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THE TOPOLOGY OF DIFFERENTIAL SPACES

BY

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1. Introduction. There are many generalizations of differential manifolds. One of the most popular among them is Sikorski's Differential Space where the differential structure is expressed by means of real valued functions on this space. Sikorski's definition [10] has recently been extended [3] to allow for functions that are only locally defined. This was necessary in order to apply the theory e.g. to cosmology and gravitational physics.

When working with local functions, the most convenient mathematical formalism is of course sheaf theory. The problem here is that sheaves are defined with respect to an a priory given topology, whereas in Sikorski's theory, many important results are connected with the fact that the initial topology of the differential structure is used.

The aim of the present paper is to show that one may begin with an arbitrary sheaf of functions, representing the differential structure. It is shown that one can always find a coarsest topology and a corresponding sheaf of functions that represents the same differential structure as our starting sheaf – independently of its special choice! The explicit construction justifies the name *initial topology*. The resulting connection between the differentiable structure and the underlying topology is necessary e.g. for the existence of a partition of unity. The complete characterization of spaces allowing a smooth partition of unity can be found in the last section.

Other authors [5], [6] work with a similar, but inequivalent generalization of Sikorski's spaces which was first introduced by Mostow [9]. Their resulting theory is – as far as the sheaf of functions is concerned – a special

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case of the one discussed here. (It is limited to the generalization of C^∞-manifolds, whereas finite differentiability classes are not possible.) So the results of the present paper are also valid there, in particular the existence of the *minimal slackening* and its properties, and the construction of the initial topology.

2. Differential Spaces. In this section, we repeat the basic definitions of Differential Spaces. They start from the observation, that many properties of a differential manifold M can be recognized by regarding the functions on M, for example the dimension, the differentiability class, and much of the cohomology. The local coordinates of M are real valued functions on an open subset $U \subseteq M$. Thus Sikorski considers a topological space M and a set C of continuous functions on M:

Definition 1. *(Sikorski). Let M be a set and C a set of functions $M \to \boldsymbol{R}$. If the following properties hold:*

a) M carries the initial topology of C

b) For all $k \in N_0$, all $\alpha \in C^\infty(\boldsymbol{R}^k, \boldsymbol{R})$, and all $f_1, \dots, f_k \in C$, also $\alpha \circ (f_1, \dots, f_k) \in C$

c) Let $g : M \to \boldsymbol{R}$ be an arbitrary function with the property that every point $x \in M$ has an open neighborhood $U \subseteq M$ satisfying

$$g|U = f|U$$

for some function $f \in C$. Then g belongs to C, then C is called **differential structure** *on M, and the pair (M, C)* **Differential Space in the sense of Sikorski.**

Axiom a) ensures that a partition of unity always exists with functions in C, provided M is paracompact. This is a great advantage over Banach and Fréchet manifolds. Axiom b) allows the usual computations such as addition, multiplication, exponentiation etc.

Although Sikorski's Differential Spaces are a very successful concept, some important problems cannot be dealt with in their framework, e.g. compact analytic manifolds and singularities in Einstein's General Relativity. This can be seen from the last axiom: While it uses ideas of sheaf theory, C contains only globally defined functions. Therefore C should be replaced by a sheaf of functions. This has been done by one of us in [3], [4]. A slightly different setting has been worked out by Mostow in [9].

In order to avoid ambiguities, we quote the definition of a sheaf of functions – technically adapted for our purposes concerning changes of topology.

Definition 2. *(sheaf of functions) Let M be a set and γ a collection of subsets $\subseteq M$, forming a topology. For every $U \in \gamma$ let $\mathcal{F}(U)$ be a set of functions $U \to \boldsymbol{R}$ such that for $V \in \gamma$ with $V \subseteq U$ and every $f \in \mathcal{F}(U)$, the restriction $f|V$ is in $\mathcal{F}(V)$. $\mathcal{F}$, regarded as $\{ \mathcal{F}(U) \mid U \in \gamma \}$ is called* **presheaf of functions**. *The local functions $f \in \mathcal{F}(U)$ are called* **(local) sections in** *$\mathcal{F}$, and the topology γ is called* **base system of** *$\mathcal{F}$. If we refer to the base system explicitly, we write $(\mathcal{F}, \gamma)$ instead of $\mathcal{F}$.*

The well known notion of a **germ** *of functions at $x \in M$ can be formulated as equivalence class by the following relation: For $f \in \mathcal{F}(V)$ (where $x \in V$) and $g \in \mathcal{F}(W)$ (where $x \in W$), we state $f \sim g \;:\Leftrightarrow$ there is an $U \in \gamma$ with $U \subseteq V \cap W$ and $f|U = g|U$. The set of all germs of $\mathcal{F}$ at x is called* **stalk** *$\mathcal{F}_x$.*

For successfully gluing local sections we need an additional property: Let $\{U_i\}_{i\in I} \subseteq \gamma$ be a collection of sets in γ; $U := \bigcup_{i\in I} U_i$ and $f : U \to \boldsymbol{R}$ any function with $f|U_i \in \mathcal{F}(U_i)$ for all $i \in I$. Then $f \in \mathcal{F}(U)$.

If a presheaf $\mathcal{F}$ has this property, it is called a **sheaf of functions**, *if in addition all sections in $\mathcal{F}$ are continuous functions (with respect to the base system), then $\mathcal{F}$ is called a* **sheaf of continuous functions**.

For example, the functions $f : \boldsymbol{R}^n \to \boldsymbol{R}$ with continuous derivatives up to order k form the sheaf $C_n^{(k)}$. For any point $y \in \boldsymbol{R}^n$ the stalk $C_{n,y}^{(k)}$ consists of all C^k-germs in n variables at the point y.

For any presheaf $\mathcal{F}$ there is (up to isomorphism) a unique sheaf $\mathcal{G}$ with the same germs, i.e. $\mathcal{F}_x = \mathcal{G}_x \;\forall x \in M$. This sheaf is called the **sheafification** of $\mathcal{F}$.

Now we are able to give the new definition [3] of a differential space:

Definition 3. *Let M be a topological space. The pair $(M, \mathcal{D})$ is called a* **differential space** *(or d*–**space** *for short) and $\mathcal{D}$ a* **differential structure**, *if $\mathcal{D}$ is a sheaf of continuous real-valued functions on M which form an algebra (w.r.t. pointwise operation).*

With germs of functions we can perform some operations similar to those with functions. Very important is the fact, that function germs can be concatenated, if they "fit" together: At any point $x \in M$, take some germs $\phi_1, \dots, \phi_n \in \mathcal{D}_x$ and any C^r-function germ $\alpha \in C_{n,y}^{(r)}$ where $y := (\phi_1(x), \dots, \phi_n(x))$. The concatenation $\psi := \alpha \circ (\phi_1, \dots, \phi_n)$ is a function germ at the same point $x \in M$. It can be obtained via representatives f_k of ϕ_k and a of α as the germ of $a \circ (f_1, \dots f_n)$ at x.

So we can do much more than only add or multiply germs. We get a richer 'algebraic' structure on our structure sheaf than one is used to from the realm of algebraic geometry. This helps us to overcome the problems with the usual definition of *tangent vectors* as derivations: It is well known that such a definition fails even for manifolds, if the differentiability class is finite.

3. Initial topology. On a differential manifold M, the topology and the differential structure are tightly interlocked, since both come locally from the model space $\boldsymbol{R}^n$ – in some cases there is even influence on the global behavior of the topology. This is the fact for the compact case (of course). But besides there is the theorem of Radó that every holomorphic manifold is second countable. This implies that there is, for instance, no *long line*, and all is paracompact, even if not stated explicitly!

On d-spaces, there is by definition only a much weaker connection between differential structure and base system. The requirement that the function germs are continuous with respect to the base system is motivated by some difficulties which come up with the concatenation of germs (cf [3]). But if we ensure that all the germs have continuous representatives, then everything works fine.

So we have a condition that makes germ concatenation possible as well as a condition for our base system: the local sections must be continuous. This yields a limit for our topology not being too coarse.

If on the other hand the topology becomes too fine, we also get some unwanted results. Consider the following examples:

Of great importance is the class of *soft* sheaves, i.e. each section over a closed subset has a global extension. The sheaf of smooth functions on a manifold is always soft. Now take $M := \boldsymbol{R}$ (with usual topology) and as d-structure the sheaf of constant germs. As closed set choose a set consisting of two different points and as germs simply take two constants with different values. We won't find a global constant function with these two values. So our d-structure is not soft. The intuitively obvious initial topology of the constant d-structure is the trivial topology, and here our sheaf is soft!

A central role for global views in differential topology plays the *partition of unity*. On manifolds (considered as hausdorff and second countable), for each open covering there is always a subordinate smooth partition of unity. For our d-space counterexample let $M := \boldsymbol{R}^2$ with usual topology. As differentiable structure we take all functions depending only on the first coordinate. Now there are open coverings without a subordinate partition of unity. However, this was not the case if M had its obvious initial topology (i.e. open sets are stripes parallel to the second coordinate axis of $\boldsymbol{R}^2$).

3.1. Local initial topologies? Consider a set or a topological space X, a space Y, and a set F of (maybe continuous) functions $X \to Y$. Changing the topology on X can change the behavior of F, for instance the continuity of its elements. But it does not influence the set F and its elements! This is trivially true – but not any longer if we leave the context of 'functions' and turn to sheaves in general. For, sheaves have the topology strictly in their definition. In the next section we will see how the differential structure can behave when the topology changes. But for now, we will try to overcome these difficulties by renouncing the structure sheaf and using only the (locally) initial topologies.

Now let M be a set, $(\mathcal{F},\gamma)$ a sheaf of (not necessarily continuous) functions. For each $U \in \gamma$ we consider the topological space (U,τ_U), where τ_U is the initial topology on U, generated by the set $\mathcal{F}(U)$ of real valued functions on U. So we've got a family

$$\{ (U,\tau_U) \mid U \in \gamma \}$$

of topological spaces. Our demanded *initial topology* σ on M for this family should be the coarsest of all those topologies, which induce τ_U on each U, i.e.

$$\sigma|U \geq \tau_U \qquad \forall U \in \gamma.$$

The usual way to proceed is the following: Take the lattice Top (M) of all topologies on M and regard the subsystem $T \subseteq$ Top (M) consisting of all those topologies $\alpha \in$ Top (M) such that $\alpha|U \geq \tau_U$ $(\forall U)$, i.e. all those topologies, whose restriction to U is at least as fine as the given topology. Then take the coarsest of all those members of T. Of course T is not empty, since the discrete topology belongs to T. But unfortunately there is in general no minimal element in T. It is not hard to find an example for a family (τ_U) and two topologies $\alpha, \beta \in T$ such that there is no element in T which is simultaneously coarser than α and β. So we have to give up this idea.

Actually we are provided with more information than just the knowledge of this directed system of topologies! There still is our structure sheaf.

3.2. Modifications of the base topologies of a sheaf of functions. What we are looking for, is a topology that serves as well as base system for the structure sheaf as for a topology that makes our function germs continuous. So we regard some suitable changes of the base system. Among them there is a distinguished base system that will serve as our initial topology, as it is the coarsest one. For a given structure sheaf the modified sheaf with this initial topology should represent the same differential structure. In the sequel we will introduce those changes and prove the existence and uniqueness of the initial topology.

Concerning changes of the base topology that respect in some sense the structure of the sheaf we have two basic modifications. The first one makes the topology coarser and is always possible, whatever kind of sheaf we have. The other refines the base system and makes use of the ability of functions of being restrictable to any subset.

Let $(\mathcal{G}, \beta)$ and $(\mathcal{F}, \gamma)$ be two sheaves on M. We call $\mathcal{G}$ **looser than** $\mathcal{F}$ (notation: $\mathcal{G} - \mathcal{F}$), if the base system of $\mathcal{G}$ is coarser than that of $\mathcal{F}$ ($\beta \leq \gamma$) and if $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for all $U \in \beta$. Let $\alpha \leq \gamma$ be any coarser topology. By $\mathcal{F}|\alpha$ we denote the presheaf $(\mathcal{H}, \alpha)$, defined by $\mathcal{H}(U) = \mathcal{F}(U)$ for $U \in \alpha$.

Note 1. Let $Sh(M)$ be the set of all sheaves of functions on the set M. Then $(Sh(M), -)$ is an ordered set. Any family $(\mathcal{F}_i)_{i \in I}$ of looser sheaves $\mathcal{F}_i - \mathcal{F}$ than a given one has an infimum in $Sh(M)$.

Proof. Let $(\mathcal{F}_i, \alpha_i)_{i \in I}$ be a family of function sheaves on M, $\mathcal{F}_i - \mathcal{F}$ for all $i \in I$. Let β be the finest topology on M that is coarser than (or equal to) each of the topologies α_i, i.e. β is the infimum of the α_i. By $\mathcal{G}(U) := \bigcap_{i \in I} \mathcal{F}_i(U)$ for each $U \in \beta$ we get a presheaf $(\mathcal{G}, \beta)$ of functions on M which is obviously the infimum of the $\mathcal{F}_i$. To show that $\mathcal{G}$ actually is a sheaf, we take any glueable collection $(U_k, f_k)_{k \in K}$ of sections in $\mathcal{G}$, i.e. $U_k \in \beta$, $f_k : U_k \to \boldsymbol{R}$, $f_k \in \mathcal{G}(U_k)$ for all $k \in K$, $U_{jk} := U_j \cap U_k$, $f_j|U_{jk} = f_k|U_{jk}$ for all $j, k \in K$. The set $U := \bigcup_{k \in K} U_k$ lies in β since it is the union of β-open sets, so $U \in \alpha_i$ for all $i \in I$. The uniquely determined function $f : U \to \boldsymbol{R}$, $f|U_k = f_k$ lies in every $\mathcal{F}_i(U)$ since each $\mathcal{F}_i$ is a sheaf, so $f \in \bigcap_{i \in I} \mathcal{F}_i(U) = \mathcal{G}(U)$. This shows that $\mathcal{G}$ is in fact a sheaf of functions. ■

Note 2. For any $(\mathcal{F}, \gamma)$ and any $\alpha \leq \gamma$ the relation $\mathcal{F}|\alpha - \mathcal{F}$ holds. According to the above note, $\mathcal{F}|\alpha$ really is a sheaf.

Now choose a sheaf $(\mathcal{F}, \gamma)$ and $\alpha \leq \gamma$. Sometimes it is possible to **recover** our original sheaf $\mathcal{F}$ from $\mathcal{G} := \mathcal{F}|\alpha$. This is the case, if for every $V \in \gamma$, every $f \in \mathcal{F}(V)$ there is $U \in \alpha$, $U \supseteq V$ and $g \in \mathcal{G}(U)$ with $g|V = f$.

Of course such a recovery is not always possible. Consider $C_1^{(1)}$ on $M := \boldsymbol{R}$ and choose α as the trivial topology $\{M, \emptyset\}$, where one can't get $1/x$ as restriction of a global function.

So we see that such base systems are special, that preserve the sheaf in a recoverable way. Thus we define:

Definition 4. *Let $(\mathcal{F}, \gamma)$ be a sheaf and $\alpha \leq \gamma$ any coarser topology. We call α a* **slackening** *of $\mathcal{F}$, if $\mathcal{F}$ can be recovered from $\mathcal{F}|\alpha$.*

For convenience we introduce the set $\int \mathcal{F}$ of all sections of $(\mathcal{F}, \gamma)$ as $\int \mathcal{F} := \bigcup_{U \in \gamma} \mathcal{F}(U)$. Since functions with different domains are distinct, the operator Def[.]: $\int \mathcal{F} \to \gamma$ that assigns to any section f its domain U

(i.e. $f \in \mathcal{F}(U)$) is well defined. Writing $g \sqsupseteq f$ iff Def[g] $\supseteq$ Def[f] and $g|$ Def $[f] = f$ we get an ordered set $(\int \mathcal{F}, \sqsupseteq)$.

Let $S(\mathcal{F})$ be the set of all slackenings of a given sheaf $(\mathcal{F}, \gamma)$. By the usual '*is-coarser*'-order $\alpha \leq \beta$ among topologies we get an ordered set $(S(\mathcal{F}), \leq)$. Of course, $S(\mathcal{F})$ is not empty, since $\gamma \in S(\mathcal{F})$. The following two lemmas will show that for any two topologies $\sigma, \tau \in S(\mathcal{F})$ there is a coarser topology $\eta \in S(\mathcal{F})$ and that any descending chain of elements in $S(\mathcal{F})$ possesses an infimum in $S(\mathcal{F})$. So by Zorn's Lemma we have a unique minimal element in $S(\mathcal{F})$. In other words:

Proposition 1. *For any sheaf of functions $(\mathcal{F}, \gamma)$, there exists a minimal slackening.*

The two lemmas are:

Lemma 1. *For any two slackenings $\sigma, \tau \in S(\mathcal{F})$ there exists an $\eta \in S(\mathcal{F})$ such that $\eta \leq \sigma$ and $\eta \leq \tau$.*

Proof. For any section $f \in \int \mathcal{F}$ we will find an extension $F_f \sqsupseteq f$ where Def $[F_f]$ is as well in σ as in τ. The collection $\{$ Def $[F_f] \mid f \in \int \mathcal{F}\}$ then yields (as a subbasis) a topology η which is coarser than σ and τ. By construction $\mathcal{F}$ can be recovered from $\mathcal{F}|\eta$, so η is our slackening.

So let $f \in \int \mathcal{F}$ be any section. There is a $g_1 \in \int \mathcal{F}|\sigma$ with $g_1 \sqsupseteq f$ (since σ is a slackening). Especially, $g_1 \in \int \mathcal{F}$. Now there is a $h_1 \in \int \mathcal{F}|\tau$ with $h_1 \sqsupseteq g_1$ (same reason) and so on. We get two sequences $(g_i) \subseteq \int \mathcal{F}|\sigma$ and $(h_i) \subseteq \int \mathcal{F}|\tau$ with $g_i \sqsubseteq h_i \sqsubseteq g_{i+1} \sqsubseteq h_{i+1} \ldots$ The consequence for the union of their domains $U := \bigcup$ Def $[g_i] \in \sigma$ and $V := \bigcup$ Def $[h_i] \in \tau$ is that $U = V$. By the sheaf property of $\mathcal{F}$ there is a section $F \in \mathcal{F}(U)$ with $F|$ Def $[g_i] = g_i$ and $F|$ Def $[h_i] = h_i$. ■

Lemma 2. *Every chain $\emptyset \neq C \subseteq S(\mathcal{F})$ has a lower bound in $S(\mathcal{F})$.*

Proof. Let $f \in \int \mathcal{F}$ be any section. We are looking for a $g \in \int \mathcal{F}$ such that $g \sqsupseteq f$ and Def $[g] \in \sigma$ for all $\sigma \in C$. If we can show that for any f there exists such a g the proof is finished since we can build a slackening out of the set of all those g in the same way as in lemma 1.

By a C-extension of f we denote a subset $I \subseteq C$ (which is also a chain) as index set, and a family $\{F_\sigma \mid \sigma \in I\} \subseteq \int \mathcal{F}$ such that Def $[F_\sigma] \in \sigma$ and $F_\tau \sqsupseteq F_\sigma \sqsupseteq f$ for all $\sigma, \tau \in I$ and $\tau \leq \sigma$. Let $\boldsymbol{E}$ denote the set of all C-extensions of f. $\boldsymbol{E}$ is not empty: take any $\sigma \in C \neq \emptyset$ and an extension $h \sqsupseteq f$ in $\mathcal{F}|\sigma$; since σ is a slackening, such a h always exists. By $I := \{\sigma\}$, $F_\sigma := h$ we get an element of $\boldsymbol{E}$. We will show, that $\boldsymbol{E}$ helps us to find an appropriate function $g \sqsupseteq f$ we are looking for. The existence will be ensured by the help of Zorn's Lemma. So we have to establish an order on $\boldsymbol{E}$: For any two families $\{F_\sigma \mid \sigma \in I\} =: F \in \boldsymbol{E}, \{G_\tau \mid \tau \in K\} =: G \in \boldsymbol{E}$ let $F \leq G$ if and only if $I \subseteq K$ and $F_\sigma = G_\sigma$ for all $\sigma \in I$, i.e. iff F is a subfamily of G.

Now let $\boldsymbol{C} \subseteq \boldsymbol{E}$ be any nonempty chain. The index set of each element in $\boldsymbol{C}$ is a subset of C, so the union K of all index sets in $\boldsymbol{C}$ is again a subset of C; especially it is a chain. For every $\sigma \in K$ there is a family in $\boldsymbol{C}$ whose index set contains this σ. We define F_σ as the corresponding element in this family. This is (due to our order) well defined. The result is a family $\{F_\sigma \mid \sigma \in K\}$ which lies in $\boldsymbol{E}$ and is an upper bound for $\boldsymbol{C}$. By Zorn's Lemma, $\boldsymbol{E}$ has maximal elements.

Now take any maximal element $\{F_\sigma \mid \sigma \in K\} \in \boldsymbol{E}$. Let $U := \bigcup_{\sigma \in K} \operatorname{Def}[F_\sigma]$. Since $\mathcal{F}$ is a sheaf there is a section $g \in \mathcal{F}(U)$ such that $g \sqsupseteq F_\sigma$ for all $\sigma \in K$. To finish the proof we show, that $\operatorname{Def}[g] \in \tau$ for all $\tau \in C$ as required. Now as a matter of fact, for each $\tau \in C$ there is an $\eta \in K$ with $\eta \leq \tau$. Because otherwise we could extend our family in the following way: the index set is $K \cup \{\tau\}$, the additional section is $F_\tau := g$. But this is a contradiction to the fact that our family was maximal. Since $\operatorname{Def}[g] \in \eta$ for all $\eta \in K$ and since for any $\tau \in C$ there is an $\eta \leq \tau$, we see that $\operatorname{Def}[g] \in \tau$. ∎

In order to get a better understanding of what this minimal slackening looks like, we give the following

Lemma 3. *Let $(M, \mathcal{D})$ be a d-space. Since $\mathcal{D}$ is a sheaf, the ordered set $(\int \mathcal{D}, \sqsupseteq)$ has maximal elements. Let $S \subseteq \int \mathcal{D}$ denote the set of all maxima. The system*

$$\{\ \operatorname{Def}[f] \mid f \in S\ \}$$

of subsets of M is a subbasis for a topology σ on M. Then σ and the minimal slackening μ are identical.

Proof. To any $f \in S \subseteq \int \mathcal{D}$ there is an extension $g \sqsupseteq f$, $\operatorname{Def}[g] \in \mu$ (since μ is a slackening). f is maximal, so $g = f$. $\operatorname{Def}[f] \in \mu$ and we conclude: $\sigma \leq \mu$. On the other hand, for any section $g \in \int \mathcal{D}$ there is a maximal extension $f \sqsupseteq g$. The topology σ we get by the subbasis $\{\operatorname{Def}[f] \mid f \in S\}$ therefore actually is a slackening; and since μ is the minimal slackening, $\mu \leq \sigma$ holds. ∎

3.3. The initial topology of a d-space. The minimal slackening μ of a sheaf of functions $\mathcal{F}$ is, in other words, the coarsest of all topologies which preserve the sectional structure of $\mathcal{F}$. $\mathcal{F}|\mu$ is a sheaf of functions but not a sheaf of *continuous* functions (with respect to μ) in general. So we can't use μ as initial topology, since for a given d-space $(M, \mathcal{D})$ we won't get a d-space $(M, \mathcal{D}|\mu)$.

Let $S^\circ(\mathcal{D})$ denote the subset of all those slackenings $\tau \in S(\mathcal{D})$ where $\mathcal{D}|\tau$ is a sheaf of continuous functions. The order relation on $S(\mathcal{D})$ induces an order on $S^\circ(\mathcal{D})$. Moreover, $(S^\circ(\mathcal{D}), \leq)$ has a minimal element, too. The existence of the minimum can be proved quite in the same manner as above (by using these two facts: If functions are continuous with respect to every topology out of a given collection, then they are continuous with respect to the infimum of this collection, too. If η is a slackening of $(\mathcal{F}, \gamma)$, then any topology σ with $\eta \leq \sigma \leq \gamma$ is a slackening too). We will show the existence in a different and more constructive way, to emphasize the closely related nature of this topology to usual initial topologies.

Definition 5. *For a given d-space $(M, \mathcal{D})$, the minimal continuous slackening μ of $\mathcal{D}$ is called* **initial topology**, *$\mathcal{D}|\mu$ is called* **initial d-structure**.

Note 3. $(M, \mathcal{D}|\mu)$ is a d-space. Obviously $\mathcal{D}|\mu$ does not carry more information about the differential structure than $\mathcal{D}$, since it is looser. On the other hand, $\mathcal{D}$ can be recovered from $\mathcal{D}|\mu$, so no information is lost. We say that $\mathcal{D}$ and $\mathcal{D}|\mu$ represent the same differential structure.

The following lemma shows explicitly, how the initial topology can be constructed. Together with lemma 3, one gets in this way some idea about its rôle in actual calculations.

Lemma 4. *Let $(M, \mathcal{D})$ be a d-space and ν the minimal slackening of $\mathcal{D}$. To any $U \in \nu$, $f \in \mathcal{D}(U)$ and any open $V \subseteq \boldsymbol{R}$, we regard $f^{-1}V \subseteq U$ as a subset of M (since $U \subseteq M$). So we get a system*

$$\{f^{-1}V \mid V \subseteq \boldsymbol{R} \text{ open},\ U \in \nu,\ f \in \mathcal{D}(U)\}$$

of subsets of M. It is a subbasis of a topology σ on M. This σ is equal to the initial topology of $(M, \mathcal{D})$.

Proof. Every slackening base system τ, where $\mathcal{D}|\tau$ is a sheaf of *continuous* functions, contains at least the sets of our given subbasis; so $\sigma \leq \tau$. By the minimality of the initial topology μ we get $\sigma = \mu$.

4. Further remarks.

4.1. Partition of the unity. d-spaces are generalizations of definition 1 in the following way: Starting with a given Differential Space (M, C), we get a d-space $(M, \mathcal{D})$ by $\mathcal{D}(U) := \{f|U \mid f \in C\}$ for all open $U \subseteq M$, accomplished by the sheafification process. Even more, $\mathcal{D}$ is an initial d-structure (due to definition 1a, cf. lemma 4). The original set C can be regained as the set of all global sections $C = \mathcal{D}(M)$ (due to definition 1c).

Obviously there are d-spaces that can't be constructed from Differential Spaces in this manner.

We say, a Differential Space (M, C) has the **PoI property**, iff for every open covering of M there is a subordinate partition of unity in C. A d-space is said to have the **PoI property**, if the partition of unity can be built out of the global sections. In [8] is shown, that every Differential Space has the PoI property. For d-spaces the following disappointing fact holds:

Lemma 5. *A d-space $(M, \mathcal{D})$ with initial d-structure has the PoI property if and only if it derives from a Differential Space.*

Proof can be found in [3]. According to the remarks about the initial topology it is necessary that the differential structure is *initial*.

4.2. Pullback of a d-space. As compared to differential manifolds, it is a great advantage of d-spaces that a pullback exists for arbitrary maps:

Let $(M, \mathcal{D})$ be a d-space, A be any set and $f : A \to M$ an arbitrary map. There exists a unique sheaf $f^{\#}\mathcal{D}$ of continuous functions such that $(A, f^{\#}\mathcal{D})$ is a d-space and behaves as the initial structure of f.

Note.: The usual sheaf theoretical pullback $f^{*}\mathcal{D}$ is (in general) quite different (not isomorphic) to our special construction $f^{\#}\mathcal{D}$.

But even if $(M, \mathcal{D})$ has its initial d-structure, the pullback $(A, f^{\#}\mathcal{D})$ in general has not.

Fortunately the following lemma holds:

Lemma 6. *Let, as in above, $(M, \mathcal{D})$ be an initial d-structure, $f : A \to M$ any mapping from an arbitrary set A to M. Denote by τ the topology of M and by σ the topology pulled back on A, i.e. the initial topology of f. If f is an open mapping between these two topological spaces, then $f^{\#}\mathcal{D}$ is an initial d-structure.*

The use of this lemma is restricted from quite a different point of view: With manifolds, the projection $\pi : TM \to M$ of the tangent bundle is an open mapping. Although one can establish the machinery of bundles quite in similar way for d-spaces (see [3]), the projection is in general not open! So lemma 6 is of no help in this case.

For a deeper examination on constructions like initial d-spaces, examples for initial, but not minimal slackening, counterexamples in the non-open case and so on, see [3].

4.3. Product of d-spaces. Also, the product of two d-spaces is a d-space in a canonical way (see [3]), and moreover, it is compatible with the initially of d-structures:

Lemma 7. *Let $(M, \mathcal{D})$ and $(N, \mathcal{D}')$ be two d-spaces with initial d-structure. Then the product structure is an initial d-structure, too.*

The proof is not straight forward. It can be found in [3].

4.4. Consequences for cosmology. The initial topology can be used to determine the topology near singularities of space-times, e.g. of black holes or the big bang. There the construction of Schmidt's b-boundary yields rather strange results [2], [7]. So the only neighborhood of the initial big bang is the whole of Friedmann's solution, including the final singularity (if it exists). In contrast, the method developed in the present paper determines a topology as expected from the behavior of particles near the singularity. This will be discussed in detail in a forthcoming paper [1].

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