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TANGENCY AND OPENNESS OF MULTIFUNCTIONS IN BANACH SPACES

BY

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Let X and Y be Banach spaces and let F be a multifunction from X into Y. For every point $(x, y) \in X \times Y$, we consider the multifunction $K_F(x, y)$ from X into Y which assigns to every point $u \in X$ the set of all points $v \in Y$ such that

$$\varnothing \neq ((x, y) + (0, r)((u, v) + W)) \cap \text{graph } (F),$$

for every neighborhood W of the origin in $X \times Y$ and for every $r > 0$. In other words graph $(K_F(x, y))$ equals $K_{\text{graph}(F)}(x, y)$, the tangent set in the sense of Bouligand [1, p. 32] and Severi [8, p. 99] to the set graph(F) at the point (x, y). According to the elementary properties of these tangent sets, graph$(K_F(x, y))$ is a closed cone which is not empty if and only if $(x, y) \in$ closure(graph(F)).

Our main results relate openness of the multifunction F at some points $(x, y) \in X \times Y$ to openness of the corresponding multifunctions $K_F(x, y)$ at the origin in $X \times Y$. Recall, on this occasion, some definitions concerning openness of a multifunction F from a topological space X into a topological space Y at a point $(x, y) \in X \times Y$. The multifunction F is said to be open at the point (x, y) if, for every neighborhood U of x, there exists a neighborhood V of y such that $V \subseteq F(U)$. Other variants of this definition make use of the inclusions $V \subseteq$ closure$(F(U))$, $V \cap F(X) \subseteq F(U)$, or $V \cap F(X) \subseteq$ closure $(F(U))$.

Now we return to our initial setting. In the statement below, for every point c of a Banach space and for every $r > 0$, $B(c, r)$ stands for the open ball with centre c and radius r. And, for every subset W of $X \times Y$, we denote

$$Z(W) = \bigcap_{(x, y) \in \text{graph}(F) \cap W} (y + \text{closure } (K_F(x, y)\,(X))).$$

Theorem 1. *Let* W *be an open subset of* $X \times Y$ *such that the set* graph$(F) \cap$ closure(W) *is closed, and let* $\lambda > 0$ *such that*

(1) $$B(0, \lambda) \cap K_F(x, y)(X) \subseteq \text{closure}(K_F(x, y)\,(B(0, 1)))$$

for every $(x, y) \in$ graph$(F) \cap W$. *Then*

(2) $$B(y, \lambda\varepsilon) \cap Z(W) \subseteq F(B(x, \varepsilon))$$

for every $(x, y) \in$ graph$(F) \cap W$ *and for every* $\varepsilon > 0$ *satisfying*

(3) $$B(x, \varepsilon) \times B(y, \lambda\varepsilon) \subseteq W.$$

Proof. Let $(x, y)\in \text{graph}(F)\cap W$ and $\varepsilon>0$ satisfying (3), and let $z\in B(y, \lambda\varepsilon)\cap Z(W)$. Consider $\mu>1$ such that $\mu||z-y||<\lambda\varepsilon$. Following an idea of Penot [7, p. 30], we endow the linear space $X\times Y$ with the norm

$$(p, q)\in X\times Y\rightarrow \max\{\lambda||p||,\ ||q||\},$$

and apply the variational principle of Ekeland [5, p. 324] to the function

$$(p, q)\in \text{graph}(F)\cap \text{closure}(W)\rightarrow \mu||z-q||\in R,$$

in order to obtain a point $(a, b)\in \text{graph}(F)\cap \text{closure}(W)$ such that

$$\mu||z-b||+||(a, b)-(x, y)||\leqslant \mu||z-y||,$$

and such that

$$\mu||z-b||\leqslant \mu||z-q||+||(p, q)-(a, b)||,$$

for every $(p, q)\in \text{graph}(F)\cap \text{closure}(W)$. Note that

$$||(a, b)-(x, y)||<\lambda\varepsilon,$$

hence $a\in B(x, \varepsilon)$, $b\in B(y, \lambda\varepsilon)$, $(a, b)\in W$,

$$z\in b+\text{closure}(K_F(a, b)(X)),$$

and, according to (1),

$$B(0, \lambda)\cap K_F(a, b)(X)\subseteq \text{closure}(K_F(a, b)(B(0, 1))).$$

Now we shall show that $z=b$. Denote $v=z-b$. Consider a sequence $(r_n)_{n\in N}$ in $(0, +\infty)$ such that it converges to 0. Then, for every $n\in N$,

$$v\in B(0, ||v||+r_n)\cap \text{closure}(K_F(a, b)(X))\subseteq \text{closure}(B(0, ||v||+r_n)\cap$$
$$\cap K_F(a, b)(X))\subseteq \text{closure}(K_F(a, b)(B(0, (||v||+r_n)/\lambda))).$$

Consider a sequence $(u_n, v_n)_{n\in N}$ in $X\times Y$ such that $(v_n)_{n\in N}$ converges to v and such that, for every $n\in N$,

$$v_n\in K_F(a, b)(u_n), \text{ and } u_n\in B(0, (||v||+r_n)/\lambda).$$

Consider, further, a sequence $(s_n, p_n, q_n)_{n\in N}$ in $(0, +\infty)\times X\times Y$ such that it converges to the origin in $R\times X\times Y$ and such that, for every $n\in N$,

$$(a, b)+s_n((u_n, v_n)+(p_n, q_n))\in \text{graph}(F).$$

We may suppose, by passing to subsequences, that, for every $n\in N$, $s_n\in(0, 1)$ and

$$(a, b)+s_n((u_n, v_n)+(p_n, q_n))\in W.$$

Since, for every $n\in N$,

$$\mu||v||\leqslant \mu||s_n(v_n+q_n)-v||+s_n||(u_n, v_n)+(p_n, q_n)||\leqslant \mu((1-s_n)||v||+s_n||v_n+$$
$$+q_n-v||)+s_n||(u_n, v_n)+(p_n, q_n)||$$

and

$$\mu||v||\leqslant\mu||v_n+q_n-v||+\max\{||v||+r_n,\ ||v_n||\}+||(p_n,q_n)||,$$

we conclude $\mu||v||\leqslant||v||$, $v=0$, $z=b\in F(a)\subseteq F(B(x,\varepsilon))$, (2) holds, and the theorem is proved.

Next we discuss two particular cases, $Z(W)=Y$ and $F(X)\subseteq Z(W)$. The first case, $Z(W)=Y$, holds if and only if

$$\text{closure}\,(K_F(x,y)(X))=Y \tag{4}$$

for every $(x, y)\in\text{graph}(F)\cap W$. Note that (1) and (4) is equivalent to

$$B(0,\lambda)\subseteq\text{closure}\,(K_F(x,y)(B(0,1))). \tag{5}$$

In that case, $Z(W)=Y$, (2) is equivalent to

$$B(y,\lambda\varepsilon)\subseteq F(B(x,\varepsilon)), \tag{6}$$

and we obtain a first corollary to Theorem 1.

Corollary 1. *Let W be an open subset of $X\times Y$ such that the set* graph$(F)\cap$ $\cap$ closure (W) *is closed, and let $\lambda>0$ such that* (5) *holds for every $(x, y)\in$* graph$F)\cap$ $\cap W$. *Then* (6) *holds for every $(x, y)\in$* graph $(F)\cap W$ *and for every $\varepsilon>0$ satisfying* (3).

Corollary 1 improves a result of Penot [7, p. 30] Where $W=U\times Y$ for some open subset U of X. The second case, $F(X)\subseteq Z(W)$, holds if and only if

$$F(X)\subseteq\text{closure}(K_F(x,y)(X)) \tag{7}$$

for every $(x, y)\in\text{graph}(F)\cap W$. Note that (7) is equivalent to

$$\text{closure}\ (K_F(x,y)(X))=K_{F(x)}(y), \tag{8}$$

and

$$F(X)\subseteq y+K_{F(x)}(y). \tag{9}$$

In that case, $F(X)\subseteq Z(W)$, (2) implies

$$B(y,\lambda\varepsilon)\cap F(X)\subseteq F(B(x,\varepsilon)), \tag{10}$$

and we obtain a second corollary to Theorem 1.

Corollary 2. *Let W be an open subset of $X\times Y$ such that the set* graph$(F)\cap$ $\cap$ closure(W) *is closed and such that* (8) *and* (9) *hold for every $(x, y)\in$* graph$(F)\cap$ $\cap W$, *and let $\lambda>0$ such that* (1) *holds for every $(x, y)\in$* graph$(F)\cap W$. *Then* (10) *holds for every $(x, y)\in$* graph$(F)\cap W$ *and for every $\varepsilon>0$ satisfying* (3).

Before, we continue, we observe that (7), hence (8) and (9), holds for every $(x, y)\in\text{graph}(F)$ if the set graph(F) is convex since, according to the properties of tangent sets to convex sets,

$$\text{graph}(F)\subseteq(x,y)+K_{\text{graph}\,(F)}(x,y),$$

for every $(x, y)\in$ closure (graph(F)). Next we prove a certain converse of Theorem 1 in case the space X is finite dimensional.

Theorem 2. *Let the space X be finite dimensional, let $(x, y)\in X\times Y$, and let $Z\subseteq Y$ and $\lambda>0$ such that there exists $\delta>0$ such that*

(11) $$B(y, \lambda\varepsilon) \cap Z \subseteq \text{closure } (F(B(x, \varepsilon))),$$

for every $\varepsilon \in (0, \delta)$. *Then*

(12) $$B(0, \lambda) \cap K_Z(y) \subseteq K_F(x, y)(B(0, 1)).$$

Proof. Let $v \in B(0, \lambda) \cap K_Z(y)$, and consider a sequence $(s_n, q_n)_{n \in N}$ in $(0, +\infty) \times Y$ such that it converges to the origin in $R \times Y$ and such that $y + s_n(v + q_n) \in Z$ for every $n \in N$. We may suppose, by passing to subsequences, that $v + q_n \in B(0, \lambda)$ and

$$B(y, \lambda s_n) \cap Z \subseteq \text{closure}(F(B(x, s_n))),$$

for every $n \in N$. Then

$$y + s_n(v + q_n) \in \text{closure}(F(B(x, s_n))),$$

for every $n \in N$, and we get a sequence $(u_n, \hat{q}_n)_{n \in N}$ in $X \times Y$ such that $(\hat{q}_n)_{n \in N}$ converges to the origin in Y and such that

$$y + s_n(v + q_n + \hat{q}_n) \in F(x + s_n u_n)$$

and $u_n \in B(0, 1)$ for every $n \in N$. We may suppose, by passing to subsequences again, that $(u_n)_{n \in N}$ converges. Denote by u its limit. Then $v \in K_F(x, y)(u)$,

$$B(0, \lambda) \cap K_Z(y) \subseteq K_F(x, y)(\text{closure}(B(0, 1))),$$

and the conclusion follows since, for every $\mu \in (0, 1)$,

$$B(0, \lambda\mu) \cap K_Z(y) \subseteq K_F(x, y)(\text{closure}(B(0, \mu))) \subseteq K_F(x, y)(B(0, 1)).$$

Again we discuss two particular cases, $Z = Y$ and $Z = F(X)$.

In the first case, $Z = Y$, (11) becomes

(13) $$B(y, \lambda\varepsilon) \subseteq \text{closure}(F(B(x, \varepsilon))),$$

and (12) becomes

(14) $$B(0, \lambda) \subseteq K_F(x, y)(B(0, 1)),$$

so we obtain a first corollary to Theorem 2.

Corollary 3. *Let the space* X *be finite dimensional, and let* $\lambda > 0$ *such that there exists* $\delta > 0$ *such that* (13) *holds for every* $\varepsilon \in (0, \delta)$. *Then* (14) *holds.*

In the second case, $Z = F(X)$, (11) becomes

(15) $$B(y, \lambda\varepsilon) \cap F(X) \subseteq \text{closure}(F(B(x, \varepsilon))),$$

and (12) implies

(16) $$B(0, \lambda) \cap K_F(x, y)(X) \subseteq K_F(x, y)(B(0, 1)),$$

since $K_F(x, y)(X) \subseteq K_{F(X)}(y)$, so we obtain a second corollary to Theorem 2.

Corollary 4. *Let the space* X *be finite dimensional, and let* $\lambda > 0$ *such that there exists* $\delta > 0$ *such that* (15) *holds for every* $\varepsilon \in (0, \delta)$. *Then* (16) *holds.*

Summing up, if the space X is finite dimensional, W is an open subset of $X \times Y$ such that the set graph$(F) \cap$ closure(W) is closed, and $\lambda > 0$, then the following four conditions are equivalent:

(5) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$;
(6) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$ and for every $\varepsilon > 0$ satisfying (3);
(13) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$ and for every $\varepsilon > 0$ satisfying (3);
(14) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$.

Moreover, if (8) and (9) hold for every $(x, y) \in \mathrm{graph}(F) \cap W$, then the following four conditions are equivalent too:

(1) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$;
(10) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$ and for every $\varepsilon > 0$ satisfying (3);
(15) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$ and for every $\varepsilon > 0$ satisfying (3);
(16) holds for every $(x, y) \in \mathrm{graph}(F) \cap W$.

To close this paper we transpose Theorems 1 and 2 in the setting of functions and of sets.

Let, first, f be a function from a subset of X into Y. For every point $x \in \mathrm{domain}(f)$, we consider the multifunction $K_f(x)$ from X into Y which assigns to every point $u \in X$ the set of all points $v \in Y$ such that

$$\emptyset \neq ((x, f(x)) + (0, r)((u, v) + W)) \cap \mathrm{graph}(f)$$

for every neighborhood W of the origin in $X \times Y$ and for every $r > 0$. In other words $\mathrm{graph}(K_f(x))$ equals $K_{\mathrm{graph}(f)}(x, f(x))$, the tangent set in the sense of Bouligand and Severi to the set $\mathrm{graph}(f)$ at the point $(x, f(x))$. The multifunction $K_f(x)$ may be easily calculated in case the function f is Severi differentiable at the point x (see Severi [9, p. 10], and [10, p. 200]) since the graph of the Severi differential of f at x equals $K_{\mathrm{graph}(f)}(x, f(x))$ too (see [10, p. 201]). A similar result has been proved earlier by Flett [6, p. 525] in the particular case when x is an interior point of the set $\mathrm{domain}(f)$ and f is a Fréchet differentiable function at the point x.

In the following, for every subset U of X, we denote

$$Z(U) = \bigcap_{x \in \mathrm{domain}(f) \cap U} (f(x) + \mathrm{closure}(K_f(x)(X))).$$

Theorem 3. *Let U be an open subset of X such that the restriction of the function f to the set* domain $(f) \cap$ closure(U) *has a closed graph, and let $\lambda > 0$ such that*

$$B(0, \lambda) \cap K_f(x)(X) \subseteq \mathrm{closure}(K_f(x)(B(0, 1)))$$

for every $x \in \mathrm{domain}(f) \cap U$. Then

$$B(f(x), \lambda\varepsilon) \cap Z(U) \subseteq f(\mathrm{domain}(f) \cap B(x, \varepsilon))$$

for every $x \in \mathrm{domain}(f) \cap U$ and for every $\varepsilon > 0$ satisfying

$$B(x, \varepsilon) \subseteq U.$$

Theorem 4. *Let the space X be finite dimensional, let $x \in \mathrm{domain}(f)$, and let $Z \subseteq Y$ and $\lambda > 0$ such that there exists $\delta > 0$ such that*

$$B(f(x), \lambda\varepsilon) \cap Z \subseteq \mathrm{closure}(f(\mathrm{domain}(f) \cap B(x, \varepsilon)))$$

for every $\varepsilon \in (0, \delta)$. Then

$$B(0, \lambda) \cap K_Z(y) \subseteq K_f(x)(B(0, 1)).$$

Proof of Theorems 3 and 4. Take $F(x)=\{f(x)\}$ for every $x\in$ domain (f), $F(x)=\emptyset$ for every $x\notin$ domain(f), $W=U\times Y$, and apply Theorems 1 and 2.

Let, finally, S be a subset of Y. For every subset V of Y, we denote

$$Z(V)=\bigcap_{y\in S\cap V}(y+K_S(y)).$$

Theorem 5. *Let V be an open subset of Y such that the set $S\cap$ closure(V) is closed. Then*

$$B(y,\varepsilon)\cap Z(V)\subseteq S,$$

for every $y\in S\cap V$ and for every $\varepsilon>0$ satisfying

$$B(y,\varepsilon)\subseteq V.$$

Theorem 6. *Let $y\in Y$, and let $Z\subseteq Y$ and $\lambda>0$ such that*

$$B(y,\lambda)\cap Z\subseteq \text{closure}(S).$$

Then

$$K_Z(y)\subseteq K_S(y).$$

Proof of Theorems 5 *and* 6. Take $X=R$, $F(x)=S$ for every $x\in X$, $W=X\times V$, and apply Theorems 1 and 2.

Theorem 3 improves a result of Cramer and Ray [2, p. 44] where f is defined and Gâteaux differentiable on X, since the graph of the Gâteaux differential of f at a point x is a subset of $K_{\text{graph}(f)}(x, f(x))$. Theorem 5 contains a result in [11, p. 183] which deals with locally closed sets and which we have proved by using the drop theorem of Daneš [3, p. 371]. The fact is that the drop theorem of Daneš and the variational principle of Ekeland are equivalent (see Daneš [4, p. 453]). Theorem 6 is elementary in the theory of tangent sets.

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