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A REPORT ON INVARIANT GAUGE FORMS ON ASSOCIATED BUNDLES OF AN ABELIAN PRINCIPAL BUNDLE

BY

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The goal of this paper is to present, in a unified way, several results which have been obtained separately by the authors in three previous papers. Basically, these results concern the structure of the algebra of gauge invariant differential forms on an abelian principal bundle: that is, a principal bundle whose structure group is a circle. The paper is divided into three parts. In the first part, the trivial bundle is considered; that is, the bundle $\pi : P = M \times U(1) \to M$, of structure group $G = U(1)$. Then, the gauge invariant differential forms on that trivial bundle are characterized. In the second step, the generalization of the above result to an arbitrary $U(1)$-bundle, not necessarily trivial, $\pi : P \to M$ with the same structure group $G = U(1)$ is studied. Finally, in the third part, the results obtained in the interacting case $\pi : K \times_M E \to M$, *i.e.*, in the case of a gauge field on K interacting with a particle field on E, are explained.

Introduction and Notations. Let us consider a principal bundle $\pi : P \to M$ over a base manifold M of dimension m, whose structure group is G, and let $p : K = K(P) \to M$ be the bundle of connections of P. Let us also consider the gauge algebra of P, gau$P$, that is, the π-vertical G-invariant vector fields on P, which can be identified with the sections of the adjoint bundle of P: gau$P = \Gamma(M, \mathrm{ad}P)$. Moreover, the action of the group of automorphisms of P on the bundle of connections, K, induces a natural representation of Lie algebras:

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$$\mathrm{gau}P \to \mathcal{X}(K), \qquad X \to \tilde{X}.$$

A Lagrangian density $\omega_m = \mathcal{L}dx_1 \wedge ...dx_m$, where $\mathcal{L} : J^1(K) \to \mathbb{R}$ is a real valued differentiable function defined on the 1-jet bundle of connections, is gauge invariant if

$$L_{\tilde{X}_{(1)}}\omega_m = 0, \forall X \in \mathrm{gau}P,$$

where

$$\mathcal{X}(K) \to \mathcal{X}(J^1(K)), \qquad X \to X_{(1)},$$

is the natural lifting of vector fields to jet bundles by infinitesimal contact transformations.

This paper was initially motivated by what is now called the Geometric Formulation of Utiyama's Theorem ([8]). This formulation states that a first order Lagrangian density $\mathcal{L}dx_1 \wedge ... \wedge dx_m$ on the bundle of connections of a principal bundle, $K(P)$, is gauge invariant if and only if the Lagrangian $\mathcal{L}$ factors through the curvature mapping: Ω, i.e., $\mathcal{L} = \mathcal{L}' \circ \Omega$, by means of a function $\mathcal{L}'$ defined on the curvature bundle $\Lambda^2 T^*(M) \otimes \mathrm{ad}P$, which must also be invariant under the gauge algebra representation on that bundle (for the details see [1], [2] and [3]).

The curvature mapping is the mapping sending the 1-jet prolongation of a connection Γ, $j^1\Gamma$, onto its curvature form Ω_Γ.

$$\Omega : J^1(K) \to \Lambda^2 T^*(M) \otimes \mathrm{ad}P,$$

where $\mathrm{ad}P$ is the adjoint bundle, that is, the bundle associated to P by the adjoint representation of G on its Lie algebra g, and $\mathcal{L}'$ is a function

$$\mathcal{L}' : \Lambda^2 T^*(M) \otimes \mathrm{ad}P \to \mathbb{R}.$$

As is well-known, Lagrangian densities are a very special kind of differential forms on the bundle of connections because they are horizontal and their degree is equal to the dimension of the base manifold. Due to the importance of this notion in Gauge Theories, it seems to be of interest to extend this idea in two directions. First, to consider arbitrary differential forms, not necessarily horizontal forms, on the bundle of connections and to apply them the invariance condition, and second, to analyze the invariance of differential forms not only under the gauge algebra, but also under the full Lie algebra of G-invariant, not necessarily π-vertical, vector fields on K.

In this paper we generalize this study to not necessarily horizontal forms of arbitrary degree.

1. First Part. In the first part, we consider the trivial principal bundle $\pi : P = M \times U(1) \to M$ of structure group $U(1)$. The starting point is to connect the two natural structures appearing in the cotangent bundle of a m-dimensional manifold M. On one hand, $T^*(M)$ is endowed with a canonical symplectic form $d\omega$, where ω stands for Liouville's form on $T^*(M)$, and, on the other, $T^*(M)$ can be canonically identified with the bundle of connections of the trivial principal bundle $P = M \times U(1)$:

$$p : K(M \times U(1)) = T^*(M) \to M.$$

This identification is as follows: every connection form can be written as: $\omega_\Gamma = (d\theta + \pi^*(\omega)) \otimes A$, where ω is a 1-form on M, A is the generator of the standard 1-parameter group in $U(1)$, i.e., the generator of the parametrization: $t \to e^{it}$, and θ is the angle in $U(1)$. So, we can identify the connection forms on the trivial $U(1)$-principal bundle with the 1-forms on M.

The gauge group of P, $\mathrm{Gau}P$, is the group of the vertical automorphisms of P, i.e., the group of all equivariant automorphisms Φ from P to P inducing the identity on M. We have the following diagram:

$$\begin{tikzcd} P \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

The gauge group of P acts on the bundle of connections in a natural way (by pulling back connections forms): $\Phi \longmapsto \tilde{\Phi}$, $\Phi \in \mathrm{Gau}P$. In particular, if P is the trivial principal bundle, $\mathrm{Gau}(M \times U(1))$ acts on $T^*(M)$.

Hence we obtain a representation of the gauge algebra of $M \times U(1)$ into the vector fields of $T^*(M)$:

$$\mathrm{gau}(M \times U(1)) \to \mathcal{X}(T^*(M)), \qquad X \to \tilde{X}$$

Remarks:

1. Locally, a vector field X belongs to the gauge algebra of $M \times U(1)$, $X \in \mathrm{gau}(M \times U(1))$, if and only if X can be written as: $X = g(q_1, ..., q_m) A^*$, where A^* is the fundamental vector field associated to A.
2. The vector field $\tilde{X}$ corresponding to the vector field X is:

$$\tilde{X} = -\frac{\partial g}{\partial q_i}\frac{\partial}{\partial p_i};$$

where (q_i, p_i) are the standard coordinates on $T^*(M)$.

Now, we shall study the invariance of the differential forms under the gauge algebra associated to the trivial principal bundle: $\mathrm{gau}(M \times U(1))$.

A differential form Ω on $T^*(M)$ is said to be $\mathrm{gau}(M \times U(1))$–invariant if and only if for every $X \in \mathrm{gau}(M \times U(1))$, we have

$$L_{\tilde{X}}\Omega = 0$$

Theorem 1. *A differential form Ω, $\Omega \in \Omega^\bullet(T^*(M))$ is $\mathrm{gau}(M \times U(1))$ -invariant if and only if there exist differential forms on the base manifold, $\eta_i \in \Omega^\bullet(M)$, such that:*

$$\Omega = \sum_{i=0}^{m} p^*(\eta_i) \wedge (d\omega)^i,$$

where ω stands for the Liouville's form on the cotangent bundle , that is, $d\omega = dp_i \wedge dq_i$.

So ([4]), the algebra of gauge invariant forms on $T^*(M)$ is the $\Omega^\bullet(M)$-algebra generated by the canonical symplectic form of the cotangent bundle, $d\omega$:

$$\{\text{gauge invariant forms on } T^*(M)\} = p^*\Omega^\bullet(M)[d\omega].$$

In this case, we can see that each differential form Ω_m on $T^*(M)$ of degree $m = \dim M$ induces a Lagrangian density. In fact, we have

$$\lambda(\Omega_m) : J^1(T^*(M)) \to \Lambda^m T^*(M), \qquad \lambda(\Omega_m)(j^1_x\alpha) = (s^*_\alpha \Omega_m)(x).$$

where $s_\alpha : M \to T^*(M)$ is the section of the cotangent bundle (tautological) induced by the differential 1-form α on M. According to the previous theorem, if Ω_m is gauge invariant, then this Lagrangian density is also gauge invariant. Moreover, as $s^*_\alpha \omega = \alpha$, we can write an explicit expression for $\lambda(\Omega_m)$:

$$\lambda(\Omega_m)(j^1_x\alpha) = \sum_{0 \le 2i \le m} \eta_i(x) \wedge (d_x\alpha)^i,$$

with $\mathrm{degree}(\eta_i) = m - 2i$.

Next, if we consider the full group of the automorphisms of P, not necessarily π-vertical, $\mathrm{Aut}P$, then its Lie algebra, $\mathrm{aut}P$, *i.e.*, the algebra of infinitesimal automorphisms of P, coincides with the G-invariant vector fields on P, not necessarily π–vertical.

The automorphism group acts naturally on the bundle of connections of P by simply pulling back connections forms ([7]). In particular, if P is the trivial principal bundle, $\mathrm{Aut}(M \times U(1))$ acts on $T^*(M)$.

If $(N; q_1, \ldots, q_m)$ is a coordinate domain of M, a vector field $X \in \mathcal{X}(M \times U(1))$ is $U(1)$-invariant if and only if on $\pi^{-1}(N)$ can be written as

$$X = f_i(q_1, \ldots, q_m)\frac{\partial}{\partial q_i} + g(q_1, \ldots, q_m)A^*.$$

Hence, we obtain a representation of the Lie algebra of infinitesimal automorphisms of $M \times U(1)$ into the vector fields of $T^*(M)$:

$$\mathrm{aut}(M \times U(1)) \to \mathcal{X}(T^*(M)), \qquad X \to \tilde{X},$$

where the local expression of $\tilde{X}$ is:

$$\tilde{X} = f_i\frac{\partial}{\partial q_i} - \left(\frac{\partial g}{\partial q_i} + \frac{\partial f_h}{\partial q_i}p_h\right)\frac{\partial}{\partial p_i}.$$

We shall study the invariance of the differential forms under the full Lie algebra of inifinitesimal automorphisms associated to the trivial principal bundle $M \times U(1)$.

A differential form Ω on $T^*(M)$ is said to be invariant under the full Lie algebra of all $U(1)$ invariant vector fields, *i.e.*, $\mathrm{aut}(M \times U(1))$–invariant, if and only if for every vector field X of this Lie algebra, we have: $L_{\tilde{X}}\Omega = 0$.

Theorem 2. *A differential form Ω on $T^*(M)$ is $\mathrm{aut}(M \times U(1))$ -invariant if and only if there exist scalars $\lambda_i \in \mathbf{R}$ such that*

$$\Omega = \sum_{i=0}^{m} \lambda_i (d\omega)^i,$$

where ω stands for Liouville's form.

2. Second Part. In the second part, we generalize the above result to an arbitrary principal bundle, not necessarily trivial, $\pi : P \to M$, of structure group $G = U(1)$ ([5]).

The first problem we have to deal with is how to endow the bundle of connections with a symplectic structure. To do this, we use the affine structure of the bundle of connections of P. In the general case, the bundle of connections of a principal bundle is an affine bundle modelled over $T^*(M)\otimes \operatorname{ad}P$, but in the present case, as the structure group is the abelian group $U(1)$, the adjoint bundle is the trivial rank-1 bundle, *i.e.*, $\operatorname{ad}P \cong M\times \boldsymbol{R}$; so one can identify $T^*(M)\otimes \operatorname{ad}P\cong T^*(M)$ with the cotangent bundle, and hence, the bundle of connections of P is an affine bundle modelled over $T^*(M)$.

Now, the question is how to generalize the symplectic structure of $T^*(M)$ to $K(P)$, in such a way that for the trivial bundle $P=M\times U(1)$, the structure thus obtained coincides with $d\omega$, that is, with the exterior differential of Liouville's form.

Theorem 3. *On $K(P)$, the bundle of connections of a $U(1)$-principal bundle P, there exists a unique symplectic form ω_2, satisfying the following properties:*

1. *The natural projection of the bundle of connections, $p: K(P)\to M$, is a Lagrangian foliation.*
2. *Let $\tau_\omega : K(P)\to K(P)$ be the translation induced by a differential form ω on M (that is, $\forall s\in K(P)_x$, $\tau_\omega(s)=\omega(x)+s$). Then, $\tau_\omega^*(\omega_2)=\omega_2+p^*(d\omega)$.*
3. *For every vector field X, $X\in\operatorname{aut}P$, we have $L_{\tilde X}\omega_2=0$, i.e., all vector fields of the natural representation are infinitesimal symmetries of ω_2.*

Remarks:

1. For $P=M\times U(1)$, $\omega_2=d\omega$ is the exterior differential of Liouville's form.
2. The cohomology class of ω_2 in $H^2(K(P),\boldsymbol{R})\cong H^2(M,\boldsymbol{R})$, coincides with the Chern class of the complex line bundle, $P\times^{U(1)}\boldsymbol{C}$, associated to P by the standard action of $U(1)$ in $\boldsymbol{C}$.

Now, we shall study the invariance of the differential forms under the gauge algebra and under the full Lie algebra associated to a non trivial principal bundle P of structure group $U(1)$.

A differential form Ω on the bundle of connections of a $U(1)$-principal bundle, $K(P)$, is said to be gauge invariant (respectively, invariant under the Lie algebra of all $U(1)$-invariant vector fields) if and only if for every vector field X of the gauge algebra (respectively, for every vector field X of the Lie algebra of all $U(1)$-invariant vector fields), we have

$$L_{\tilde X}\Omega=0.$$

Theorem 4. *Let ω_2 be the symplectic form defined above on the bundle of connections of a $U(1)$-principal bundle P. Then,*

1. *The algebra of gauge invariant differential forms on the cotangent bundle of P is generated over $p^*\Omega^\bullet(M)$ by ω_2:*

$$\{\operatorname{gau}P-\text{invariant forms}\}=p^*\Omega^\bullet(M)[\omega_2]$$

2. *The algebra of invariant differential forms under the full Lie algebra of infinitesimal automorphisms of P is generated over $\boldsymbol{R}$ by ω_2:*

$$\{\operatorname{aut}P-\text{invariant forms}\}=\boldsymbol{R}[\omega_2]$$

3. Third Part. In the third step, we shall try to extend the previous results to the case of a gauge field interacting with a particle field ([6]). We thus consider the fibered product $K\times_M E\to M$ of the bundle of connections of P, K, *i.e.*, the bundle of the gauge field, with the associated bundle to P by a linear representation of $U(1)$ into a vector space V, *i.e.*, the bundle E of the particle field.

Then, we have: $E=P\times^{U(1)}V=(P\times V)/U(1)$ and the linear representation is $\rho: U(1)\to Gl(V)$.

We denote by $[u,v]$ the orbit of $(u,v)\in P\times V$ in E. The gauge group of P, $\operatorname{Gau}P$, acts on E as follows: $\Phi\in\operatorname{Gau}P$, $\Phi_E[u,v]=[\Phi(u),v]$, where we have

$$\begin{array}{ccc} E & \xrightarrow{\Phi_E} & E\\ \downarrow & & \downarrow\\ M & \xrightarrow{\text{id}} & M \end{array}$$

Hence, we obtain a representation of the gauge algebra of P into the vector fields on E:

$$\operatorname{gau}P\to\mathcal{X}(E),\qquad X\to X_E,$$

and finally, we obtain a representation of the gauge algebra of P into the Lie algebra of vector fields on $K\times_M E$, by simply adding both representations:

$$\operatorname{gau}P\to\mathcal{X}(K\times_M E),\qquad X\to\bar X,$$

where $\bar X=\tilde X+X_E$.

A differential form Ω on $K\times_M E$ is said to be gau P-invariant if for every vector field X of the gauge algebra of P, gau P, we have

$$L_{\bar X}\Omega=0.$$

The basic procedure to get new differential forms is the following:

Theorem 5. *Let $\mathcal{A}(V)$ be the algebra of differential forms on V given by:*

$$\mathcal{A}(V) = \{\Omega \in \Omega^{\bullet}(M)\,;\, i_{A^*}\Omega = 0, i_{A^*}d\Omega = 0\},$$

where A^* is the fundamental vector field associated to the representation $\rho : U(1) \to Gl(V)$.

1. *Each differential form $\Omega \in \mathcal{A}(V)$ induces a differential form Ω_E on the vector bundle E, uniquely determined by the following condition: for every $X_1, ..., X_r \in T_{(u,v)}(P \times V)$*

$$\Omega_E(q_* X_1, ..., q_* X_r) = \Omega((\pi_2)_* X_1, ..., (\pi_2)_* X_r),$$

where q is the canonical projection and π_2 is the projection on the second factor:

$$\begin{array}{ccc} P \times V & \xrightarrow{q} & E = (P \times V)/U(1) \\ \pi_2 \downarrow & & \\ V & & \end{array}$$

2. *Ω_E is gauge invariant, that is, for every vertical automorphism $\Phi \in$Gau P, is*

$$\Phi_E^* \Omega_E = \Omega_E.$$

Notation: We call $\mathcal{A}(E)$ the induced invariant differential forms on E by the forms in $\mathcal{A}(V)$: $\mathcal{A}(E) = \{\Omega_E : \Omega \in \mathcal{A}(V)\}$.

Remarks:

1. The invariance under the gauge group implies the invariance under the gauge algebra. Hence, Ω_E is a gauge invariant differential form on E and also on $K \times_M E$.
2. Gauge invariant forms on K and gauge invariant forms on E do not generate all gauge invariant forms on $K \times_M E$, *i.e.*, gau P invariant differential forms on $K \times_M E$ are not generate by ω_2 in $\mathcal{A}(E)$. Although the previous procedure provides a way to obtain all gauge invariant forms on E.

For the sake of simplicity, in what follows we assume that the vector space V is the complex plane, $V = \boldsymbol{C}$, and that ρ is the representation given by the formula:

$$\rho : U(1) \to Gl(\boldsymbol{C}), \qquad \rho(e^{i\theta})(z) = e^{ir\theta} z,$$

where $r \in \boldsymbol{Z}, r > 0$.

This restriction is not essential since every $U(1)$-representation is equivalent to a direct sum of some copies of the above representation plus, eventually, the trivial representation.

Now, the local coordinates in $K \times_M E$ are the following: in K these coordinates are (q_i, p_i) and in E the coordinates are $(q_i, z = x + y\sqrt{-1})$.

Moreover, the local expression of the representation is as follows:

$$X = \tilde{X} + X_E = -\frac{\partial g}{\partial q_i}\frac{\partial}{\partial p_i} - rg\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right).$$

Then we have:

Theorem 6. *With the above assumptions, let $\alpha : E \to \boldsymbol{R}$, be the function given by*

$$\alpha([u, v]) = z\,\bar{z} = x^2 + y^2,$$

and let β be the 1- form on $K \times_M E$ given by

$$\beta = x\,dy - y\,dx + r\left(x^2 + y^2\right) p_i\, dq_i.$$

Then, $\alpha, d\alpha, \beta, d\beta$ and ω_2 span over $C^{\infty}(M)$ the algebra of gauge invariant differential forms on $K \times_M E$.

Remarks:

1. $\alpha, d\alpha \in \mathcal{A}(E)$, but $\beta, d\beta$ do not belong to $\mathcal{A}(E)$;
2. β contains the characteristic exponent r of the representation. So, different representations induce different gauge invariant forms;
3. β has an intrinsic meaning.

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RIEMANNIAN CONVEXITY IN PROGRAMMING (I)

BY

CONSTANTIN UDRIŞTE

Introduction. The techniques and analysis presented in this paper for convex programs on Riemannian manifolds are generalizations of some ideas on Euclidean spaces. These generalizations are important because many optimization problems require naturally the structure of a diferentiable manifold endowed with a Riemannian metric.

§**1** is devoted to the Newton algorithm on Riemannian manifolds for finding zeros of a C^∞ differential 1-form (or of a vector field).

§**2** deals with properties of the Newton method near the central path of a convex program (one unit Newton step stays inside the fesible set, quadratic convergence results, upper bound for the difference of two barrier function values, and upper bound for the difference of two objective function values) obtained using simultaneously the original Riemannian metric and a Hessian Riemannian metric.

§**3** gives upper bounds for the total number of outer and inner iterations for the logarithmic barrier algorithm applied to a convex program on a Riemannian manifold.

The theorems in §§**2 - 3** have their origin in the Euclidean variants exposed in [1], [2] and in the Riemannian point of view about convex programming developed in [6], [7].

1. Newton method on Riemannian manifolds. The generalizations of numerical techniques on Euclidean space to a Riemannian manifold are realized via an intrinsic approach which leads one from the extrinsic idea of vector addition to the exponential map and parallel translation, from the search along straight lines to the search along geodesics, and from partial

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