where V^0 and H^0 are the projectors corresponding to D^0. For the torsion $\mathcal{T}^{\nu_0}$ and the curvature $\mathcal{R}^{\nu_0}$ of $\mathcal{D}^{\nu_0}$ we obtain

$$(52)\qquad \mathcal{T}^{\nu_0}(X^{h_0}, Y^{h_0}) = T^{\nabla}(X,Y)^{h_0} + \gamma R^{D^0}_{XY},$$

$$\mathcal{T}^{\nu_0}(X^{h_0}, u^v) = S(X,u)^v\ ,\quad \mathcal{T}^{\nu_0}(u^v, v^v) = 0,$$

where $S = D - D^0$,

$$(53)\qquad \mathcal{R}^{\nu_0}_{X^{h_0}Y^{h_0}} Z^{h_0} = (R^{\nabla}_{XY}Z)^{h_0}, \mathcal{R}^{\nu_0}_{X^{h_0}Y^{h_0}} u^v = (R^{D}_{XY}u)^v$$

and zero in rest.

The case of tangent bundle, where the difference between our definitions and those given before is more important, will be treated in another work.

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POISSON–NIJENHUIS MANIFOLDS

BY
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This expository paper is a summary of the author's talk at the 25^{th} Rumanian National Conference on Geometry and Topology held at the "Al.I.Cuza" University, Jassy (Rumania) in September 1995.

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The results discussed here either belong to other authors, quoted in references, or were published earlier by the present author. The difference from our previous survey [30] is that, here, we concentrate on Poisson–Nijenhuis structures of finite dimensional differentiable manifolds only. But, for the benefit of the reader, we will add an extensive (but not exhaustive) list of references, which also includes Poisson–Nijenhuis structures in the most general algebraic setting.

1. The Basic Theorem. Poisson–Nijenhuis manifolds were studied by Magri [18], [19], [6], and Y. Kosmann-Schwarzbach and Magri [13] as a method in the theory of completely integrable Hamiltonian dynamical systems. Simultaneously, a corresponding general algebraic theory, and many applications, were developed by Gelfand and Dorfman [8], [7]. We believe that all these studies motivate a purely differential geometric study of finite dimensional Poisson–Nijenhuis manifolds.

In what follows, we assume that the reader is familiar with Poisson manifolds. Otherwise, he should read about them first (e.g., [25]).

The starting point of Magri's studies was to look for first integrals in involution of a Hamiltonian dynamical system in the form of eigenvalues of

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(1,1)-tensor fields which, for technical reasons, should have a vanishing *Nijenhuis torsion*. More exactly, if the system X_H is defined by a *Hamiltonian* H on a Poisson manifold (M,Π), we want it to preserve a (1,1) tensor field A of M (i.e., $L_{X_H}A = 0$, where L denotes the Lie derivative), such that

$$\mathcal{N}_A(X,Y) := [AX,AY] - A[X,AY] - A[AX,Y] + A^2[X,Y] = 0, \tag{1.1}$$

where X, Y are vector fields on M.

Of course, we cannot expect to have a good theory unless there is a good compatibility relation between A and Π. Geometers know such relations are provided by natural concomitants on the tensor fields involved. In the case of a bivector and a (1,1)-tensor, such a concomitant was discovered much earlier by Schouten (who used to study concomitants systematically [23]). First, we will assume that the compositions of A and Π yields again a bivector field Π_1, characterized by $\sharp\Pi_1 = A \circ \sharp\Pi$. (Generally, $\sharp_\Pi$ is defined by $<\sharp_\Pi\alpha,\beta>= \Pi(\alpha,\beta), \alpha,\beta \in T^*(M)$). This condition is equivalent to

$$\Pi(\alpha,\beta\circ A) = \Pi(\alpha\circ A,\beta), \tag{1.2}$$

and it is always assumed in what follows. With this condition, the *Schouten concomitant* is a T^*M-valued bivector given by

$$\begin{aligned} C_{(\Pi,A)}(\alpha,\beta) = \beta\circ L_{\sharp_\Pi\alpha}A - \alpha\circ L_{\sharp_\Pi\beta}A + \\ d(\Pi(\alpha,\beta))\circ A - d(\Pi(\alpha\circ A,\beta)). \end{aligned} \tag{1.3}$$

Using this concomitant, Magri gave

Definition 1.1. *(Π, A) above is a Poisson–Nijenhuis structure of M, and (M, Π, A) is a Poisson–Nijenhuis manifold, if (1.2) holds, and* $C_{(\Pi,A)} = 0$.

In particular, if the Poisson structure Π belongs to a symplectic manifold (M,σ) (i.e., σ is a closed non degenerate 2-form of M, and $\sharp_\Pi\circ\flat_\sigma = Id., \flat_\sigma(X) := i(X)\sigma$) a Poisson–Nijenhuis structure (M,Π,A) is called a *symplectic–Nijenhuis structure* (M,σ,A).

For a Poisson–Nijenhuis structure it is enough to ask only that $C_{(\Pi,A)}(df,dg) = 0$ for any $f,g\in C^\infty(M)$. Computations will put this defining property under the form [26]

$$dg\circ L_{X_f}A - i(X_g)d(df\circ A) = 0. \tag{1.4}$$

where $X_f = \sharp\Pi(df)$ (same for X_g) as usual. From (1.4) we see that, if a Hamiltonian f is such that $df\circ A$ is a closed 1-form, we have $L_{X_f}A = 0$. It is for such Hamiltonian systems that Magri studied complete integrability via Poisson–Nijenhuis structures.

The fundamental property of a Poisson–Nijenhuis structure is

Theorem 1.2. *(The Poisson Hierarchy Theorem [19], [13].) Let (M, Π, A) be a Poisson–Nijenhuis manifold, and put* $\Pi_0 := \Pi$. *Then, there exists a sequence of bivectors* Π_k *defined by* $\sharp\Pi_k = A\circ\sharp\Pi_{k-1}, (k = 1,0,\ldots)$, *which are pairwise compatible Poisson structures on M. Moreover, all the pairs* $(\Pi_k, A^p)(k,p = 0,1,2,\ldots)$ *are again Poisson–Nijenhuis structures, and we say that they form the Poisson hierarchy of M.*

The proof involves rather lenghty computations and we refer either to [19] or to [26] for it. Let us only recall that two Poisson structures Π, Π′ are said to be compatible if Π + Π′ is again a Poisson structure or, equivalently, $[\Pi,\Pi'] = 0$, where the bracket is the *Schouten–Nijenhuis bracket* (e.g., [25]). (Remember also that the characteristic property of a Poisson bivector is $[\Pi,\Pi] = 0$.)

For a symplectic–Nijenhuis structure there are also some other nice characteristic properties.

Theorem 1.3. *[19] i) Let (M,σ) be a symplectic manifold, and A a Nijenhuis tensor of M. Define*

$$\theta(X,Y) = \sigma(AX,Y) \quad (X,Y\in TM). \tag{1.5}$$

Then (M,σ,A) is a symplectic–Nijenhuis manifold iff θ is skew symmetric, and dθ = 0. ii) Let (M,σ) be a symplectic manifold, θ a closed 2-form of M, and $A = -\sharp\Pi\circ\flat\theta$ ($\sharp_\Pi\circ\flat_\sigma = -Id.$) Then (M,Π,A) is a symplectic–Nijenhuis manifold iff the 2-form $\tilde\theta$ defined by

$$\tilde\theta(X,Y) = \theta(AX,Y) = \sigma(AX,AY) \tag{1.6}$$

is a closed 2-form.

Proof. i) The skew symmetric of θ is equivalent to that of Π_1, of the Poisson hierarchy since it follows easily that $\Pi_1 = \sharp\Pi\theta$ for the natural extension of ♯Π. When skew symmetric, θ is called the *associated form* of Π. Then, the definition of θ yields

$$<AX,\alpha>= -\theta(X,\sharp_\Pi\alpha), \tag{1.7}$$

whence computations lead to

$$<dg\circ L_{X_f}A - i(X_g)d(df\circ A),Y>= -d\theta(X_f,X_g,Y). \tag{1.8}$$

Thus, $C_{(\Pi,A)} = 0$ iff $d\theta = 0$.

ii) Using again (1.7), one gets

$$(1.9)\qquad <\alpha, \mathcal{N}_A(X,Y)>= d\theta(AX, Y, \sharp_\Pi\alpha) - d\theta(AY, X, \sharp_\Pi\alpha)+$$

$$+d\theta(X, Y, \sharp_\Pi\alpha) + [\Pi,\Pi](i(X)\theta, i(Y)\theta, \alpha).$$

Since, $d\theta = 0, [\Pi,\Pi] = 0$, and $\sharp_\Pi$ is an isomorphism, $\mathcal{N}_A = o$ iff $d\theta = 0$.

2. Complementary 2–Forms. We defined and studied the notion of a complementary 2-forms of a Poisson structure in [27]. This notion is very helpful in the theory of Poisson–Nijenhuis structures.

It is well known (e.g., [25]) that the cotangent bundle of a Poisson manifold (M,Π) has a natural structure of a *Lie algebroid* of *anchor map* $\sharp_\Pi$, with the Lie bracket of 1-form given by

$$(2.1)\qquad \{\alpha,\beta\}_\Pi = L_{\sharp_\Pi\alpha}\beta - L_{\sharp_\Pi\beta}\alpha - d(\Pi(\alpha,\beta)).$$

This bracket is a natural extension of the natural definition $\{df, dg\} = d\{f,g\}$, where the bracket of functions is the Poisson bracket of (M,Π).

Then, if recall that the usual formulas of calculus on manifolds can be transferred to any Lie algebroid (e.g., [13]), we may extend the bracket (2.1) to Schouten–Nijenhuis bracket of the algebroid T^*M, and get what is called the *Koszul bracket* of any two differential forms of M [14].

Definition 2.1. *[27] The 2-form ω of the Poisson manifold (M,Π) is a complementary form of Π if*

$$(2.2)\qquad \{\omega,\omega\}_\Pi = 0.$$

The main result about complementary forms is

Theorem 2.2. *[27] If ω is a complementary 2-form of the Poisson manifold (M,Π), the formula*

$$(2.3)\qquad [X,Y]' = [AX,Y] + [X,AY] - A[X,Y] + \sharp_\Pi i(X)s(Y)d\omega$$

defines a Lie algebroid structure of anchor $A = \sharp_\Pi \circ \flat_\omega$ on TM. Furthermore, if ω is closed, (Π, A) is a Poisson–Nijenhuis structure of M.

Proof. Formula (2.1) can be used for any "bivector" $B \in \Gamma\wedge^2 E$, where E is an arbitrary Lie algebroid, in order to define a *dual bracket* of B on ΓE^*. We just have used the operations of the E–calculus instead of

same operations of usual calculus encountered in (2.1). And, it is known that, if B is a E–Poisson bivector (i.e., $[B,B]_E = 0$, for the Schouten–Nijenhuis bracket of E), the dual bracket makes E^* into a Lie algebroid of anchor $(anchor\ of\ E) \circ \sharp_B$ [13].

If we use this result for $E = T^*M, B = \omega$, and in view if (2.2), we obtain a Lie algebroid bracket of anchor $\sharp_\Pi \circ \flat_\omega$ on TM. Then, by lengthy computations which use E–calculus, one will check that this bracket defined by formula of type (2.1) is just the bracket $[,]'$ of (2.3). We refer to [27] for the actual computations.

Furthermore, if $d\omega = 0$, the last term of (2.3) vanishes, and the Lie algebroid condition $A[X,Y]' = [AX, AY]$ is exactly $\mathcal{N}_A = 0$. Finally, if we use (1.8) for ω instead of θ, we see that $d\omega = 0$ implies $C_{(\Pi,A)} = 0$. Q.e.d.

In particular, let (M,σ,A) be a symplectic–Nijehuis manifold, and let θ be the associated form of $\Pi(\sharp_\Pi \circ \flat_\sigma = -Id.)$ defined in Theorem 1.3. Since $\Pi_1 = \sharp_\Pi\theta$ belongs to the Poisson hierarchy of (Π, A), we have

$$(2.4)\qquad \sharp_\Pi\{\theta,\theta\}_\Pi = [\Pi_1,\Pi_1] = 0,$$

and, because $\sharp_\Pi$is an isomorphism, $\{\theta,\theta\}_\Pi = 0$ hence, is a complementary form of Π.

In other words, it follows that *the symplectic–Nijenhuis structures are in a 1–1 correspondence with pairs which consist of a symplectic form, and a closed complementary form of the former.*

Furthermore, on the same symplectic manifold, we have a 1–1 correspondence between complementary forms and Poisson structures namely, $\theta \leftrightarrow (\sharp_\Pi\theta)$. Moreover, on symplectic manifolds (M,σ), the following nice formula can be checked (using local coordinates for instance):

$$(2.5)\qquad d\lambda = \{\sigma,\lambda\}_\Pi,$$

where λ is an arbitrary differential form. Hence, $d\theta = 0$ is equivalent with $\{\sigma,\theta\}_\Pi = 0$ i.e., with $[\Pi,\Pi_1] = 0$. Hence, our previous conclusion may be turned around as follows: *the symplectic– Nijenhuis structures are in 1–1 correspondence with pairs of compatible Poisson structures, one of which is symplectic.* For this reason, many authors call the symplectic–Nijenhuis structures *bi–Hamiltonian structures*.

The advantage of complementary forms is functoriality:

Theorem 2.3. *[27] If $\varphi : (M_1,\Pi_1) \to (M_2,\Pi_2)$ is a Poisson morphism, and if ω_2 is a (closed) complementary form of Π_2, then $\omega_1 := \varphi^*\omega_2$ is a (closed) complementary form of Π_1.*

Proof. By definition, a Poisson morphism preserves the Poisson brackets of functions. One can see that this implies the preservation of the Koszul brackets of forms, and the conclusion becomes obvious. Q.e.d.

3. Examples of Poisson–Nijenhuis Manifolds. The case $A = \pm Id.$ is trivial, and will be excluded. As interesting examples we give:

a) ([26]) Let $M = T^*N$ be a cotangent bundle with the canonical symplectic form $\omega_0 = \Sigma d\xi_i \wedge dx^i$, where x^i are coordinates on N, and ξ_i are corresponding convector coordinates. The functions of the form $\varphi = m(X) + f \circ \Pi$, where $f \in C^\infty(N), \Pi : T^*N \to N, X = \Sigma X^i(\partial/\partial x^i)$ is a vector field on N,and $m(X)$ is the *momentum* of X defined by $m(X) = \Sigma X^i \xi_i$ are called *affine functions* on M.

A Poisson structure Π on M such that the bracket of two affine functions is an affine function is called an *affine Poisson structure*, and it is defined by the following brackets:

$$\{f \circ \Pi, g \circ \Pi\}_\Pi = 0, \{m(X), f \circ \Pi\}_\Pi = (AX)f. \tag{3.1}$$

$$\{m(X), m(Y)\}_\Pi = \Phi(X, Y) + m(\Psi(X, Y)),$$

where $f, g \in C^\infty(N), X, Y$ are vector fields on N, Φ is a scalar 2–form on N, Ψ is a TM–valued 2–form on $N, [X, Y]' := \Psi(X, Y)$ makes TM into a Lie algebroid of anchor map A, and Φ is closed with respect to the exterior differential d' of this new Lie algebroid structure.

In particular, Π above is symplectic iff A is non singular and, then, the corresponding symplectic form is

$$\omega_\Pi = \Pi^*\nu + d\mu, \tag{3.2}$$

where $\nu \in \wedge^2 N$ is closed and given by

$$\nu(X, Y) = -\Phi(A^{-1}X, A^{-1}Y),$$

and μ is the 1–form of M defined by

$$\mu_\xi(\chi) = \xi(A^{-1}\Pi_*\chi) \ (\xi \in T^*N, \chi \in T_\xi T^*N).$$

Based on these results, we can prove

Proposition 3.1. *The affine Poisson structure Π of $M = T^*N$ is compatible with the Poisson structure Π_0 of ω_0, and makes (M, ω_0) into a symplectic–Nijenhuis structure, iff $d\Phi = 0$, and*

$$\Psi(X, Y) = [AX, Y] + [X, AY] - A[X, Y]$$

(thereby implying $N_A = 0$).

The proof follows by noticing that, the Poisson structure $\Pi_0 + \Pi$ must also be affine, hence, defined by formulas of the type (3.1). Then, the corresponding conditions for (3.1) imply the conclusion of the present Proposition.

Proposition 3.1 shows us how to define an affine Poisson–Nijenhuis structure on T^*N if we have a closed 2–form ν, and a Nijenhuis tensor A on N. In particular, if N is a Kähler manifold, and we take ν to be the Kähler form, and A the complex structure, we get a pair of symplectic forms of T^*N which have compatible Poisson structures, and make T^*N into a symplectic Nijenhuis manifold [26].

b) ([27]) In a different direction, we give

Proposition 3.2. *Let M be a compact Hermitian symmetric space with metric g and Kähler form σ; then, any harmonic 2–form ω of M is a closed complementary 2–form, and defines a symplectic–Nijenhuis structure of (M, σ).*

Proof. The Koszul bracket of forms can also be expressed by *the Koszul formula* [14]

$$\{\lambda, \mu\}_\Pi = \delta\lambda \wedge \mu + (-1)^{deg\lambda}\lambda \wedge \delta\mu - \delta(\lambda \wedge \mu), \tag{3.3}$$

where: $\delta := i(\Pi)d - di(\Pi)$. Moreover, in the symplectic case δ is Libermann's operator [15], and in the Kähler case $\delta = C'\delta_g C'$, where δ_g is the Riemannian codifferential and C is the Weil operator

$$(C\lambda)(X_1, \ldots, X_k) = \lambda(JX_1, \ldots, JX_k),$$

where J is the complex structure of the manifold.

In our case,

$$\{\omega, \omega\} = 2(C'\delta_g C'\omega) \wedge \omega - (C'\delta_g C')(\omega \wedge \omega) = 0,$$

since ω and, therefore, $C\omega, \omega \wedge \omega, C(\omega \wedge \omega)$ are harmonic forms as well. (On a compact, Riemannian, symmetric space the harmonic forms are the invariant forms, and the exterior product of harmonic forms is harmonic. And, on any Kähler manifold, λ and $C\lambda$ are harmonic simultaneously.) Q.e.d.

c) ([27]) Let $\mathcal{G}$ be a finite dimensional Lie algebra. Then, the dual space $\mathcal{G}^*$ has the *Lie–Poisson structure* Λ defined by

$$\{f, g\}_\Lambda(\gamma) = \langle \gamma, [d_\gamma f, d_\gamma g] \rangle \qquad (f, g \in C^\infty(\mathcal{G}^*), \gamma \in \mathcal{G}^*. \tag{3.4}$$

where the right hand side bracket is in $\mathcal{G}$ (e.g., [25]).

On the other hand, one has the famous *classical Yang–Baxter equation* (e.g., [25])

$$[r, r] = 0, \quad r \in \wedge^2 \mathcal{G}, \tag{3.5}$$

where, [,] is the Schouten–Nijenhuis bracket of $\mathcal{G}$.

Now, what we get is

Proposition 3.3. *Any solution r of (3.5) may be seen as a closed complementary 2–form of $(\mathcal{G}^*, \Lambda)$. Furthermore, if G is a Lie group of Lie algebra $\mathcal{G}$, which acts on the symplectic manifold (M, σ) by a Hamiltonian action with the equivariant momentum $J : M \to \mathcal{G}^*, J^*r$ is a closed, complementary form, and it yields a symplectic–Nijenhuis structure of (M, σ).*

Proof. r may be seen as a 2–form with constant coefficients on $\mathcal{G}^*$; hence, a closed form. Then, one can see that $\{r, r\}_\Lambda = [r, r]$, and (3.5) shows that we have a complementary form. For the second assertion, it is enough to remember that J is a Poisson morphism, and use the functoriality of complementary forms. Q.e.d.

d) [29] The *symplectic reduction* can be extended to Poisson manifolds by the following definition of Marsden–Raţiu [20]. Let (M, Π) be a Poisson manifold, $\iota : N \to M$ a submanifold, and $E \subseteq TM|N$ a subbundle such that: i) $E \cap TN$ foliates N by the fibres of a submersion $\Pi := N \to Q$, ii) if for $\varphi, \psi \in C^\infty(M)$ one has $d\varphi|E = 0, d\psi|E = 0$, then $d\{\varphi, \psi\}|E = 0$. Then, if there exists a Poisson structure Π' on Q such that

$$\{f, g\}' \circ \Pi = \{\varphi, \psi\} \circ \iota, \tag{3.6}$$

$\forall f, g \in C^\infty(Q)$, and where $\varphi, \psi \in C^\infty(M)$ extend f, g, respectively, such that $d\varphi|E = 0, d\psi|E = 0$. (Q, Π') is the *reduction of* (M, Π) *via* (N, E).

Now, for Poisson–Nijenhuis manifolds, one can prove

Theorem 3.4. *Let (M, Π, A) be a Poisson–Nijenhuis manifold, and let (N, E) be as above. Suppose that: $A(TN) \subseteq TN, A(E) \subseteq E$. $A|N$ sends Π–projectable vector fields to Π–projectable vector fields, $\sharp_\Pi$ (Annihilator $E) \subseteq TN$. Then, the reduced Poisson structure Π' of Q exists, A indices a Nijenhuis tensor field A' on Q, and (Q, Π, A') is a Poisson–Nijenhuis manifold (the reduction of (M, Π, A)).*

Proof. The hypotheses imply the Marsden–Raţiu existence condition of Π' [20], and ensure the existence of A'. Then, one can check that the Schouten concomitant $C_{(\Pi', A')} = 0$. Q.e.d. (See [29] for details.)

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ASYMPTOTIC SPECTRA FOR WEYL GEOMETRIES

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1 Preliminaries on Weyl structures. A Weyl structure $\mathcal{W}$ on a smooth (C^∞) manifold M of dimension $m \geq 2$ consists of a torsion free connection $\hat{\nabla}$, called the Weyl connection, a conformal class $\mathcal{C} = \{h\}$ of semi-Riemannian metrics (here and in the following we identify metrics in $\mathcal{C}$ which merely differ by a constant positive factor), and a class $\mathcal{T} := \{\hat{\theta}_h \mid h \in \mathcal{C}\}$ of 1-forms satisfying the compatibility conditions

$$\hat{\nabla} h = 2\hat{\theta} \odot h := 2\hat{\theta}_h \odot h \tag{1.1}$$

The compatibility condition described in equation (1.1) is invariant under so-called gauge transformations

$$h \mapsto h^\# = \beta h, \quad \hat{\theta} \mapsto \hat{\theta}^\# = \hat{\theta} + \tfrac{1}{2} d\, lg\, \beta, \tag{1.2}$$

where $0 < \beta \in C^\infty(M)$. We call $(M, \mathcal{W}) := (M, \hat{\nabla}, \mathcal{C}, \mathcal{T})$ a Weyl manifold.

It is well known that a Weyl structure $\mathcal{W}$ can be generated from a given pair $\{h, \hat{\theta}\}$ (where h is a semi-Riemannian metric and where $\hat{\theta}$ is a 1-form) in the following way. Let $u, v, w, \ldots$ be vector fields on M, let $\hat{\nabla} := \nabla(h)$ be the Levi-Civita connection of h, and let θ be the vector field defined by

$$h(\theta, v) := \hat{\theta}(v) \tag{1.3}$$

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