INVARIANT SUBMANIFOLDS OF CODIMENSION 2 OF FRAMED MANIFOLDS [1])

BY

SAMUEL I. GOLDBERG

Presented at the „A. Myller" Memorial Scientific Session, Iaşi, August 20—25, 1970

1. Introduction. The complex hypersurfaces of the simply connected complex space forms have been classified under the conditions that in the induced metric they are complete Einstein spaces [9]. The submanifold M is then totally geodesic, or else the holomorphic sectional curvature of the ambient space $\widetilde{M}$ is positive and M is a complex quadric. A local version was subsequently given in [1]. Various generalizations then followed both on the global and local levels by assuming that either the Ricci tensor of M is parallel [8], [10] or its contraction is a constant [7].

Recently [2], the author showed that if a hypersurface M of a Kaehler manifold $\widetilde{M}$ is immersed as an invariant hypersurface of an orientable hypersurface P of $\widetilde{M}$, and if the unit normal field of the immersion in P is a Killing vector field, then M is a totally geodesic submanifold of $\widetilde{M}$. No assumption on either the metric structure of $\widetilde{M}$ or M was made. Indeed, it was not assumed that the ambient space is a complex space form or that the Ricci tensor of the submanifold is a parallel field.

An odd dimensional analogue was provided in a more recent paper by assuming that the ambient space is either a cosymplectic or a normal contact manifold [3], the latter corresponding to the odd dimensional analogue of Chern's result [6].

The purpose below is to extend the results in [2] and [3] to framed metric manifolds thereby unifying the treatments of the odd and even dimensional cases which continues a recently inaugurated program.

---

[1]) Research supported by the National Science Foundation.

2. Framed manifolds. An m-dimensional C^∞ manifold M, carrying a tensor field f of type (1,1) and class C^∞ which satisfies the algebraic relation

$$f^3 + f = 0,$$

is called an *f-manifold* provided the *f-structure* f is of constant rank r at each point of M. The existence of these structures is equivalent to a reduction of the structural group of the tangent bundle of M to $U(r/2) \times 0(m-r)$. Examples are afforded by the almost complex structures for $m = 2n$ and the almost contact structures for $m = 2n+1$, the former having maximal rank, whereas the latter has rank $2n$. Indeed, by setting

$$s = -f^2, \quad t = f^2 + I,$$

where I is the identity field,

$$s + t = I,$$

$$s^2 = s, \quad t^2 = t,$$

$$f^2 s = -s, \quad ft = 0.$$

The operators s and t acting in the tangent space at each point of M are therefore complementary projection operators defining distributions S and T in M corresponding to s and t, respectively. The distribution S is r-dimensional and T has dimension $m - r$.

If there are $m - r$ vector fields E_a spanning the distribution T at each point of M, and if there exist $m - r$ linear differential forms η_a satisfying

$$\eta^a(E_b) = \delta^a_b, \tag{2.1}$$

where δ^a_b, $a, b = 1, \dots, m-r$, is the "Kronecker delta", and furthermore, if

$$f^2 = -I + \eta^a \otimes E_a, \tag{2.2}$$

where $\otimes$ denotes the tensor product, the summation convention being employed here and in the sequel, then M is called a *globally framed f-manifold* or, simply, *framed manifold*. As examples, we again have the almost complex manifolds for m even and the almost contact spaces for m odd. (Strictly speaking, because of the first example, the indices a, b should run through $0, 1, \dots, m-r$ with $E_0 = 0$ and $\eta^0 = 0$.) The framed structure on M will be denoted by $M(f, E_a, \eta^a)$. From (2.1) and (2.2), one easily notes that

$$fE_a = 0, \quad \eta^a \circ f = 0, \quad a = 1, \dots, m-r. \tag{2.3}$$

The manifold $M(f, E_a, \eta^a)$ is called a *framed metric manifold* if M carries a Riemannian metric g such that and

(ii) f is skew symmetric with respect to g.

It can be shown that a framed manifold carries a metric with these properties. We put $F(X, Y) = g(fX, Y)$, where $X, Y \in \mathfrak{X}(M)$ — the module of C^∞ vector fields on M, and call it the *fundamental* 2-*form* of the framed metric structure.

If $m = 2n$ an almost complex structure

$$f' = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}, \quad i = 1,\dots, n - r/2$$

is defined on M in terms of which g is hermitian. Setting $F'(X, Y) = g(f'X, Y)$, we obtain

$$F' = F + 2 \sum_i \eta^{2i} \wedge \eta^{2i-1}. \tag{2.4}$$

If F and the η^a are closed forms, the almost hermitian structure on M is almost Kaehlerian. It is Kaehlerian, if and only if, f' has vanishing covariant derivative with respect to the metric g, that is, if the structure tensors f and E_a are covariant constant with respect to g. (In this case, the fields E_a are infinitesimal automorphisms of the Kaehlerian structure.)

If $m = 2n + 1$ the framed structure gives rise to the almost contact metric structure $M(f', \eta^{2n-r+1}, g)$. For, $g(f'X, f'Y) = g(X, Y) - \eta^{2n-r+1}(X)\eta^{2n-r+1}(Y)$. If we set $\Phi(X, Y) = g(f'X, Y)$, then

$$\Phi = F + 2 \sum_i \eta^{2i} \wedge \eta^{2i-1}.$$

If the fundamental 2-form Φ and the 1-form η^{2n-r+1} are closed, the almost contact structure on M is almost cosymplectic [4]. It is cosymplectic if, in addition, $[f', f']$ vanishes, where for any linear transformation field h, $[h, h](X, Y) = [hX, hY] - h[hX, Y] - h[X, hY] + h^2[X, Y]$.

A framed metric manifold is called a *K-manifold* if its structure tensors are parallel fields [5]. As examples we therefore have the Kaehlerian and cosymplectic spaces.

3. Hypersurfaces of framed manifolds. Let $\widetilde{M}(\tilde{f}, \widetilde{E}_a, \tilde{\eta}^a)$ be an m-dimensional framed metric manifold, $(m \geqq 4)$, with Riemannian metric $\tilde{g}$. Then,

$$\tilde{\eta}^a = \tilde{g}(\widetilde{E}_a, \cdot), \; a = 1,\dots, m - r,$$

where rank $\tilde{f} = r$. Consider an $(m-1)$-dimensional manifold P immersed in $\widetilde{M}$ with immersion $i: P \to M$ having the property

(T): *For each $p \in P$, the vectors $\widetilde{E}_a$ at $i(p)$ belong to the tangent hyperplane $i(P)_{i(p)}$.*

Then, if $\tilde{N}$ is the field of unit normals to $i(P)$ with respect to $\tilde{g}$,

$$\tilde{f} i_* X = i_* \bar{f} X + \alpha(X)\tilde{N}, \tag{3.1}$$

where

$$\alpha(X) = \tilde{g}(\tilde{f} i_* X, \tilde{N})$$

and

$$\tilde{f}\tilde{E}_a = 0, \tag{3.2}$$

$$\tilde{\eta}^a(\tilde{N}) = 0, \tag{3.3}$$

$\bar{f}$ and α being tensor fields on P of types $(1,1)$ and $(0,1)$, respectively, i_* is the induced tangent map and $X \in \mathfrak{X}(P)$. Since i is a regular map, there are vector fields $\bar{E}_a$, $a = 1,\dots,m-r$ on P such that

$$\tilde{E}_a = i_* \bar{E}_a. \tag{3.4}$$

Hence, by (3.1) and (3.2), $\bar{f}\bar{E}_a = 0$ and $\alpha(\bar{E}_a) = 0$, $a = 1,\dots,m-r$. Putting $\bar{\eta}^a = i^* \tilde{\eta}^a$, we have

$$\bar{\eta}^a(\bar{E}_b) = \delta^a_b. \tag{3.5}$$

Since $\tilde{f}\tilde{N}$ is orthogonal to $\tilde{N}$ with respect to $\tilde{g}$, it is tangent to the hypersurface, so there is a vector field A on P, of unit length by (3.3), such that

$$\tilde{f}\tilde{N} = -i_* A. \tag{3.6}$$

Applying $\tilde{f}$ to both sides of (3.1) and (3.6) yields

$$\bar{f}^2 X = -X + \bar{\eta}^a(X)\bar{E}_a + \alpha(X)A, \quad \alpha(\bar{f}X) = 0 \tag{3.7}$$

and

$$\alpha(A) = 1, \quad \bar{f}A = 0. \tag{3.8}$$

Theorem 1. *Let P be an $(m-1)$-dimensionul hypersurface immersed in the framed metric manifold $\tilde{M}$ of rank r with immersion i. Then, P is endowed with a framed metric structure of rank $r-2$.*

Proof. From (3.1) and (3.2), we obtain

$$\bar{f}\bar{E}_a = 0, \quad \alpha(\bar{E}_a) = 0, \tag{3.9}$$

and from (3.7),

$$\bar{\eta}^a(A) = 0, \quad \alpha(\bar{E}_a) = 0, \tag{3.10}$$

$a=1,\ldots,m-r$. If $\bar{f}X=0$, then $X=\bar{\eta}^a(X)\bar{E}_a+\alpha(X)A$, so rank $\bar{f}=(m-1)-(m-r+1)=r-2$.

Let $\widetilde{\nabla}$ be the Riemannian connection of $(\widetilde{M},\widetilde{g})$ and let D be the induced connection on P, that is, the Riemannian connection of $\bar{g}=i^*\widetilde{g}$. Then, the equations of Gauss and Weingarten are

$$\widetilde{\nabla}_{i_*X}i_*Y=i_*D_XY+h(X,Y)\widetilde{N} \tag{3.11}$$

and

$$\widetilde{\nabla}_{i_*X}\widetilde{N}=-i_*HX, \tag{3.12}$$

respectively, h and H being the second fundamental tensors of the immersion of types (0,2) and (1,1), respectively, where $h(X,Y)=g(HX,Y)$.

4. **Invariant immersions.** A submanifold (M,ι) of codimension 2 of a framed manifold $\widetilde{M}(\widetilde{f},\widetilde{E}_a,\widetilde{\eta}^a)$ is said to be *invariantly immersed* in $\widetilde{M}$ and M is then called an *invariant submanifold* if there exists a linear transformation field f on M such that

(i) $\widetilde{f}\iota_*x=\iota_*fx, \quad x\in\mathfrak{X}(M)$

and

(ii) For each $m\in M$, the vectors $\widetilde{E}_a$ at $\iota(m)$ belong to $\iota(M)_{\iota(m)}$.

By (ii), since ι is a regular map, there exist vector fields E_a on M $a=1,\ldots,m-r$, such that

$$\widetilde{E}_a=\iota_*E_a. \tag{4.1}$$

Hence, by (i) and (3.2),

$$fE_a=0. \tag{4.2}$$

Putting $\eta^a=\iota^*\widetilde{\eta}^a$, we have

$$\eta^a(E_b)=\delta^a_b. \tag{4.3}$$

Applying $\widetilde{f}$ to both sides of (i) yields

$$f^2x=-x+\eta^a(x)E_a \tag{4.4}$$

by virtue of (4.1) and the regularity of ι. Thus.

Theorem 2. *An invariant submanifold of a framed manifold of rank r is a framed manifold of rank $r-2$.*

Corollary. *An invariant submanifold of an almost complex (almost contact) manifold is an almost complex (almost contact) manifold.*

Assume now that M is an invariant submanifold of the framed metric manifold $\widetilde{M}$ immersed as an orientable hypersurface (M,j) of a hypersurface (P, i) satisfying the property (T). Then, by (3.4) and (4.1), the vector fields $\bar{E}_a$ are tangent to $i(P)$, that is

(4.5) $$\bar{E}_a = j_* E_a.$$

Denoting by N the unit normal field to $j(M)$ with respect to the metric $\bar{g}$ (with orientation determined by P),

$$\tilde{f} j_* x = j_* f_1 x + \beta(x) N,$$

where

$$\beta(x) = \bar{g}(\bar{f} j_* x, N).$$

Moreover,

$$\bar{f}\, \bar{E}_a = 0$$

and

(4.6) $$\bar{\eta}^a(N) = 0,$$

the latter being a consequence of the fact that

$$\bar{\eta}^a = \bar{g}(\bar{E}_a, \cdot\,), \quad a = 1, \dots, m-r\,.$$

Since

$$\tilde{f} \iota_* x = \tilde{f} i_*(j_* x) = i_* \bar{f} j_* x + \alpha(j_* x) \tilde{N} = i_*\{j_* f_1 x + \beta(x) N\} + (j^* \alpha)(x) \tilde{N} = \iota_* f_1 x + \beta(x) i_* N + (j^* \alpha)(x) \tilde{N},$$

we conclude that

$$f_1 = f, \quad \beta = 0, \quad j^* \alpha = 0.$$

Lemma 1. *Let M be an invariant submanifold of the framed metric manifold $\tilde{M}$ immersed as an orientable hypersurface (M, j) of a hypersurface (P, i) satisfying the property* (T). *Then, the vector field A on P coincides with the unit normal field N.*

Proof. Since $\bar{f}$ is skew symmetric with respect to $\bar{g}$, $\bar{f}N$ is orthogonal to N with respect to $\bar{g}$. Hence, $\bar{f}N = j_* u$ for some vector field u on M, from which $0 = \bar{g}(\bar{f} j_* x, N) = -\bar{g}(j_* x, \bar{f}N) = -\bar{g}(j_* x, j_* u) = -g(x, u)$, where $g = j^* \bar{g}$. Since g is definite, u must vanish, so $\bar{f}N = 0$. Thus, $N = \bar{\eta}^a(N) \bar{E}_a + \alpha(N) A = \alpha(N) A$ by (4.6). The vector fields A and N each being of unit length, $A = N$ by choosing $\alpha(N) = 1$.

Let ∇ be the Riemannian connection of (M, g). Then,

(4.7) $$D_{j_* x} j_* y = j_* \nabla_x y + k(x, y) N$$

and

$$D_{j_*x}N = -j_*Kx,$$

where k and K are the second fundamental tensors of the immersion j. From (3.6), (3.11) and (4.7), we derive

$$\tilde{\nabla}_{\iota_* x}\iota_* y = \iota_* \nabla_x y - k(x,y)\tilde{f}\tilde{N} + h'(x,y)\tilde{N},$$

where $h'(x,y) = h(j_*x, j_*y)$. Putting

$$Hj_*x = j_*H'x + \omega(x)N,$$

where ω is a 1-form on M, we see that

$$h'(x,y) = g(H'x,y).$$

We require the following lemma.

Lemma 2. *If the ambient space is a K-manifold,*

$$(D_X\bar{f})Y = \alpha(Y)HX - h(X,Y)A,$$

$$D_X\bar{E}_a = 0, \quad (D_X\alpha)(Y) = -h(X,\bar{f}Y),$$

$$h(X,\bar{E}_a) = 0 \;\; or \;\; \eta^a(HX) = 0.$$

Proof. Differentiate both sides of the relation (3.1) and take note of (3.11) and (3.12).

Theorem 3. *Let $\tilde{M}(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ be a K-manifold. If M is an invariant submanifold immersed as an orientable hypersurface (M,j) of a hypersurface (P,i) satisfying the property* (T), *then*

$$K = -fH' = H'f.$$

Proof. By differentiating the function $\alpha(j_*y)$ in the x-direction applying Lemma 2 and observing that $j^*\alpha$ vanishes, we get

$$\begin{aligned} x(\alpha(j_*y)) &= (D_{j_*x}\alpha)(j_*y) + \alpha(i_*\nabla_x y + k(x,y)N) \\ &= -h(j_*x, \bar{f}j_*y) + k(x,y) = -h(j_*x, j_*fy) + k(x,y) \\ &= -h'(x,fy) + k(x,y) = -g(H'x,fy) + g(Kx,y) \\ &= g(fH'x,y) + g(Kx,y), \end{aligned}$$

so that

$$g(Kx,y) = -g(fH'x,y)$$

and

$$g(x,Ky) = g(x,H'fy).$$

Corollary. *Under the conditions of the theorem*

$$K^2 = H'^2$$

and

$$\text{trace } H' = \text{trace } K = 0,$$

so M is a minimal submanifold of $\tilde{M}$.

Proof. By the theorem, $K^2x = -H'f^2H'x = -H'\{-H'x + \eta^a(H'x)E_a\} = H'^2x - \eta^a(H'x)H'E_a$. But, by Lemma 2, $0 = h(X, \bar{E}_a) = \bar{g}(HX, \bar{E}_a) = \bar{g}(X, H\bar{E}_a) = \bar{g}(X, Hj_*\bar{E}_a)$, so $\bar{g}(j_*x, Hj_*\bar{E}_a) = \bar{g}(j_*x, j_*H'E_a) = g(x, H'E_a) = 0$. That M is a minimal submanifold is a consequence of the fact that the second fundamental tensors are symmetric and $\bar{f}$ is skew symmetric with respect to $\bar{g}$.

Theorem 4. *Let* (M, ι) *be an invariant submanifold of a K-manifold* $\tilde{M}$ *immersed as an orientable hypersurface* (M, j) *of a hypersurface* (P, i) *with the property* (T), $\iota = i \circ j$. *If the field of unit normals to* $j(M)$ *with respect to the induced metric is a Killing vector field, then M is a framed totally geodesic submanifold of* $\tilde{M}$.

Proof. By Lemma 2, $h(X, \bar{f}Y) + h(Y, \bar{f}X) = 0$ which is equivalent to the statement that H' and $\bar{f}$ commute. This implies that H' and f also commute. For,

$$H\bar{f}j_*x = Hj_*fx = j_*H'fx + \omega(fx)N$$

and

$$\bar{f}Hj_*x = \bar{f}\{j_*H'x + \omega(x)N\} = j_*fH'x$$

since $\bar{f}N = fA = 0$. Thus,

$$H'f = fH', \quad \omega \circ f = 0.$$

Applying Theorem 3, we see that $K = 0$ and $H' = 0$.

Remark. Under the conditions in the theorem, P is also a K-manifold. We omit the proof of this fact.

Corollary 1. *Let M be a complex hypersurface of a Kaehler manifold. If M is immersed in* $\tilde{M}$ *as a hypersurface* (M, j) *of an orientable hypersurface of* $\tilde{M}$ *and if the unit normal field to* $j(M)$ *is a Killing vector field, then M is a totally geodesic submanifold of* $\tilde{M}$.

Corollary 2. *Let M be an invariant submanifold of a cosymplectic manifold* $\tilde{M}$. *If M is immersed in* $\tilde{M}$ *as an orientable hypersurface* (M, j) *of a hypersurface with the property* (T), *and if the field of unit normals to* $j(M)$ *is a Killing vector field, then M is a totally geodesic submanifold of* $\tilde{M}$.

Corollaries 1 and 2 could have been obtained by utilizing the underlying Kaehlerian (cosymplectic) structure in the even (odd) dimensional case and then applying [2], Proposition 3 and [3], Theorem 2 (see § 2).

By an appropriate modification of Theorem 4 we can obtain [3], Theorem 2, namely, Corollary 2 above with "cosymplectic" replaced by "normal contact".

University of Illinois
Urbana, Illinois

BIBLIOGRAPHY

1. Chern S. S. — *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature.* J. of Diff. Geom., 1(1967), pp. 21—31.
2. Goldberg S. I. — *On immersions of complex hypersurfaces of Kaehler manifolds* To appear in Revue Roumaine de Math. Pures et Appl. (Dedication volume honouring Professor G. Vrănceanu).
3. — — *Invariant submanifolds of codimension 2 of almost contact manifolds.* To appear in Annali della Scuola Normale Superiore.
4. Goldberg S.I. and Yano K. — *Noninvariant hypersurfaces of almost contact manifolds.* J. Math. Soc. of Japan, 22 (1970), pp. 25—34.
5. — *Globally framed f-manifolds.* To appear in Illinois J. of Math.
6. Kenmotsu K. — *Invariant submanifolds in a Sasakian manifold.* Tôhoku Math. J., 21 (1969), pp. 495—500.
7. Kobayashi S. — *Hypersurfaces of complex projective space with constant scalar curvature.* J. of Diff. Geom., 1 (1967). pp. 369—370.
8. Nomizu K. and Smyth B. — *Differential geometry of complex hypersurfaces. II.* J. Math. Soc. of Japan, 20 (1968), pp. 498—521.
9. Smyth B. — *Differential geometry of complex hypersurfaces.* Ann. of Math. 85 (1967) pp. 246—266.
10. Takahashi T. — *Hypersurfaces with parallel Ricci tensor in a space of constant holomorphic sectional curvature.* J. Math. Soc. of Japan, 19 (1967), 199—204.

SUBVARIETĂŢI INVARIANTE DE CODIMENSIUNE 2 ALE VARIETĂŢILOR REPERATE

Rezumat

Recent [2], autorul a arătat că dacă o hipersuprafaţă M a unei varietăţi kähleriene $\tilde{M}$ este imersă ca hipersuprafaţă a unei hipersuprafeţe orientabile P a lui $\tilde{M}$ şi dacă cîmpul unitar normal al imersiunii în P este un cîmp de vectori Killing, atunci M este o subvarietate total geodezică a lui $\tilde{M}$.

Într-o lucrare mai recentă [3] s-a obţinut un rezultat analog pentru dimensiunile impare, presupunîndu-se că spaţiul ambiant este o varietate cu structură cosimplectică sau cu structură de contact normală.

În lucrare se extind rezultatele din [2] şi [3] la o clasă nouă de varietăţi, pe care le numim varietăţi metrice reperate.