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ON THE BOCHNER TYPE CURVATURE TENSOR OF $CR-$ STRUCTURES

BY

V. OPROIU

Introduction. In the study of differential geometry there are some tensor fields obtained from the curvature tensor fields of linear connections on real or complex manifolds, with remarkable invariance properties. First of all, I should point out the Weyl projective tensor field obtained from the curvature tensor field of a torsion free linear connection on a real manifold as an invariant tensor field under the projective transformations of the connection (see e.g. [3]). Then, the conformal Weyl tensor field is obtained in a quite similar manner from the curvature tensor field of the Riemannian connection as an invariant under the conformal transformations of the metric on a Riemann space (see e.g. [18]).

In the case of the complex manifolds we have the $H-$ projective curvature tensor field of a torsion free adapted connection obtained from the curvature tensor field as the invariant under the complex projective transformations (see e.g [17]). The Bochner curvature tensor field on a Kaehler manifold is related in a certain way with the conformal changes of the metric (see [19]).

In the case of the quaternion manifolds a projective curvature tensor field arises in a natural way from the integrability problem for the almost quaternal structures (see [8],[9]).

In all the cases presented above, all the traces of the invariant curvature tensor fields are zero.

In the case of the normal almost contact structures we have the $C-$ projective curvature tensor field of a torsion free connection adapted to a

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normal almost contact structure ([4], see also [2],[13], where a similar tensor field is introduced in the case of the cosymplectic manifolds).

This is a survey paper in which I should intend to present some aspects of my joint work with dott-ssa P.Matzeu from the University of Cagliari, Italy [5],[6] as well as some results obtained by my students G.Mitric, [7] and M.Munteanu, [8], concerning the Bochner type tensor of a pseudoconvex $CR-$structure. The interested reader can find the corresponding proofs of the results stated here in the cited papers. The idea of the introducing of a Bochner type tensor field started from the finding of a class of almost contact structures subordinated to a pseudoconvex $CR-$structure. Then we found some torsion free connections associated with these almost contact structures and, finally, we studied the changes of the curvature tensor fields of these associated connections when one changes the subordinate almost contact structure (by the so called gauge transformations). Next we looked for some examples of pseudoconvex $CR-$structures whose Bochner type curvature tensors have some remarkable properties (e.g. are trivial).

Remark that K.Sakamoto and Y.Takemura [11],[12] obtained too a Bochner type curvature tensor for a pseudoconvex $CR-$structure in terms of a connection, called the Tanaka connection [14]. This connection has the property that all the tensor fields involved in the definition of a subordinated almost contact structure are parallel, but it is not torsion free. The expression of the Bochner type tensor obtained by Sakamoto and Takemura is quite complicated and is presented as a sum of two tensors involving not only the Ricci tensor field but, also, other traces of the curvature tensor field of the Tanaka connection. The expression found by the author and P.Matzeu, in terms of the curvature tensor field of a torsion free associated connection is very much similar to that of the Weyl and Bochner curvature tensor fields from the theory of the real and complex manifolds. Moreover, we have been able to show that this tensor does coincide with that obtained by Sakamoto and Takemura.

1. $CR-$structures. Let M be a real $m-$dimensional manifold, TM its tangent bundle and T^cM the complexification of TM. A $CR-$structure on M is defined by a complex vector subbundle $H(M)$ in T^cM such that:

i) $H(M) \cap A(M) = 0$ where $A(M) = \overline{H(M)}$

ii) $H(M)$ is complex involutive .i.e. the bracket $[Z, W]$ of two $H(M)-$valued complex vector fields Z, W is $H(M)-$valued too.

Denote by the same letter $H(M)$ the decomplexification of $H(M)$ and by J the operator on $H(M)$, corresponding to the multiplication by

the imaginary unit i ($J^2 = -1_{H(M)}$). Denote by $\Gamma(H(M))$ the set of the sections in $H(M)$. Then the above conditions are expressed as follows:

$$\Gamma(H(M)) = \{X - iJX \mid X \in \Gamma(H(M))\}$$

$$[X, Y] - [JX, JY] \in \Gamma(H(M)), \ \forall X, Y \in \Gamma(H(M))$$

$$N_J(X, Y) = [JX, JY] - [X, Y] - J\{[JX, Y] + [X, JY]\} = 0, \forall X, Y \in \Gamma(H(M))$$

The $CR-$structures are structures induced naturally on the real submanifolds of complex manifolds. If $p \in M \subset \widetilde{M}$ is a point in the real submanifold M of the complex manifold $\widetilde{M}$, then the tangent space T_pM is a real vector subspace in the complex vector space $T_p\widetilde{M}$ and we may define

$$H(T_pM) = \{v \in T_pM \mid iv \in T_pM\}$$

The dimension of $H(T_pM)$ may vary with the point p. If this dimension is constant on M then we obtain a $CR-$structure $H(TM)$ on M induced from the complex structure of $\widetilde{M}$.

The most interesting case is obtained when M is a real hypersurface in the complex manifold $\widetilde{M}$. If $\dim_{\mathbb{C}}\widetilde{M} = n + 1$ then $\dim_{\mathbb{R}} M = 2n + 1$ and the (real) fibre dimension of $H(TM)$ is constant equal to $2n$. It is obtained a $CR-$structure of hypersurface type.

In the following we shall deal with $CR-$structures of hypersurface type only. The Levi form, which can be defined for a general $CR-$structure, can be expressed, in this case, in a very convenient way. Let $H(M)$ define a $CR-$structure of hypersurface type on the $(2n + 1)-$dimensional manifold M. Let Z, W be two $H(M)-$valued complex vector fields. The bracket $[Z, \overline{W}]$ is a complex vector field on M and it is important to see whether it is $H(M)-$valued or $A(M)-$valued or $H(M) \oplus A(M)-$valued. Then it is natural to consider the projection of $[Z, \overline{W}]$ on the quotient $T^cM/H(M) \oplus A(M)$. The result of this projection is just the Levi form of the considered $CR-$structure. This form can be expressed easily in terms of the decomplexification of $H(M)$. Let η be a (locally defined) $1-$form defining the distribution $H(M)$. If M is orientable then η can be chosen globally on M. Then we have:

$$H(M) = \{v \in TM \mid \eta(v) = 0\}$$

The Levi form can be expressed by the form $i\eta([Z, \overline{W}])$ which is a complex valued sesquilinear form on $H(M)$. The $CR-$structure $H(M)$

is pseudoconvex if its Levi form is nondegenerate. The condition for $H(M)$ to be pseudoconvex can be expressed simply by the condition :

$$\eta \wedge d\eta^n \neq 0$$

Choosing a particular $1-$form η, defining the distribution $H(M)$, we obtain an almost contact structure subordinated to the pseudoconvex $CR-$structure $H(M)$. This almost contact structure is defined by: the tensor field φ, of type $(1,1)$, the $1-$form η, considered above and the vector field ξ (called the Reeb vector field) given by the conditions:

$$\varphi = Jh \; , \; \eta(\xi) = 1 \; , \; i_\xi d\eta = 0$$

where $h = I - \eta \otimes \xi$ is the projection operator on $H(M)$. Remark that we have also:

$$\varphi^2 = -I + \xi \otimes \eta, \quad \varphi\xi = 0, \quad \eta\,\varphi = 0$$

The complex involutivity condition for $H(M)$ is equivalent to

$$S = 0$$

(see [5]), where the tensor field S of type (1,2) is defined by:

$$(1.1)\quad S(X,Y) = N_\varphi(X,Y) + d\eta(X,Y)\xi + \eta(X)\varphi(\mathcal{L}_\xi\varphi)Y - \eta(Y)\varphi(\mathcal{L}_\xi\varphi)X.$$

Here $\mathcal{L}$ denotes the Lie derivative and N_φ is the Nijenhuis tensor field of φ. The tensor field S has some interesting properties, expressed by the formulas:

$$(1.2)\quad \begin{cases} \eta(S(X,Y)) = d\eta(X,Y) - d\eta(\varphi X, \varphi Y), \\ S(X,\xi) = 0, \\ S(\varphi X, \varphi Y) = -S(X,Y), \\ \varphi S(X,\varphi Y) = S(X,Y) - \{d\eta(X,Y) - d\eta(\varphi X,\varphi Y)\}\xi, \end{cases}$$

2. The canonical torsion-free connections of pseudo-convex CR-structures.

Using an almost contact structure subordinated to a given $CR-$structure $(M, H(M))$, we obtain:

Theorem 2.1. *If (φ,ξ,η) is the almost contact structure subordinated to the pseudo-convex CR-structure $(M, H(M))$, then there exists one and only one torsion-free connection ∇ such that for every $X, Y \in \mathcal{X}(M)$*

$$(2.1)\quad \begin{cases} (\nabla_X\eta)(Y) = \frac{1}{2}d\eta(X,Y), \qquad \nabla_X d\eta = 0, \qquad \nabla_X\xi = 0 \\ (\nabla_X\varphi)Y = 2\eta(X)\varphi Y - \frac{1}{2}d\eta(X,\varphi Y)\xi. \end{cases}$$

The connection ∇ is determined by the formulas:

$$(2.2)\quad 2\eta(\nabla_X Y) = 2X(\eta(Y)) - d\eta(X,Y).$$

$$(2.3)\quad \begin{aligned} 2d\eta(\nabla_X Y, Z) = {}& 2\eta(X)d\eta(\varphi Y, \varphi Z) + 2\eta(Y)d\eta(\varphi X, \varphi Z) + \\ & + \varphi Z(d\eta(X,\varphi Y)) + d\eta([X,\varphi Z],\varphi Y) + d\eta([Y,\varphi Z],\varphi X) + \\ & + X(d\eta(Y,Z)) + Y(d\eta(X,Z)) + d\eta([X,Y],Z). \end{aligned}$$

The obtained connection ∇ is called the canonical torsion free linear connection associated with the almost contact structure (φ,η,ξ).

3. Gauge transformations of almost contact structures.

Remark that, instead of the $1-$form η defining the distribution $H(M)$, we may use another $1-$form η' obtained from η by the multiplication with a nonvanishing function g on M, i.e. $\eta' = g\,\eta$, $g \neq 0$. We have:

Proposition 3.1. *(see [11]) The two almost contact structures (φ,ξ,η) and (φ',ξ',η') are associated to the same pseudo-convex CR-structure iff for some function $f \in C^\infty(M)$ they satisfy*

$$(3.1)\quad \begin{cases} \eta' = \varepsilon e^f \eta \qquad d\eta' = \varepsilon e^f\{d\eta + df \wedge \eta\} \\ \varphi' = \varphi + \eta \otimes A \qquad \xi' = \varepsilon e^{-f}\{\xi + \varphi A\} \\ \varepsilon = \pm 1 \end{cases}$$

the vector field A is defined by the conditions

$$(3.2)\quad \eta(A) = 0, \qquad d\eta(\varphi A, X) = df(hX)$$

and $df(X)=X(f)$.

A transformation of η, φ, ξ as above is called a gauge transformation. The transformation of the canonical torsion free connections associated to the two subordinated almost contact structures obtained by a gauge transformation is given by:

Theorem 3.2. *A gauge transformation between two almost contact structures (φ,ξ,η) and (φ',ξ',η') induces a transformation between their associated torsion-free canonical connections ∇ and ∇' given by*

$$\nabla'_X Y = \nabla_X Y + P(X,Y) \qquad X,Y \in \mathcal{X}(M) \tag{3.3}$$

where P is the symmetric tensor field of type (1,2) with the following expression

$$\begin{aligned}2P(X,Y) = {} & df(hX)hY + df(hY)hX + d\eta(X,\varphi Y)A- \\ & - df(\varphi X)\varphi Y - df(\varphi Y)\varphi X + \eta(X)\{df(hY)\xi + df(hY)\varphi A- \\ & - df(A)\varphi Y - 2df(\varphi Y)A - 2\varphi\nabla_{hY}A\} + \eta(Y)\{df(hX)\xi+ \\ & + df(hX)\varphi A - df(A)\varphi X - 2df(\varphi X)A - 2\varphi\nabla_{hX}A\}+ \\ & + 2\eta(X)\eta(Y)\{df(\xi)\xi + df(\xi)\varphi A - [\xi,\varphi A] - \frac{3}{2}df(A)A + \varphi\nabla_{\varphi A}A\}.\end{aligned}$$

4. Curvature of torsion free canonical connections. The curvature tensor field of a canonical torsion free linear connection associated with an almost contact structure subordinated to a pseudo-convex $CR-$structure has some interesting properties described by the formulas:

$$\begin{cases} d\eta(R_{XY}Z,W) = -d\eta(Z,R_{XY}W) \\ R_{XY}\xi = 0 \\ \eta(R_{XY}Z) = 0 \\ \varphi R_{XY}Z = R_{XY}\varphi Z - 2d\eta(X,Y)\psi Z + \eta(X)d\eta(Y,\psi Z)\xi- \\ \qquad -\eta(Y)d\eta(X,\psi Z)\xi + 2\eta(X)(\nabla_Y\psi)Z - 2\eta(Y)(\nabla_X\psi)Z. \end{cases} \tag{4.1}$$

where $\psi = \frac{1}{2}\mathcal{L}\varphi$ and

$$\mathcal{R}(W,Z,X,Y) = g(W,R_{XY}Z) = d\eta(\varphi W,R_{XY}Z); \tag{4.2}$$

Now, studying the change of the curvature tensor fields of the canonical torsion free connections, associated to almost contact structure subordinated to a given $CR-$structure, we obtain the main result:

Theorem 4.1. *Let (φ,ξ,η) be an almost contact structure on M associated with the pseudo-convex CR-structure $(M,H(M))$. Then for $X,Y,Z \in \Gamma(H(M))$ the tensor field*

$$\begin{aligned}B(R)_{XY}Z = {} & R_{XY}Z + L(X,Z)Y - L(Y,Z)X + L(Y,\varphi Z)\varphi X- \\ & - L(X,\varphi Z)\varphi Y - \{L(X,\varphi Y) - L(Y,\varphi X)\}\varphi Z- \\ & - \frac{1}{2}d\eta(X,Z)KY + \frac{1}{2}d\eta(Y,Z)KX + \frac{1}{2}d\eta(X,\varphi Z)\varphi KY- \\ & - d\eta(X,Y)KZ - \frac{1}{2}d\eta(Y,\varphi Z)\varphi KX\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}L(X,Y) & = \frac{n+1}{2n(n+2)}\rho(R)(X,Y) - \frac{1}{2n(n+2)}\rho(R)(\varphi X,\varphi Y)+ \\ & + \frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\varphi Y) \\ & = \frac{1}{2(n+2)}\{\rho(R)(X,Y) + 2d\eta(\varphi X,\psi Y)\}+ \\ & + \frac{1}{8(n+1)(n+2)}\tau(R)d\eta(X,\varphi Y),\end{aligned} \tag{4.4}$$

and

$$\frac{1}{2}d\eta(KX,Y) = L(X,Y)$$

is invariant under gauge transformations.

Next, we tried to find, in [6], the expression of the Bochner type tensor of $CR-$structures in the case of some classes of hypersurfaces in complex or almost complex manifolds.

Let $(\widetilde{M},J)$ be a Kaehelerian manifold with $\dim_{\mathbb{C}}\widetilde{M} = n+1$; denote by $\widetilde{g}$ and $\widetilde{\nabla}$ the metric tensor field and the Levi Civita connection of $\widetilde{M}$ respectively. Assume that $\widetilde{M} = \widetilde{M}^{n+1}(c)$. Consider an orientable $(2n+1)-$dimensional real hypersurface $M \subset \widetilde{M}^{n+1}(c)$; if N denotes the unit normal of M, an almost contact structure (φ,ξ,η) is induced on M by the formulas (see [19])

$$\xi = -JN, \quad \eta(X) = \widetilde{g}(X,\xi), \quad JX = \varphi X + \eta(X)N, \qquad X \in \mathcal{X}(M).$$

Denote by A and $\dot\nabla$ the Weingarten mapping and the Levi Civita connection on M. We have:

$$\begin{cases} \dot\nabla_X \xi = \varphi A X, \quad (\dot\nabla_X \eta)(Y) = g(\varphi A X, Y) \\ (\dot\nabla_X \varphi)Y = \eta(Y) A X - g(AX, Y)\xi, \qquad X, Y \in \mathcal{X}(M). \end{cases} \tag{4.5}$$

Generally, the vector field ξ defined by the almost contact structure induced on the hypersurface M, does not coincide with the Reeb vector field associated with the induced $CR-$ structure on M. In fact, the induced $CR-$ structure could be not pseudo-convex. We shall consider some cases where the induced $CR-$ structure is pseudo-convex and the vector field ξ is an eigenvector of the Weingarten mapping. In these cases the vector field ξ does coincide with the Reeb vector field defined by the $1-$ form η.

The Levi Civita connection $\dot\nabla$ and the canonical torsion free connection ∇ are related by the formula:

$$\nabla = \dot\nabla + Q$$

where Q is a tensor field of type $(1,2)$ on M, which will be determined in the considered cases.

The first class of hypersurfaces we considered is the class of the hypersurfaces for which:

$$A\varphi = \varphi A; \tag{4.6}$$

Then:

$$A\xi = \nu\xi, \tag{4.8}$$

and $\eta(AX) = \nu\eta(X)$ It follows:

$$A^2 X - \nu A X - c(X - \eta(X)\xi) = 0 \qquad X \in \mathcal{X}(M). \tag{4.8}$$

Then we obtain the expression of the tensor field Q:

$$Q(X, Y) = \eta(X)\varphi A Y - \eta(Y)\varphi A X$$

It follows that, in this case, $(M, H(M))$ is Bochner flat.

The second class we considered is the class of hypersurfaces satisfying the condition:

$$A\varphi + \varphi A = \alpha\varphi, \qquad \alpha \neq 0. \tag{4.9}$$

We have again $A\xi = \nu\xi$ and $\eta(AX) = \nu\eta(X)$. Then:

$$2A^2 X - 2\alpha A X + (2c + \nu\alpha)X = 0. \tag{4.10}$$

true for every $X \in \Gamma(H(M))$.

It follows that, in this case, the tensor field Q is given by:

$$Q(X, Y) = \eta(X)\varphi A Y - \eta(Y)\varphi A X + \{\frac{1}{2}\alpha g(X, \varphi Y) - g(AX, \varphi Y)\}\xi$$

Doing the necessary computations, we find that, in this case, $(M, H(M))$ is never Bochner flat.

Another class of real hypersurfaces in almost complex manifolds is given by the unit tangent bundle of a Riemannian manifold. Let (M, g) be an $(n+1)$ dimensional Riemannian manifold. Denote by TM the tangent bundle of the manifold M. Then we have the splitting

$$TTM = VTM \oplus HTM$$

defined by the Levi Civita connection and the usual almost complex structure on TM, defined by:

$$JX^H = X^V, \; JX^V = -X^H ; X \in \chi(M)n \tag{4.11}$$

it is well known that TM with this almost complex structure and the Sasaki metric has a structure of almost Kaehlerian manifold.

Consider the unit tangent bundle T_1M as the bundle of the unit tangent vectors at M. So, if $v \in TM$, then $v \in T_1M \iff g(v, v) = 1$

Theorem ([7],[16]). *The unit tangent bundle T_1M of a Riemannian manifold (M, g) has a Cauchy-Riemann structure induced from the usual almost Kählerian structure of TM, if and only if the base manifold M has constant sectional curvature.*

Theorem ([8],[16]). *Let (M, g) be a Riemannian manifold of dimension $n+1 > 2$ and T_1M be the unit tangent bundle with the standard contact Riemannian structure. Then the gauge invariant B of $(1,3)$-type (the Bochner type tensor associated to the CR-structure induced on T_1M) vanishes, if and only if (M, g) has constant curvature -1.*

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THE TOPOLOGY OF DIFFERENTIAL SPACES

BY

MANFRED GERSTNER AND KLAUS BUCHNER

1. Introduction. There are many generalizations of differential manifolds. One of the most popular among them is Sikorski's Differential Space where the differential structure is expressed by means of real valued functions on this space. Sikorski's definition [10] has recently been extended [3] to allow for functions that are only locally defined. This was necessary in order to apply the theory e.g. to cosmology and gravitational physics.

When working with local functions, the most convenient mathematical formalism is of course sheaf theory. The problem here is that sheaves are defined with respect to an a priory given topology, whereas in Sikorski's theory, many important results are connected with the fact that the initial topology of the differential structure is used.

The aim of the present paper is to show that one may begin with an arbitrary sheaf of functions, representing the differential structure. It is shown that one can always find a coarsest topology and a corresponding sheaf of functions that represents the same differential structure as our starting sheaf – independently of its special choice! The explicit construction justifies the name *initial topology*. The resulting connection between the differentiable structure and the underlying topology is necessary e.g. for the existence of a partition of unity. The complete characterization of spaces allowing a smooth partition of unity can be found in the last section.

Other authors [5], [6] work with a similar, but inequivalent generalization of Sikorski's spaces which was first introduced by Mostow [9]. Their resulting theory is – as far as the sheaf of functions is concerned – a special

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