ξ depend of the normalization N.
For the vertical lift of a 1-form $\mu \in \mathcal{T}_1(\xi)$, one has

$$\begin{aligned} &\mathcal{L}_K \mu^v = \mu^v, \quad d\mu^v(X^h, Y^h) = \gamma(\mu \circ R^D_{XY}), \\ &d\mu^v(X^h, u^v) = (D_X\mu)(u)^v, \quad d\mu^v(u^v, v^v) = 0. \end{aligned} \tag{18}$$

It follows from here

Proposition 2.1. *The vertical lift μ^v of $\mu \in \mathcal{T}_1(\xi)$ is closed if and only if μ is covariant constant. In this case μ^v is exact and one has* $\mu^v = d(\gamma\mu)$.

Definition 2.5. *A horizontal 1-form, on the total space E of ξ, is a 1-form which vanishes on each vertical vector field.*

Hence, a horizontal 1-form is a section on the subbundle $V^\perp E \subset T^*E$. Such a 1-form has the local expression $\alpha(z) = \alpha_i(x,y)d^i$.

Definition 2.6. *The horizontal lift of a 1-form $\omega \in \mathcal{T}^1(M)$, is the 1-form $\omega^h \in \mathcal{T}_1(E)$, given by the relation*

$$\omega^h = T^*\pi(\omega). \tag{19}$$

It follows

Proposition 2.2. *The horizontal lift of a 1-form ω on M is the 1-form ω^h on E given by*

$$\omega^h(X^h) = \omega(X)^v, \quad \omega^h(u^v) = 0, \quad \forall X \in \mathcal{T}^1(M), \quad u \in \mathcal{T}^1(\xi). \tag{20}$$

If $\omega = \omega_i(x)d^i$, then $\omega^h(z) = \omega_i(x)d^i$. For $\omega = d^i$, we obtain

$$(d^i)^h = d^i, i = 1,2,\dots,m. \tag{21}$$

Remark. The horizontal 1-form on E and the horizontal lift for a 1-form on M are independent of the normalization N.

From the previous considerations it results that the following systems of local sections (δ_i, ∂_a), (d^i, δ^a) represent the dual bases adapted to normalization N, defined by the linear connection D on ξ and the adapted charts on E. We shall use these bases in the sequel to simplify the calculations.

3. N–Decomposable Tensor Fields and ν–Lifts.

A normalization N for the vertical foliation determines a direct sum decomposition of the bundles TE and T^*E.

$$TE = HE \oplus VE, \qquad T^*E = V^\perp E \oplus H^\perp E. \tag{22}$$

We denote by H and V the horizontal and the vertical projectors and by $F = V - H$ the almost product structure associated to this decomposition.

For each $A \in \mathcal{T}^1(E)$ and $\alpha \in \mathcal{T}_1(E)$ we obtain

$$A = HA + VA, \alpha = H\alpha + V\alpha = \alpha \circ H + \alpha \circ V. \tag{23}$$

Definition 3.1. *A N-decomposable tensor field of type (p,q,r,s) on the total space E of the vector bundle ξ, with respect to the linear connection D, is a section of the vector bundle $\otimes^p HE \otimes^r VE \otimes^q V^\perp E \otimes^s H^\perp E$.*

We denote by $\mathcal{T}^{p,r}_{q,s}(E,N)$ and $\mathcal{T}(E,N)$ the $\mathcal{F}(E)$– module of N–decomposable tensor fields of type (p,q,r,s) and the corresponding four-graded tensor algebra.

Considering a tensor field $T \in \mathcal{T}^{p+r}_{q+s}(E)$ as a $\mathcal{F}(E)$–multilinear application $T:(\mathcal{T}_1E)^{p+r} \times (\mathcal{T}^1E)^{q+s} \to \mathcal{F}(E)$, it follows

Proposition 3.1. *A tensor field $\widetilde{T} \in \mathcal{T}^{p+r}_{q+s}(E)$ is N-decomposable of type (p,q,r,s) if and only if*

$$\widetilde{T} = \widetilde{T} \circ (H^p \times V^r \times H^q \times V^s). \tag{24}$$

Such a tensor field has the local expression in adapted basis

$$\widetilde{T}(z) = \widetilde{T}^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x,y)\delta_{i_1} \otimes \dots \partial_{a_1} \otimes \dots d^{j_1} \otimes \dots \delta^{b_1} \otimes \dots \otimes \delta^{b_s}. \tag{25}$$

Hence, a N-decomposable tensor field is a d–tensor field in the sense used in [6]. Evidently, it depends of the normalization N, defined by the linear connection D on ξ.

From 23) and 24) it results that each tensor field $T \in \mathcal{T}^i_j(E)$ can be decomposed in 2^{i+j} N-decomposable tensor fields of type (p,q,r,s) with $p+r=i$ and $q+s=j$. Therefore, we have

$$\mathcal{T}^i_j(E) = \bigoplus_{\substack{p+r=i \\ q+s=j}} \mathcal{T}^{p,r}_{q,s}(E,N).$$

Using this decomposition, the bigraded algebra $\mathcal{T}(E)$, can be replaced by the fourgraded algebra $\mathcal{T}(E,N)$.

Definition 3.2. *The N-lift for a d-tensor field T of type (p,q,r,s) on E, given by (4), is the N-decomposable tensor field $\widetilde{\mathrm{T}}$ of the same type, given by (25), where*

$$\widetilde{T}^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x,y) = T^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x,y). \tag{26}$$

It follows that the $\mathcal{N}$-lift is an isomorphism between the fourgraded algebras $\mathcal{T}(\xi, E)$ and $\mathcal{T}(E, N)$.

Definition 3.3. *The ν-lift, for a d-tensor field t of type (p,q,r,s) on M, is the N-decomposable tensor field t^ν on E, given by*

$$(27)\qquad t^\nu(\omega_1^h, \dots, \omega_p^h, \mu_1^v, \dots, \mu_r^v, X_1^h, \dots, X_q^h, u_1^v, \dots, u_s^v) = $$
$$= t(\omega_1, \dots, \omega_p, \mu_1, \dots, \mu_r, X_1, \dots, X_q, u_1, \dots, u_s)^v,$$

where $\omega_i \in \mathcal{T}_1(M), \mu_a \in \mathcal{T}_1(\xi), X_j \in \mathcal{T}^1(M),\ u_b \in \mathcal{T}^1(\xi)$.

Locally, if t is given by 3), for t^ν we obtain

$$(28)\qquad t^\nu(z) = t^{i_1\dots i_p a_1 \dots a_r}_{j_1 \dots j_q b_1 \dots b_s}(x)\delta_{i_1} \otimes \dots \partial_{a_1} \otimes \dots d^{j_1} \otimes \dots \delta^{b_1} \otimes \dots \otimes \delta^{b_s}.$$

Remark. The lifts d, $\mathcal{N}$ and ν satisfy the relation

$$(29).\qquad \nu = \mathcal{N} \circ d$$

It is not difficult to check the following characteristic property for the ν-lift.

Proposition 3.2. *A N-decomposable tensor field $\widetilde{\mathrm{T}}$, of type (p,q,r,s) on E, is the ν-lift of a d-tensor field of the same type on M if and only if*

$$(30)\qquad \mathcal{L}_K \widetilde{T} = (s-r)\widetilde{T}.$$

In the following table, we consider certain important classes of N-decomposable tensor fields of type (p,q,r,s) on the total space E.

Name	Characterisation	Local expresion
Vertical	$p = q = 0$	$T = T^{a_1 \dots a_r}_{b_1 \dots b_s}\partial_{a_1} \otimes \dots \partial_{a_r} \otimes \delta^{b_1} \otimes \dots \otimes \delta^{b_s}$
Horizontal	$r = s = 0$	$T = T^{i_1 \dots i_p}_{j_1 \dots j_q}\delta_{i_1} \otimes \dots \otimes \delta_{i_p} \otimes d^{j_1} \otimes \dots \otimes d^{j_q}$
Vertical-horizontal	$p = s = 0$	$T = T^{a_1 \dots a_r}_{j_1 \dots j_q}\partial_{a_1} \otimes \dots \otimes \partial_{a_r} \otimes d^{j_1} \otimes \dots \otimes d^{j_q}$
Horizontal-vertical	$r = q = 0$	$T = T^{i_1 \dots i_p}_{b_1 \dots b_s}\delta_{i_1} \otimes \dots \otimes \delta_{i_p} \otimes \delta^{b_1} \otimes \dots \otimes \delta^{b_s}$

These tensor fields determines four bigraded subalgebras of the algebra $\mathcal{T}(E, N)$, which will be called: *vertical, horizontal, vertical–horizontal and horizontal–vertical subalgebras*, respectively. We remark that the vertical, horizontal and horizontal-vertical subalgebras depend of the normalization. The vertical-horizontal subalgebra is independent of the normalization and coincides with the subalgebra of $\mathcal{T}(\xi, E)$ given by the d-tensor fields of the type (o,q,r,o), $q,r \in N$ on E.

Definition 3.4. *The vertical (v), horizontal (h), vertical–horizontal (vh) and horizontal–vertical (hv)-lifts are the restrictions of the ν-lift to the following submodules of tensor fields of type (p,q,r,s) on M.*

$$v = \nu_{/\mathcal{T}^{o,r}_{o,s}(M,\xi) = \mathcal{T}^r_s(\xi)},\quad h = \nu_{/\mathcal{T}^{p,o}_{q,o}(M,\xi) = \mathcal{T}^p_q(M)},$$

$$(31)\qquad vh = \nu_{/\mathcal{T}^{o,r}_{q,o}(M,\xi)},\quad hv = \nu_{/\mathcal{T}^{p,o}_{o,s}(M,\xi)}.$$

Examples. 1) For I_ξ and I_{TM}, the identical automorphism of ξ and TM, we obtain

$$(32)\qquad (I_\xi)^v = V,\quad (I_{TM})^h = H.$$

2) Considering algebraic combinations of the vertical and horizontal lifts for certain structures on the base manifold M and the vector bundle ξ, we can obtain interesting structures on the total space E.

a) For I_ξ and I_{TM}, we get

$$(33)\qquad (I_\xi)^v - (I_{TM})^h = F.$$

b) For two metrics g on M and γ on ξ, we obtain

$$(34)\qquad G = g^h + \gamma^v,$$

which is a metric on E, generalizing the metric of Sasaki on the tangent bundle [4].

Remark. Denoting by $\widetilde{v}$ and $\widetilde{h}$ the vertical and horizontal lifts, defined in [4], for $h \in \mathcal{T}_1^1(\xi)$ and $k \in \mathcal{T}_1^1(M)$, we have

$$h^{\widetilde{v}} = h^v + H,\ k^{\widetilde{h}} = k^h + V.$$

Hence, $\widetilde{h^v}$ is not vertical and $\widetilde{k^h}$ is not horizontal. After that, $\widetilde{h^v}$ and $\widetilde{k^h}$ are more complicated than h^v and k^h.

4. Derivation Laws in the Algebras of d–Tensor Fields.

If we give a connection ∇ on M and a connection D on ξ, then we obtain connections in the bundles TM, T^*M, ξ^* and so, in each of the vector bundle $\otimes^p TM \otimes^r \xi \otimes^q T^*M \otimes^s \xi^*$. Hence,

Proposition 4.1. *A pair (∇, D) of linear connections, on M and ξ, determines a law of derivation in the fourgraded algebra $\mathcal{T}(M,\xi)$ of d–tensor fields on M.*

But if we give a linear connection $\mathcal{D}$ on the total space E of the vector bundle ξ, generally it does not determine a law of derivation in the algebra of d–tensor fields on E. However we have

Proposition 4.2. *A linear connection $\mathcal{D}$ on the total space E of the vector bundle ξ determines a law of derivation in the fourgraded algebra $\mathcal{T}(\xi, E)$ of d–tensor fields on E if and only if it preserves the vertical subbundle.*

Proof. In fact, $\mathcal{D}$ preserving the type of d–tensor fields, it has to preserve the type $(0,0,1,0,)$, that is the vertical subbundle. Conversely, if $\mathcal{D}$ preserves the vertical subbundle, it comes out that it preserves also the orthogonal dual $V^\perp E$ and so it induces by restriction, linear connections on VE and $V^\perp E$, denoted by $\widetilde{\mathcal{D}}$. Putting then

$$\overline{\mathcal{D}}_A \overline{B} = \overline{\mathcal{D}_A B} \text{ and } \overline{\mathcal{D}}_A \overline{\alpha} = \overline{\mathcal{D}_A \alpha}, \tag{35}$$

for $A, B \in \mathcal{T}^1(E)$ and $\alpha \in \mathcal{T}_1(E)$, we obtain linear connections $\widetilde{\mathcal{D}}$ in the bundles WE and $W^\perp E$ and so a law of derivation in the algebra $\mathcal{T}(\xi, E)$. It follows also

Proposition 4.3. *A connection $\mathcal{D}$ on the total space E of the vector bundle ξ determines a law of derivation in the algebra of d–tensor fields on E if and only if*

$$p \circ \mathcal{D}_A \circ i = 0, \qquad \forall A \in \mathcal{T}^1(E). \tag{36}$$

5. Derivation Laws in the Algebra on N–Decomposable Tensor Fields.

Let D be a linear connection in ξ and N the corresponding normalisation on E.

Definition 5.1. *A vertical connection is a linear connection $\mathcal{D}$ on the total space E of the vector bundle ξ, which preserves the vertical subbundle i.e.*

$$\mathcal{D}_A B \in V\mathcal{T}^1(E), \forall A \in \mathcal{T}^1(E), B \in V\mathcal{T}^1(E). \tag{37}$$

It follows

Proposition 5.1. *A linear connection $\mathcal{D}$ on the total space E is vertical if and only if*

$$H \circ \mathcal{D}_A \circ V = 0, \ \forall A \in \mathcal{T}^1(E). \tag{38}$$

Hence, a vertical connection induces by restriction, a linear connection on the vertical subbundle VE and also on the orthogonal dual $V^\perp E$. It follows that a vertical connection defines a law of derivation in the algebra of vertical–horizontal tensor fields on E.

Definition 5.2. *The vertical lift of a linear connection D in the vector bundle ξ, is the linear connection D^v on the vertical subbundle VE, given by*

$$D^v_{X^h} u^v = (D_X u)^v, \ D^v_{u^v} v^v = 0, \ \forall X \in \mathcal{T}^1(M), \ \ u, v \in \mathcal{T}^1(\xi). \tag{39}$$

For the curvature $\widetilde{\mathcal{R}}$ of D^v, we obtain

$$\widetilde{\mathcal{R}}_{X^h Y^h} u^v = (R^D_{XY} u)^v \text{ and zero in rest }. \tag{40}$$

Hence, $\widetilde{\mathcal{R}} = 0$ if and only if $R^D = 0$.

Definition 5.3. *A horizontal connection is a linear connection $\mathcal{D}$ on the total space E of the vector bundle ξ which preserves the horizontal subbundle i.e.*

$$\mathcal{D}_A B \in H\mathcal{T}^1(E), \forall A \in \mathcal{T}^1(E), \ B \in H\mathcal{T}^1(E). \tag{41}$$

Proposition 5.2. *A linear connection $\mathcal{D}$ on the total space E of the vector bundle ξ is horizontal if and only if*

$$V \circ \mathcal{D}_A \circ H = 0, \ \forall A \subset \mathcal{T}^1(E). \tag{42}$$

A horizontal connection on E induces, by restrictions, linear connections on the horizontal subbundle HE and on its dual orthogonal $H^{\perp}E$. Hence, a horizontal connection defines a law of derivation in the algebra of horizontal-vertical tensor fields on E.

Definition 5.4. *The* horizontal *lift of a linear connection ∇ on the base manifold M of ξ, with respect to the linear connection D on ξ, is the linear connection ∇^h on the horizontal subbundle HE, given by*

$$\nabla^h_{X^h} Y^h = (\nabla_X Y)^h, \ \nabla^h_{u^v} Y^h = 0, \ \ \forall X, Y \in \mathcal{T}^1(M), \ \ u \in \mathcal{T}^1(\xi). \tag{43}$$

For the curvature $\overset{\approx}{\mathcal{R}}$ of ∇^h we get

$$\overset{\approx}{\mathcal{R}}_{X^h Y^h} Z^h = (R^{\nabla}_{XY} Z)^h \text{ and zero in rest.} \tag{44}$$

Hence, $\overset{\approx}{\mathcal{R}} = 0$ if and only if $R^{\nabla} = 0$.

A vertical connection $\mathcal{D}$ on E, generally does not carry a vertical 1-form on E to a vertical one. In fact, if $\alpha \in \mathcal{T}_1(E)$, then we have

$$(\mathcal{D}_A \alpha)(X^h) = -\alpha(\mathcal{D}_A X^h), \ \forall A \in \mathcal{T}^1(E), X \in \mathcal{T}^1(M). \tag{45}$$

Hence, $\mathcal{D}_A \alpha$ is vertical if and only if $\mathcal{D}_A X^h$ is horizontal for each $X \in \mathcal{T}^1(M)$, that is if $\mathcal{D}$ is also horizontal.

Definition 5.5. *A* N-decomposable *connection is a linear connection $\mathcal{D}$ on the total space E of the vector bundle ξ which induces a derivation law in the fourgraded algebra of N-decomposable tensor fields.*

From the previous considerations it follows

Proposition 5.3. *A connection $\mathcal{D}$, on the total space E of a vector bundle ξ, determines a law of derivation in the algebra of N-decomposable tensor fields on E if and only if it satisfies one of the following conditions:*

1) $\mathcal{D}$ is in the same time vertical and horizontal connection.
2) $\mathcal{D}$ preserves the subalgebra of vertical tensor fields,
3) $\mathcal{D}$ preserves the subalgebra of horizontal tensor fields.

4) $\mathcal{D}$ is a F-connection, that is $\mathcal{D}F = 0$.
5) There exists a pair of connection $(\widetilde{\mathcal{D}}, \overset{\approx}{\mathcal{D}})$ on VE and HE so that

$$\mathcal{D}_A = \widetilde{\mathcal{D}}_A \circ V + \overset{\approx}{\mathcal{D}}_A \circ H, \qquad \forall A \in \mathcal{T}^1(E). \tag{46}$$

These connections were studied in [1,5,6] and called d-connections. In a more general setting they were considered in [2,3].

Definition 5.6. *The ν-lift of a pair (∇, D) of linear connections, on the base manifold M and the vector bundle ξ, with respect to the normalization N defined by D, is the N-decomposable connection $\mathcal{D}^{\nu}$ on the total space E, given by*

$$\mathcal{D}^{\nu}_A = D^{v}_A \circ V + \nabla^h_A \circ H, \qquad \forall A \in \mathcal{T}^1(E). \tag{47}$$

For the torsion $\mathcal{T}^{\nu}$ and the curvature $\mathcal{R}^{\nu}$ of $\mathcal{D}^{\nu}$, we obtain

$$\mathcal{T}^{\nu}(X^h, Y^h) = T^{\nabla}(X, Y)^h + \gamma R^D_{XY} \quad \text{and zero in rest.} \tag{48}$$

$$\mathcal{R}^{\nu}_{X^h Y^h} Z^h = (R^{\nabla}_{XY} Z)^h, \mathcal{R}^{\nu}_{X^h Y^h} u^v = (R^D_{XY} u)^v \quad \text{and zero in rest.} \tag{49}$$

where T^{∇} and R^{∇} are the torsion and the curvature of ∇ and R^D is the curvature of D.

The lift $\mathcal{D}^{\nu}$ was considered in [4] and called the horizontal lift of the pair (∇, D), but it determines a law of derivation in all the algebra of N-decomposable tensor fields.

These considerations can be extended as follows. Let D^0 be a fixed connection on the vector bundle ξ. Considering a connection D on ξ we can see that putting

$$D^{\nu_0}_{X^{h_0}} u^v = (D_X u)^v, D^{\nu_0}_{u^v} v^v = 0, \tag{50}$$

we obtain a connection D^{ν_0} on the vertical subbundle, called the ν_0-vertical lift of D. Now we can give

Definition 5.7. *The ν_0-lift of a pair (∇, D) of connections on M and ξ, with respect to the normalization N_0, defined by the fixed connection D^0 on ξ, is the connection $\mathcal{D}^{\nu_0}$ on E given by*

$$\mathcal{D}^{\nu_0}_A = D^{\nu_0}_A \circ V^0 + \nabla^{h_0}_A \circ H^0. \tag{51}$$

where V^0 and H^0 are the projectors corresponding to D^0. For the torsion $\mathcal{T}^{\nu_0}$ and the curvature $\mathcal{R}^{\nu_0}$ of $\mathcal{D}^{\nu_0}$ we obtain

$$\mathcal{T}^{\nu_0}(X^{h_0}, Y^{h_0}) = T^{\nabla}(X,Y)^{h_0} + \gamma R^{D^0}_{XY}, \tag{52}$$

$$\mathcal{T}^{\nu_0}(X^{h_0}, u^{v}) = S(X,u)^{v}, \quad \mathcal{T}^{\nu_0}(u^{v}, v^{v}) = 0,$$

where $S = D - D^0$,

$$\mathcal{R}^{\nu_0}_{X^{h_0} Y^{h_0}} Z^{h_0} = (R^{\nabla}_{XY} Z)^{h_0}, \mathcal{R}^{\nu_0}_{X^{h_0} Y^{h_0}} u^{v} = (R^{D}_{XY} u)^{v} \tag{53}$$

and zero in rest.

The case of tangent bundle, where the difference between our definitions and those given before is more important, will be treated in another work.

REFERENCES

1. Anastasiei, M. – *Vector bundles. Einstein Equations.* An.şt. Univ.Iaşi, 32, s.I.a Mat. 1986, 17–24.
2. Cruceanu, V. – *Connections compatibles avec certaines structures sur un fibré vectoriel banachique.* Czechosl.Math.J. 24, 99, 1974, 126–142.
3. Cruceanu, V. – *Sur la théorie des sous-fibrés vectoriels.* C.R.Acad.Sc.Paris, t.302, Série I, no.20, 1986, 705–708.
4. Duc, V.T. – *Sur la géométrie differentielle des fibrés vectoriels.* Kodai Math.Sem.Rep. 26(1975), 349–408.
5. Miron, R. – *Techniques of Finsler Geometry in the theory of vector bundles.* Acta Sci.Math. 49(1985), 119–129.
6. Miron, R. and Anastasiei, M. – *The Geometry of Lagrange Spaces: Theory and Applications.* Kluwer Academic Publishers, vol.59, 1994.
7. Yano, K. and Kobayashi, S. – *Prolongations of tensor fields and connections on tangent bundles.* I.J.Math.Soc.Japan, 18(1966), 194–210.
8. Yano, K. and Ishihara, S. – *Tangent and cotangent bundles.* M.Dekker Inc. New York, 1973.

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POISSON–NIJENHUIS MANIFOLDS

BY

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This expository paper is a summary of the author's talk at the 25^{th} Rumanian National Conference on Geometry and Topology held at the "Al.I.Cuza" University, Jassy (Rumania) in September 1995.

The author is grateful to the organizers of the Conference for their kind invitation and support.

The results discussed here either belong to other authors, quoted in references, or were published earlier by the present author. The difference from our previous survey [30] is that, here, we concentrate on Poisson–Nijenhuis structures of finite dimensional differentiable manifolds only. But, for the benefit of the reader, we will add an extensive (but not exhaustive) list of references, which also includes Poisson–Nijenhuis structures in the most general algebraic setting.

1. The Basic Theorem. Poisson–Nijenhuis manifolds were studied by Magri [18], [19], [6], and Y. Kosmann-Schwarzbach and Magri [13] as a method in the theory of completely integrable Hamiltonian dynamical systems. Simultaneously, a corresponding general algebraic theory, and many applications, were developed by Gelfand and Dorfman [8], [7]. We believe that all these studies motivate a purely differential geometric study of finite dimensional Poisson–Nijenhuis manifolds.

In what follows, we assume that the reader is familiar with Poisson manifolds. Otherwise, he should read about them first (e.g., [25]).

The starting point of Magri's studies was to look for first integrals in involution of a Hamiltonian dynamical system in the form of eigenvalues of

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