ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I.CUZA" IAŞI
Tomul XLVIII, s.I a, Matematică, 2002, f.2.

BIHARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

BY

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Abstract. We characterise the biharmonic submanifolds of the unitary Euclidean sphere $\mathbb{S}^n$ and the biharmonic Riemannian submersions. Then, we study the biharmonicity of the canonical projection $\pi : TM \to M$.

1991 Mathematics Subject Classification: 58E20.

Key words and phrases: Harmonic maps, biharmonic maps, Jacobi operator, Riemannian submanifolds, Riemannian submersions.

1. Introduction. A harmonic map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is a critical point of the energy $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \, v_g$. The corresponding Euler-Lagrange equation for the energy is given by the vanishing of the *tension field* $\tau(\phi) = \operatorname{trace} \nabla d\phi$. The *bienergy* of a map ϕ is defined by $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g$, and ϕ is said to be *biharmonic* if it is a critical point of the bienergy.

In [4, 5] G.Y. Jiang has obtained the Euler-Lagrange equation for E_2; that is

$$\tau_2(\phi) = J(\tau(\phi)) = 0, \tag{1.1}$$

where J is the Jacobi operator of ϕ. The equation $\tau_2(\phi) = 0$ will be called the *biharmonic equation.* Also, G.Y. Jiang has observed that the generalized Clifford torus

$$\mathbb{S}^p(\frac{1}{\sqrt{2}}) \times \mathbb{S}^q(\frac{1}{\sqrt{2}}) \hookrightarrow \mathbb{S}^{m+1}$$

with $p+q = m$ and $p \neq q$ is a nonminimal biharmonic submanifold of $\mathbb{S}^{m+1}$.

In the first part of this paper we give some results about the nonexistence of nonharmonic biharmonic maps into a manifold with nonpositive sectional curvature, or with nonpositive Ricci curvature.

In the second part we characterize the biharmonic submanifolds of $\mathbb{S}^n$.

The last part is devoted to the study of the biharmonicity of Riemannian submersions. First we give some results about the nonexistence of nonharmonic biharmonic Riemannian submersions onto a manifold with nonpositive Ricci curvature. Then we prove a theorem which gives a class of nonharmonic biharmonic Riemannian submersions. We finish with the study of the biharmonicity of the canonical projection $\pi : (TM, S) \to (M, g)$, where S is a Riemannian metric of Sasaki type, and $\pi : (TM, S) \to (M, g)$ is a Riemannian submersion.

The manifolds, maps, vector fields etc. considered in this work are assumed to be smooth, i.e. differentiable of class C^∞. By (M, g) and (N, h) we shall mean connected Riemannian manifolds, of dimensions m and n, respectively, without boundary. We shall denote by ∇ the Levi-Civita connection on (M, g). For vector fields X, Y, Z on M we define the Riemann curvature operator by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$. We shall say that M has nonpositive sectional curvature, and write $\text{Riem} \leq 0$, if for every $p \in M$ and 2-plane P in T_pM, $\text{Riem}_p(P) =< R(X, Y)Y, X >\leq 0$, where $\{X, Y\}$ is any orthonormal basis of P. The indices i, j, k, l take the values $1, 2, \ldots, m$, and the indices α, β, γ take the values $1, 2, \ldots, n$.

2. Biharmonic maps. Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. Assume that M is compact and orientable. The tension field of ϕ is given by $\tau(\phi) = \text{trace}\,\nabla d\phi$, and the *bienergy* is defined by

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, v_g.$$

Then we call *biharmonic* a smooth map ϕ which is a critical point of the bienergy functional. As we have said in the introduction, we have for the bienergy the following first variation formula:

$$\left.\frac{dE_2(\phi_t)}{dt}\right|_{t=0} = \int_M < \tau_2(\phi), v > v_g,$$

where v_g is the volume element, v is the variational vector field along ϕ, and

$$\tau_2(\phi) = -\Delta\tau(\phi) - \text{trace } R^N(d\phi -, \tau(\phi))d\phi - . \tag{2.1}$$

When M is not compact and orientable, the map ϕ is biharmonic if $\tau_2(\phi) = 0$. In the sequel we shall specify if we are assuming M to be compact and orientable.

Of course, every harmonic map is biharmonic. Further we shall give some results about the nonexistence of nonharmonic biharmonic maps.

First, we recall the following result of G.Y. Jiang.

Theorem 2.1 *([5]). Let $\phi : (M, g) \to (N, h)$ be a smooth map. Suppose that M is compact and orientable, and* $\text{Riem}^N \leq 0$*. Then ϕ is biharmonic if and only if it is harmonic.*

If we give up the hypothesis M compact and orientable, we can get

Proposition 2.2. *Let $\phi : (M, g) \to (N, h)$ be a Riemannian immersion such that $|\tau(\phi)| = constant$. Suppose that* $\text{Riem}^N \leq 0$*. Then ϕ is biharmonic if and only if it is harmonic.*

Proof. Assume that $\tau_2(\phi) = 0$. From the Weitzenböck formula we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\tau(\phi)|^2 &= < \Delta\tau(\phi), \tau(\phi) > -|d\tau(\phi)|^2 \\ &= < \text{trace}\, R^N(\tau(\phi), d\phi -)d\phi -, \tau(\phi) > -|d\tau(\phi)|^2. \end{aligned}$$

If $|\tau(\phi)| = constant$, as $\text{Riem}^N \leq 0$, from the above equation it results that $d\tau(\phi) = 0$. Using now the equation

$$-|\tau(\phi)|^2 =< d\phi, d\tau(\phi) >,$$

which is true for any Riemannian immersion, we conclude that ϕ is harmonic.

When $\dim M = \dim N - 1$, we can replace the hypothesis $\text{Riem}^N \leq 0$ with the hypothesis $\text{Ricci}^N \leq 0$, and we obtain

Theorem 2.3. *Let $\phi : (M, g) \to (N, h)$ be a Riemannian immersion. Suppose that M is compact and orientable,* $\text{Ricci}^N \leq 0$ *and $m = n - 1$. Then ϕ is biharmonic if and only if it is harmonic.*

Proof. Since ϕ is a Riemannian immersion and $m = n-1$, we have

$$\begin{aligned}\operatorname{trace} R^N(d\phi -, \tau(\phi))d\phi - &= \sum_{i=1}^{m} R^N(d\phi(X_i), \tau(\phi)), d\phi(X_i) \\ &= -\operatorname{Ricci}^N(\tau(\phi)),\end{aligned}$$

where $\{X_i\}_{i=1}^m$ is an orthonormal frame field. Assume that ϕ is biharmonic. From the Weitzenböck formula and from the maximum principle we obtain $d\tau(\phi) = 0$. Now, the result follows applying the same argument as in the proof of Proposition 2.2. □

Proposition 2.4. *Let $\phi : (M, g) \to (N, h)$ be a Riemannian immersion such that $|\tau(\phi)| = constant$. Suppose $m = n - 1$, and* $\operatorname{Ricci}^N \leq 0$. *Then ϕ is biharmonic if and only if it is harmonic.*

Proposition 2.5. *Let $\phi : (M, g) \to (N, h)$ be a smooth map with the property that there is $p \in M$ such that* $\operatorname{rank} \phi(p) \geq 2$ *and* $|\tau(\phi)| = constant$. *Suppose that* $\operatorname{Riem}^N < 0$. *Then ϕ is biharmonic if and only if it is harmonic.*

Proof. Assume that ϕ is biharmonic. Again, from the Weitzenböck formula we obtain

$$R^N(\tau(\phi), d\phi(X_i), \tau(\phi), d\phi(X_i)) = 0, \ \forall i = 1, \cdots, m,$$

where $\{X_i\}_{i=1}^m$ is an orthonormal frame field. Suppose $\tau(\phi)(p) \neq 0$; then, as $\operatorname{Riem}^N < 0$, it follows that $\tau(\phi)(p) \parallel d\phi_p(X_i), \forall i$, i.e. $\operatorname{rank} \phi(p) \leq 1$. So $\tau(\phi)(p) = 0$ and since $|\tau(\phi)| = constant$, the proposition follows.

3. Biharmonic submanifolds of $\mathbb{S}^n$. Let M be an $m-$dimensional submanifold of $\mathbb{S}^n$, and let $\mathbf{i} : M \to \mathbb{S}^n$ be the canonical inclusion. We denote by B the second fundamental form of the submanifold $M \subset \mathbb{S}^n$, by A the shape operator, by H the mean curvature vector field of M, by $\nabla^\perp$ the normal connection and by $\Delta^\perp$ the Laplacian in the normal bundle of M.

Then we have

Theorem 3.1. *The map* $\mathbf{i}$ *is biharmonic if and only if*

$$(3.1)\qquad \begin{cases} -\Delta^{\perp}H - \operatorname{trace} B(-,A_H-) + mH &= 0 \\ 2\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) + \frac{m}{2}\operatorname{grad}(|H|^2) &= 0. \end{cases}$$

Proof. Since

$$\operatorname{trace} R^{\mathbb{S}^n}(d\mathbf{i}-,\tau(\mathbf{i}))d\mathbf{i}- = -m\tau(\mathbf{i}),$$

the map $\mathbf{i}$ is biharmonic if and only if

$$(3.2)\qquad \tau_2(\mathbf{i}) = \operatorname{trace}\nabla d\tau(\mathbf{i}) + m\tau(\mathbf{i}) = m\{\operatorname{trace}\nabla dH + mH\} = 0.$$

To prove this theorem we can choose, without loss of generality, a system of normal coordinates $\{x^i\}_{i=1}^m$ around an arbitrary point $p \in M$. If we put $e_i = \dfrac{\partial}{\partial x^i}$, at p we have

$$\begin{aligned} \operatorname{trace}\nabla dH &= \sum_{i=1}^m \nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} H = \sum_{i=1}^m \{\nabla^{\mathbb{S}^n}_{e_i}[\nabla^{\perp}_{e_i}H - A_H(e_i)]\} \\ &= \sum_{i=1}^m \{\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}H - A_{\nabla^{\perp}_{e_i}H}(e_i) - \nabla_{e_i}A_H(e_i) - B(e_i, A_H(e_i))\} \\ &= -\Delta^{\perp}H - \operatorname{trace} B(-,A_H-) - \sum_{i=1}^m [A_{\nabla^{\perp}_{e_i}H}(e_i) + \nabla_{e_i}A_H(e_i)]. \end{aligned}$$

Moreover, a straightforward computation shows that at p one has

$$\begin{aligned} \sum_{i=1}^m [A_{\nabla^{\perp}_{e_i}H}(e_i) + \nabla_{e_i}A_H(e_i)] &= 2\sum_{i=1}^m A_{\nabla^{\perp}_{e_i}H}(e_i) + \frac{m}{2}(d|H|^2)^{\sharp} \\ &= 2\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) + \frac{m}{2}\operatorname{grad}(|H|^2), \end{aligned}$$

where $\sharp : T^*M \to TM$ is the musical isomorphism that through the metric identifies 1-forms with vectors. Therefore, replacing the value of $\operatorname{trace}\nabla dH$ in (3.2), we have that $\mathbf{i}$ is biharmonic if and only if

$$(3.3)\qquad \begin{aligned} -\Delta^{\perp}H - \operatorname{trace} B(-,A_H-) + mH = \\ = 2\operatorname{trace} A_{\nabla^{\perp}_{(-)}H}(-) + \frac{m}{2}\operatorname{grad}(|H|^2). \end{aligned}$$

Since the left-hand side of (3.3) is normal to M and the right-hand side of (3.3) is tangent, the theorem follows. $\square$

Proposition 3.2. *Let M be a compact orientable manifold which is pseudo-umbilical, i.e. $A_H = |H|^2 I$. Assume that $|H|^2 \geq 1$. Then* **i** *is biharmonic if and only if $|H| = 1$ and $\nabla^{\perp} H = 0$.*

Proof. Assume that M is a biharmonic submanifold in $\mathbb{S}^n$. Then, from the first equation of (3.1) we obtain that

$$\Delta^{\perp} H = (1 - |H|^2) m H,$$

and applying the Weitzenböck formula, we get

$$\begin{aligned} \frac{1}{2}\Delta |H|^2 &= < \Delta^{\perp} H, H > - |\nabla^{\perp} H|^2 \\ &= m\left(1 - |H|^2\right)|H|^2 - |\nabla^{\perp} H|^2 \leq 0. \end{aligned}$$

Now, from the maximum principle, we conclude.

The converse is immediate.

Proposition 3.3. *Let M be a pseudo-umbilical submanifold of $\mathbb{S}^n$ with mean curvature constant $|H| = 1$. Then it is nonharmonic biharmonic if and only if $\nabla^{\perp} H = 0$.*

4. Biharmonic Riemannian submersions. Let $\phi : (M, g) \to (N, h)$ be a Riemannian submersion and let $p \in M$; we have $T_p M = T_p^H M \oplus T_p^V M$, where $T_p^V M = \ker\ d\phi_p$ and $T_p^H M$ is the orthogonal complement of $T_p^V M$ with respect to the metric g. Let W be an open subset in N such that $\phi(p) \in W$, $U = \phi^{-1}(W)$ and let $\{\widetilde{X}_\alpha\}$ be an orthonormal frame field on W. We consider $X_\alpha = (\widetilde{X}_\alpha)^H$ and $\{X_a\}_{a=n+1}^m$ an orthonormal frame field on $T^V U$. We have the well-known formula

$$\tau(\phi)(p) = - \sum_{a=n+1}^{m} d\phi_p(\nabla_{X_a} X_a) = -(m-n) d\phi_p(H(p)),$$

where $H(p)$ is the mean curvature vector field in p of the submanifold $\phi^{-1}(\phi(p))$ of M.

In the following we shall suppose that the tension field of ϕ is basic, i.e. $\tau(\phi)(p) = \tau(\phi)(q)$ whenever $\phi(p) = \phi(q)$. Thus $\tau(\phi)$ can be thought of as a vector field on manifold N.

Theorem 4.1. *Let $\phi : (M, g) \to (N, h)$ be a Riemannian submersion with basic tension field. Then we have*

$$\tau_2(\phi) = \operatorname{trace}^N \nabla^2 \tau(\phi) +^N \nabla_{\tau(\phi)} \tau(\phi) + \operatorname{Ricci}^N \tau(\phi). \tag{4.1.}$$

Proof. Using the above orthonormal frame fields $\{X_i\}_{i=1}^m$ and $\{\widetilde{X}_\alpha\}_{\alpha=1}^n$, we obtain

$$\begin{aligned}
\operatorname{trace} \nabla^2 \tau(\phi) &= \sum_{\alpha=1}^{n} \left\{ \nabla^{\phi^{-1}(TN)}_{X_\alpha} \nabla^{\phi^{-1}(TN)}_{X_\alpha} \tau(\phi) - \nabla^{\phi^{-1}(TN)}_{\nabla_{X_\alpha} X_\alpha} \tau(\phi) \right\} \\
&\quad + \sum_{a=n+1}^{n} \left\{ \nabla^{\phi^{-1}(TN)}_{X_a} \nabla^{\phi^{-1}(TN)}_{X_a} \tau(\phi) - \nabla^{\phi^{-1}(TN)}_{\nabla_{X_a} X_a} \tau(\phi) \right\} \\
&= \sum_{\alpha=1}^{n} \left\{ {}^N\nabla_{\widetilde{X}_\alpha} {}^N\nabla_{\widetilde{X}_\alpha} \tau(\phi) -^N \nabla_{d\phi(\nabla_{X_\alpha} X_\alpha)} \tau(\phi) \right\} \\
&\quad + \sum_{a=n+1}^{n} \left\{ {}^N\nabla_{-d\phi(\nabla_{X_a} X_a)} \tau(\phi) \right\} \\
&= \operatorname{trace}^N \nabla^2 \tau(\phi) +^N \nabla_{\tau(\phi)} \tau(\phi),
\end{aligned}$$

and

$$\operatorname{trace} R^N(d\phi -, \tau(\phi)) d\phi - = - \operatorname{Ricci}^N(\tau(\phi)).$$

Therefore, replacing the values of $\operatorname{trace} \nabla^2 \tau(\phi)$ and $\operatorname{trace} R^N(d\phi-, \tau(\phi))d\phi-$ in the expression of $\tau_2(\phi)$, we have the theorem.

Next we give some results about the nonexistence of nonharmonic biharmonic Riemannian submersions.

Theorem 4.2. *Assume that M is compact and orientable and* $\operatorname{Ricci}^N \leq 0$*, Then ϕ is biharmonic if and only if it is harmonic.*

Proof. If ϕ is biharmonic, then, from the Weitzenböck formula, we obtain

$$\begin{aligned}
\frac{1}{2} \Delta |\tau(\phi)|^2 &= < \Delta \tau(\phi), \tau(\phi) > - |d\tau(\phi)|^2 \\
&= < \operatorname{Ricci}^N \tau(\phi), \tau(\phi) > - |d\tau(\phi)|^2 \leq 0.
\end{aligned}$$

Since M is compact, applying the maximum principle, we obtain $d\tau(\phi) = 0$. Now, from

$$0 = \int_M < d\tau(\phi), d\phi > v_g = \int_M < \tau(\phi), d^\star d\phi > v_g = -\int_M |\tau(\phi)|^2 v_g,$$

we conclude.

Proposition 4.3. *Assume that* $\mathrm{Ricci}^N \leq 0$*, and* $\exists q \in N$ *such that* $\mathrm{Ricci}^N(q) < 0$*. If* $|\tau(\phi)| =$ *constant, then* ϕ *is biharmonic if and only if it is harmonic.*

Now, we replace the hypothesis M compact and orientable with the hypothesis N compact. We obtain

Proposition 4.4. *Assume that* N *is compact and orientable, and* $\mathrm{Ricci}^N \leq 0$*. Then* ϕ *is biharmonic if and only if* ${}^N\nabla\tau(\phi) = 0$*.*

Proof. Suppose that ϕ is biharmonic. Again, from the Weitzenböck formula, we obtain

$$\frac{1}{2}\Delta|\tau(\phi)|^2 \leq 0,$$

where Δ acts over smooth functions on M. Since $\tau(\phi)$ can be thought as a vector field on N, and N is compact, then $\exists p_0 \in M$ such that $|\tau(\phi)|(p_0) \geq |\tau(\phi)|(p)$, $\forall p \in M$. So, using the maximum principle and the Weitzenböck formula, we obtain $d\tau(\phi) = 0$, which is equivalent with ${}^N\nabla\tau(\phi) = 0$.

The converse results from Theorem 4.1.

Corollary 4.5. *Assume that* N *is compact and orientable,* $\mathrm{Ricci}^N \leq 0$ *and* $\exists q \in N$ *such that* $\mathrm{Ricci}^N(q) < 0$*. Then* ϕ *is biharmonic if and only if it is harmonic.*

Corollary 4.6. *Assume that* N *is compact and orientable,* $\mathrm{Ricci}^N \leq 0$*, and the Euler-Poincare characteristic* $\chi(N) \neq 0$*. Then* ϕ *is biharmonic if and only if it is harmonic.*

Now, looking for nonharmonic biharmonic Riemannian submersions, we can get

Theorem 4.7. *If* $\tau(\phi)$ *is an unitary Killing vector field on* N*, then* ϕ *is a biharmonic map.*

Proof. The hypothesis $\tau(\phi)$ Killing implies

$$\mathrm{trace}^N \nabla^2 \tau(\phi) + \mathrm{Ricci}^N(\tau(\phi)) = 0,$$

and the hypothesis $\tau(\phi)$ unitary Killing vector field on N implies ${}^N\nabla_{\tau(\phi)}\tau(\phi) = 0$. Now, we conclude.

Corollary 4.8. *Let $\phi : (M, g) \to S^1$ be a Riemannian submersion such that its fibres have constant mean curvature field, i.e. $|H(p)| = 1$, $\forall p \in M$, and $\tau(\phi)$ is basic. Then ϕ is biharmonic.*

Proposition 4.9. *Suppose that N is compact and orientable, $|\tau(\phi)| = 1$ and $\operatorname{div} \tau(\phi) = 0$. Then ϕ is biharmonic if and only if $\tau(\phi)$ is Killing.*

Proof. It follows from the formula

$$\int_N \{-h(\mathrm{trace}^N \nabla^2 \tau(\phi) + \mathrm{Ricci}^N(\tau(\phi)), \tau(\phi)) - \frac{1}{2}|L_{\tau(\phi)} h|^2 + (\operatorname{div} \tau(\phi))^2\} v_g = 0,$$

(see [10]).

4.1. Application. In this subsection we shall study the biharmonicity of the canonical projection $\pi : TM \to M$.

Let (M, g) be an m-dimensional Riemannian manifold and let $\pi : TM \to M$ be its tangent bundle. A local chart $(U; x^i)$, $i = 1, \ldots, m$, on M induces a local chart $(\pi^{-1}(U); x^i, y^j)$, $i, j = 1, \ldots, m$, on TM, where we denote, by abuse, x^i instead of $\pi^\star x^i = x^i \circ \pi$, and y^j are the vector space coordinates of the element $v \in \pi^{-1}(U) \subset TM$, with respect to the natural basis $\{(\frac{\partial}{\partial x^i})_{\pi(v)}\}_{i=1}^m$.

We have the vertical distribution $V(TM)$ on TM, defined by $V_v(TM) = \ker d\pi_v$, $v \in TM$. We consider a nonlinear connection on TM defined by the distribution $H(TM)$ on TM, complementary to $V(TM)$, i.e. $H_v(TM) \oplus V_v(TM) = T_v(TM)$, $v \in TM$. The distribution $H(TM)$ is the horizontal distribution. For any induced local chart $(\pi^{-1}(U); x^i, y^j)$ we have a local adapted frame in $H(TM)$ defined by the local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}, \quad i = 1, \ldots, m,$$

where the local functions $N^i_j(x,y)$ are the connection coefficients of the nonlinear connection defined by $H(TM)$. The vector fields $\{\frac{\partial}{\partial y^i}\}_{i=1}^m$ define a local frame for the vertical distribution $V(TM)$.

Let $\xi = \xi^i \frac{\partial}{\partial x^i}$ be a (local) vector field on M. The horizontal and the vertical lifts of ξ are defined by

$$\xi^H = \xi^i \frac{\delta}{\delta x^i}, \quad \xi^V = \xi^i \frac{\partial}{\partial y^i}.$$

We consider the Riemannian metric S of Sasaki type on TM, defined by

$$S(X^V, Y^V) = S(X^H, Y^H) = g(X,Y), \quad S(X^V, Y^H) = 0.$$

The canonical projection $\pi : (TM, S) \to (M, g)$ is a Riemannian submersion. By computing its Levi-Civita connection ${}^S\nabla$, we obtain

$$\text{(4.2)} \qquad \begin{cases} {}^S\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \frac{1}{2}(-\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial N^l_k}{\partial y^i} g_{lj} + \frac{\partial N^l_k}{\partial y^j} g_{li}) g^{kh} \frac{\delta}{\delta x^h} \\ \\ {}^S\nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \Gamma^h_{ij} \frac{\delta}{\delta x^h} + \frac{1}{2} R^h_{ji} \frac{\partial}{\partial y^h}, \end{cases}$$

where Γ^i_{jk} are the Christoffel symbols and $R^i_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k}$ (see [6]).

We shall consider the following three cases

A) $N^i_j = \Gamma^i_{jk} y^k + T^i_j(x)$, where $T^i_j(x)$ are the components of a tensor on M of type $(1,1)$. By a straightforward computation we obtain

Proposition 4.10. *The map π is harmonic and therefore it is biharmonic.*

B) $N^i_j = (\Gamma^i_{jk} + \delta^i_j \xi_k + \delta^i_k \xi_j) y^k$, where ξ is a vector field on M and $\xi_i = g_{ij}\xi^j$ (a projective change of the Levi-Civita connection ∇). By a direct computation and applying Theorem 4.1, we obtain

$$\text{(4.3)} \qquad \begin{cases} \tau(\pi) = -(m+1)\xi, \\ \\ \tau_2(\pi) = -(m+1)\{\text{trace}\, \nabla^2 \xi + \text{Ricci}(\xi) - (m+1)\nabla_\xi \xi\}. \end{cases}$$

So we conclude

Proposition 4.11. *a) If ξ is an unitary Killing vector field, then π is nonharmonic biharmonic.*
b) If $\nabla\xi = 0$ and $\xi \neq 0$, then π is a nonharmonic biharmonic map.

C) $N^i_j = (\Gamma^i_{jk} + \delta^i_j \alpha_k + \delta^i_k \alpha_j - g_{jk}\alpha^i)y^k$, where $\alpha_k = \dfrac{\partial \rho}{\partial x^k}$ and $\rho \in C^\infty(M)$, $\rho \neq$ *constant* (a conformal change of the connection ∇). Again, applying Theorem 4.1, we get

$$\begin{cases} \tau(\pi) = -m \operatorname{grad}\rho, \\ \\ \tau_2(\pi) = -m\{\operatorname{trace}\nabla^2 \operatorname{grad}\rho + \operatorname{Ricci}(\operatorname{grad}\rho) - m\nabla_{\operatorname{grad}\rho}\operatorname{grad}\rho\}. \end{cases} \tag{4.4}$$

Thus, we have

Proposition 4.12. *If ρ is a Killing potential, then π is a nonharmonic biharmonic map.*

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Received: 11.VII.2001

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