Abstract. The geometry of a nonconservative mechanical system is determined by its associated semispray and the corresponding nonlinear connection. The semispray is uniquely determined by the symplectic structure and the energy of the corresponding Lagrange space and the external force field. We prove that the corresponding nonlinear connection is uniquely determined by its compatibility with the metric tensor and the symplectic structure of the Lagrange space. We study the variation of the energy and Lagrangian functions along the evolution curves and the horizontal curves and give necessary and sufficient conditions by which these variations vanish. We provide examples of mechanical systems which are dissipative and for which the evolution nonlinear connection is either metric or symplectic.

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nonconservative mechanical system, semispray, nonlinear connection, external force field, dissipative system

Introduction

The use of nonlinear connections for the geometry of systems of second order differential equation has been proposed by Crampin [6] and Griffone [12]. Since then nonlinear connections were intensively studied together with some associated geometric structures such as Berwald connections, dynamical covariant derivative, horizontal differentiation, horizontal lift by Abraham and Marsden [1], Crampin and Pirani [7], De León and Rodrigues [9], Krupkova [14], Miron and Anastasiei [18], Szenthe [20], Szilasi and Muzsnay [21], Yano and Ishihara [22].

The geometry of a Lagrange space is the geometry of its canonical semispray and the associated nonlinear connection as it has been developed by Miron and Anastasiei [18]. In this case, it has been shown in [4] that the canonical nonlinear connection is uniquely determined by its compatibility with the metric structure and the symplectic structure. Using techniques that are specific to Lagrange geometry, Miron [16] introduces and investigates some geometric aspects of nonconservative mechanical systems by means of the corresponding semispray and nonlinear connection.

The geometry of nonconservative mechanical systems, where the external force field depends on both position and velocity, was rigorously investigated by Klein [13] and Godbillon [11]. The dynamical system of a nonconservative mechanical system is a second order vector field, or a semispray, and it has been uniquely determined by Godbillon [11] using the symplectic structure and the energy of the Lagrange space and the external force field. Using the external force field of the nonconservative mechanical system, Klein [13] introduces a force tensor, which is a second rank skew symmetric tensor. Aspects regarding first integrals for nonconservative mechanical
systems were investigated by Djukic and Vujanovic [10] and Cantijn [5]. For the particular case when the external force field is the vertical derivation of a dissipation function, such systems were studied by Bloch [3].

In this work we extend the geometric investigation of nonconservative mechanical systems, using the associated evolution nonlinear connection. We prove that the evolution nonlinear connection is uniquely determined by two compatibility conditions with the metric structure and the symplectic structure of the Lagrange space. The covariant derivative of the Lagrange metric tensor with respect to the evolution nonlinear connection is a second rank symmetric tensor, which uniquely determines the symmetric part of the connection. The difference between the symplectic structure of the Lagrange space and the almost-symplectic structure of the nonconservative mechanical system is the force tensor introduced by Klein [13], and used recently by Miron [16]. The force tensor, which is the vertical derivation of the external force, uniquely determines the skew-symmetric part of the evolution nonlinear connection. The force tensor vanishes in the work of Bloch [3] and therefore the symplectic geometry of the nonconservative mechanical system coincides with the symplectic geometry of the underlying Lagrange space as it has been developed by Abraham and Marsden [1].

If the Lagrangian function is not homogeneous of second order with respect to the velocity-coordinates, as it happens in the Riemannian and Finslerian framework, the energy of the system is different from the Lagrangian function and the evolution curves (solution of the Lagrange equations) are different from the horizontal curves of the system.

In the last part of the paper a special attention is paid to the particular case of Finslerian mechanical systems. Examples of dissipative mechanical systems are given, as well as examples of nonconservative mechanical systems that are compatible with the metric structure and the symplectic structure of the space.

1. Geometric structures on tangent bundle

In this section we introduce the geometric structures that live on the total space of tangent (cotangent) bundle, which we use in this work such as: Liouville vector field, semispray, vertical and horizontal distribution.

For an \( n \)-dimensional \( C^\infty \)-manifold \( M \), we denote by \( (TM, \pi, M) \) its tangent bundle and by \( (T^*M, \pi, M) \) its cotangent bundle. The total space \( TM \) (\( T^*M \)) of the tangent (cotangent) bundle will be the phase space of the coordinate velocities (momenta) of our mechanical system. Let \((U, \phi = (x^i))\) be a local chart at some point \( q \in M \) from a fixed atlas of \( C^\infty \)-class of the differentiable manifold \( M \). We denote by \( (\pi^{-1}(U), \Phi = (x^i, y^i)) \) the induced local chart at \( u \in \pi^{-1}(q) \subset TM \).

The linear map \( \pi_{*,u} : T_uTM \to T_{\pi(u)}M \) induced by the canonical submersion \( \pi \) is an epimorphism of linear spaces for each \( u \in TM \). Therefore, its kernel determines a regular, \( n \)-dimensional, integrable distribution \( V_uTM \) induced by the canonical submersion \( \pi \), which is called the vertical distribution. For every \( u \in TM \), \( \{ \partial/\partial y^i|_u \} \) is a basis of \( V_uTM \), where \( \{ \partial/\partial x^i|_u, \partial/\partial y^i|_u \} \) is the natural basis of \( T_uTM \) induced by a local chart. Denote by \( F(TM) \) the ring of real-valued functions over \( TM \) and by \( \mathcal{X}(TM) \) the \( F(TM) \)-module of vector fields on \( TM \). We also consider \( \mathcal{X}^v(TM) \) the \( F(TM) \)-module of vertical vector fields on \( TM \). An important vertical vector field is \( C = y^i(\partial/\partial y^i) \), which is called the Liouville vector field.
The mapping \( J : \mathcal{X}(TM) \rightarrow \mathcal{X}(TM) \) given by \( J = (\partial/\partial y^i) \otimes dx^i \) is called the tangent structure and it has the following properties: \( \operatorname{Ker} J = \operatorname{Im} J = \mathcal{X}^v(TM) \); \( \operatorname{rank} J = n \) and \( J^2 = 0 \). One can consider also the cotangent structure \( J^\ast = dx^i \otimes (\partial/\partial y^i) \) with similar properties.

A vector field \( S \in \chi(TM) \) is called a semispray, or a second order vector field, if \( JS = C \). In local coordinates a semispray can be represented as follows:

\[
S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i}.
\]

Integral curves of a semispray \( S \) are solutions of the following system of SODE:

\[
d^2x^i dt^2 + 2G^i \left( x, \frac{dx}{dt} \right) = 0.
\]

A nonlinear connection \( N \) on \( TM \) is an \( n \)-dimensional distribution \( N : u \in TM \mapsto N_u TM \subset T_u TM \) that is supplementary to the vertical distribution. Therefore, \( N \) is called also a horizontal distribution. This means that for every \( u \in TM \) we have the direct sum

\[
T_u TM = N_u TM \oplus V_u TM.
\]

We denote by \( h \) and \( v \) the horizontal and the vertical projectors that correspond to the above decomposition and by \( \mathcal{X}^h(TM) \) the \( \mathcal{F}(TM) \)-module of horizontal vector fields on \( TM \). For every \( u = (x,y) \in TM \) we denote by \( \delta/\delta x^i|_u = h(\partial/\partial x^i|_u) \). Then \( \{ \delta/\delta x^i|_u, \partial/\partial y^i|_u \} \) is a basis of \( T_u TM \) adapted to the decomposition (3).

With respect to the natural basis \( \{ \partial/\partial x^i|_u, \partial/\partial y^i|_u \} \) of \( T_u TM \), we have the expression:

\[
\frac{\delta}{\delta x^i} \bigg|_u = \frac{\partial}{\partial x^i} \bigg|_u - N^i_j(u) \frac{\partial}{\partial y^j} \bigg|_u, \quad u = (x,y) \in TM.
\]

The functions \( N^i_j(x,y) \), defined on domains of induced local charts, are called the local coefficients of the nonlinear connection. The corresponding dual basis is \( \{ dx^i, \delta y^i = dy^i + N^i_j dx^j \} \).

It has been shown by Crampin [6] and Grifone [12] that every semispray determines a nonlinear connection. The horizontal projector \( h \) that corresponds to this nonlinear connection is given by:

\[
h(X) = \frac{1}{2} \left(\mathcal{L}_S J \right)(X) = \frac{1}{2} \left( X - [S, JX] - J[S, X] \right).
\]

Local coefficients of the induced nonlinear connection are given by \( N^i_j = \partial G^i/\partial y^j \).

2. Geometric structures on a Lagrange space

The presence of a regular Lagrangian on the tangent bundle \( TM \) determines the existence of some geometric structures such as: semispray, nonlinear connection and symplectic structure.

Consider \( L^n = (M, L) \) a Lagrange space. [17, 18]. This means that \( L : TM \rightarrow \mathbb{R} \) is differentiable of \( C^\infty \)-class on \( TM = TM \setminus \{0\} \) and only continuous on the null section. We also assume that \( L \) is a regular Lagrangian. In other words, the \( (0,2) \)-type, symmetric, \( d \)-tensor field with components

\[
g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \text{ has rank } n \text{ on } TM.
\]
The Cartan 1-form $\theta_L$ of the Lagrange space can be defined as follows:

$$
\theta_L = J^* (dL) = d_J L = \frac{\partial L}{\partial y^i} \, dx^i.
$$

For a vector field $X = X^i \left( \frac{\partial}{\partial x^i} \right) + Y^i \left( \frac{\partial}{\partial y^i} \right)$ on $TM$, the following formulae are true:

$$
\theta_L (X) = dL (JX) = d_J L (X) = (JX) (L) = \frac{\partial L}{\partial y^i} X^i.
$$

The Cartan 2-form $\omega_L$ of the Lagrange space can be defined as follows:

$$
\omega_L = \mathrm{d} \theta_L = \mathrm{d} \left( J^* (dL) \right) = \left( \frac{\partial L}{\partial y^i} \right) \, dx^i.
$$

In local coordinates, the Cartan 2-form $\omega_L$ has the following expression:

$$
\omega_L = 2 g_{ij} \, dy^j \wedge dx^i + \frac{1}{2} \left( \frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j} \right) \, dx^j \wedge dx^i.
$$

We can see from expression (10) that the regularity of the Lagrangian $L$ is equivalent with the fact that the Cartan 2-form $\omega_L$ has rank $2n$ on $\tilde{T}M$ and hence it is a symplectic structure on $\tilde{T}M$. With respect to this symplectic structure, the vertical subbundle is a Lagrangian subbundle of the tangent bundle.

The canonical semispray of the Lagrange space $L^n$ is the unique vector field $\hat{S}$ on $TM$, [1], that satisfies the equation

$$
\hat{S} \omega_L = -dE_L.
$$

Here $E_L = C(L) - L$ is the energy of the Lagrange space $L^n$. The local coefficients $\hat{G}^i$ of the canonical semispray $\hat{S}$ are given by the following formula, [18]:

$$
\hat{G}^i = \frac{1}{4} g^{ik} \left( \frac{\partial^2 L}{\partial y^k \partial x^i} y^k - \frac{\partial L}{\partial x^i} \right).
$$

Using the canonical semispray $\hat{S}$ we can associate to a regular Lagrangian $L$ a canonical nonlinear connection with local coefficients given by expression $N^i_j = \partial \hat{G}^i / \partial y^j$.

The horizontal subbundle $NTM$ that corresponds to the canonical nonlinear connection is a Lagrangian subbundle of the tangent bundle $TTM$ with respect to the symplectic structure $\omega_L$. This means that $\omega_L (hX, hY) = 0$, $\forall X, Y \in \chi(TM)$. For a more detailed discussion regarding symplectic structures in Lagrange geometry we recommend [2]. In local coordinates this implies the following expression for the symplectic structure $\omega_L$:

$$
\omega_L = 2 g_{ij} \, dy^j \wedge dx^i.
$$

The dynamical derivative that corresponds to the pair $\left( \hat{S}, \hat{N} \right)$ is defined by $\nabla : \chi^v (TM) \rightarrow \chi^v (TM)$ through:

$$
\nabla \left( X^i \frac{\partial}{\partial y^i} \right) = \left( \hat{S} (X^i) + X^i \hat{N}^j \frac{\partial}{\partial y^j} \right) \frac{\partial}{\partial y^i}.
$$

In terms of the natural basis of the vertical distribution we have

$$
\nabla \frac{\partial}{\partial y^i} = \hat{N}^j \frac{\partial}{\partial y^j}.
$$
Hence, $N^j_i$ are also local coefficients of the dynamical derivative. Dynamical derivative $\nabla$ is the same with the covariant derivative $D$ in [8] or $D$ in [14], where it is called the $\Gamma$-derivative. Dynamical derivative $\nabla$ has the following properties:

1. $\nabla (X + Y) = \nabla X + \nabla Y, \forall X, Y \in \chi^c (TM)$;
2. $\nabla (f X) = S (f) X + f \nabla X, \forall X \in \chi^c (TM), \forall f \in F (TM)$.

It is easy to extend the action of $\nabla$ to the algebra of d-tensor fields by requiring for $\nabla$ to preserve the tensor product. For the metric tensor $g$, its dynamical derivative is given by

$$\nabla g (X, Y) = S (g (X, Y)) - g (\nabla X, Y) - g (X, \nabla Y).$$

In local coordinates, we have:

$$g_{ij} := (\nabla g) \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = S (g_{ij}) - g_{im} N^m_j - g_{mj} N^m_i.$$

The canonical nonlinear connection $\tilde{N}$ is metric, which means that $\nabla g = 0$. In [4] it is shown that the canonical nonlinear connection of a Lagrange space is the unique nonlinear connection that is metric and symplectic.

### 3. Geometric structures of nonconservative mechanical systems

The dynamical system of a nonconservative mechanical system is a semispray, which we call the evolution semispray of the system. Such semispray is uniquely determined by the symplectic structure and the energy of the underlying Lagrange space and the external force field, [11], [5]. Based on such result, one can easily prove that the energy of the system is decreasing if and only if the force field is dissipative.

The nonlinear connection we associate to the evolution semispray is called the evolution nonlinear connection. We prove that the evolution nonlinear connection of a nonconservative mechanical system is uniquely determined by two compatibility conditions with the metric structure and the symplectic structure of the Lagrange space. Conditions by which such nonlinear connection is either metric or symplectic are studied.

A nonconservative mechanical system is a triple $\Sigma_L = (M, L, \sigma)$, where $(M, L)$ is a Lagrange space and $\sigma = \sigma^i (x, y) \left( \partial / \partial y^i \right)$ is a vertical vector field, which is called the external force field of the system. Using the metric tensor $g_{ij}$ of the Lagrange space one can define the vertical one-form $\sigma = \sigma_i dx^i$, where $\sigma_i (x, y) = g_{ij} (x, y) \sigma^j (x, y)$. In this paper we use the same notation $\sigma$ for the vertical vector field $\sigma^i (x, y) \left( \partial / \partial y^i \right)$ and the vertical one-form $\sigma_i dx^i$ and it will be clear from the context to which one we refer to. The covariant force field $\sigma$ induces two second rank d-tensor fields, which are the skew-symmetric and the symmetric part of the second rank d-tensor field $\partial \sigma_i / \partial y^j$. The helicoidal tensor, [16], or the generalized force tensor, [13], is defined as the vertical derivative of the covariant force field.

$$P = d_j \sigma = \frac{1}{2} \left( \frac{\partial \sigma_i}{\partial y_j} - \frac{\partial \sigma_j}{\partial y_i} \right) dx^i \wedge dx^j. $$

We consider also the second rank symmetric d-tensor field

$$Q = \frac{1}{2} \left( \frac{\partial \sigma_i}{\partial y_j} + \frac{\partial \sigma_j}{\partial y_i} \right) dx^i \otimes dx^j.$$
The external force field $\sigma$ is dissipative if $g(\mathcal{C}, \sigma) = g_{ij} y^i \sigma^j \leq 0$.

The evolution equations of the mechanical system $\Sigma_L$ are given by the following Lagrange equations, [13], [11], [3], [16]:

\begin{equation}
\frac{d}{dt} \left( \frac{\partial L}{\partial q^i} \right) - \frac{\partial L}{\partial x^i} = \sigma^i, \quad y^i = \frac{dx^i}{dt}.
\end{equation}

For a regular Lagrangian, the Lagrange equations (20) are equivalent to the following system of second order differential equations:

\begin{equation}
\frac{d^2 x^i}{dt^2} + 2 \dot{G}^i \left( x, \frac{dx}{dt} \right) = \frac{1}{2} \sigma^i \left( x, \frac{dx}{dt} \right).
\end{equation}

Here the functions $\dot{G}^i$ are local coefficients of the canonical semispray of the Lagrange space, given by expression (12). Since $(1/2) \sigma^i$ are components of a $d$-vector field on $TM$, from expression (21) we obtain that the functions

\begin{equation}
2\dot{G}^i (x, y) = 2\dot{G}^i (x, y) - \frac{1}{2} \sigma^i (x, y)
\end{equation}

are coefficients of a semispray $S$, to which we refer to as the evolution semispray of the nonconservative mechanical system, [16]. Therefore, the evolution semispray is given by $S = \dot{S} + (1/2) \sigma$. Integral curves of the evolution semispray $S$ are solutions of the SODE given by expression (21). In [11], Godbillon proves that the evolution semispray $S$ is the unique vector field on $TM$, solution of the equation

\begin{equation}
i_{\dot{S}} \omega_L = -dE_L + \sigma.
\end{equation}

One can use the above expression that uniquely defines the evolution semispray to prove that the energy of the Lagrange space $L^n$ is decreasing along the evolution curves of the mechanical system if and only if the external force field is dissipative. Indeed, using expression (23) we obtain $S (E_L) = dE_L (S) = \sigma (S) = \sigma y^i$. Along the evolution curves of the mechanical system, one can write this expression as follows:

\begin{equation}
\frac{d}{dt} (E_L) = \sigma^i \left( x, \frac{dx}{dt} \right) \frac{dx^i}{dt}.
\end{equation}

Therefore, the energy is decreasing along the evolution curves if and only if the external force field is dissipative, which means $\sigma y^i \leq 0$.

Expression (24) has been obtained by Munoz-Lecanda and Yaniz-Fernandez in [19] for the particular case of a Riemannian mechanical system. For the general case of nonconservative mechanical systems, expression (24) has been also obtained by Miron in [16], using different techniques. If the covariant force field is the vertical differential of a dissipation function, $\sigma = -d_J D$, the corresponding form of expression (24) has been obtained by Bloch in [3].

The evolution nonlinear connection of the mechanical system $\Sigma_L$ has the local coefficients $N^i_j$ given by

\begin{equation}
N^i_j = \frac{\partial \dot{G}^i}{\partial y^j} - \frac{1}{4} \frac{\partial \sigma^i}{\partial y^j} = \dot{N}^i_j - \frac{1}{4} \frac{\partial \sigma^i}{\partial y^j}.
\end{equation}

**Theorem 3.1.** For a nonconservative mechanical system $\Sigma_L = (M, L, \sigma)$, the evolution nonlinear connection is the unique nonlinear connection that satisfies the following two conditions

\begin{equation}
\nabla g = \frac{1}{2} Q
\end{equation}
\[ \omega_L(hX, hY) = \frac{1}{2} P(X, Y), \forall X, Y \in \chi(TM). \]

**Proof.** The dynamical covariant derivative of the metric tensor \( g_{ij} \) with respect to the pair \((S, N)\) is given by

\[
\begin{align*}
\nabla g_{ij} &= \left( \dot{S} + \sigma \right) (g_{ij}) - g_{ik}N^k_j - g_{kj}N^k_i \\
&= \dot{S} (g_{ij}) - g_{ik} \dot{N}^k_j - g_{kj} \dot{N}^k_i + \frac{1}{4} \left( 2 \sigma (g_{ij}) + g_{ik} \frac{\partial \sigma}{\partial y^j} + g_{kj} \frac{\partial \sigma}{\partial y^i} \right) \\
&= \frac{V^k}{4} \left( 2 \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{ik}}{\partial y^j} - \frac{\partial g_{kj}}{\partial y^i} \right) + \frac{1}{4} \left( \frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right) \\
&= \frac{1}{4} \left( \frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right).
\end{align*}
\]

In the above calculations we did use the fact that the canonical nonlinear connection \( \dot{N} \) is metric and the Cartan tensor

\[
C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} - \frac{1}{4} \frac{\partial^3 L}{\partial y^i \partial y^j \partial y^k}
\]

is totally symmetric. Therefore, we have obtained that the dynamical covariant derivative of the metric tensor \( g_{ij} \) with respect to the pair \((S, N)\) is given by \( \nabla g = \frac{1}{2} Q \). Moreover, this uniquely determine the symmetric part \( N_{(ij)} \) of the evolution nonlinear connection. First and last line of equation (26) can be rewritten as follows

\[
2N_{(ij)} := N_{ij} + N_{ji} := g_{ik}N^k_j + g_{kj}N^k_i = S (g_{ij}) - \frac{1}{4} \left( \frac{\partial \sigma_i}{\partial y^j} + \frac{\partial \sigma_j}{\partial y^i} \right).
\]

Let us consider the almost-symplectic structure, which is called by Klein, [13], the fundamental two-form of the mechanical system:

\[
\omega = 2g_{ij} \delta y^j \wedge dx^i,
\]

with respect to which both horizontal and vertical subbundles are Lagrangian sub-bundles. Using expressions (13) and (25) the canonical symplectic structure can be expressed as follows:

\[
\begin{align*}
\omega_L &= 2g_{ij} \delta y^j \wedge dx^i = 2g_{ij} \left( \delta y^j + \frac{1}{4} \frac{\partial \sigma^j}{\partial y^i} dx^k \right) \wedge dx^i \\
&= 2g_{ij} \delta y^j \wedge dx^i + \frac{1}{4} \left( \frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right) dx^j \wedge dx^i \\
&= \omega + \frac{1}{2} P_{ij} dx^j \wedge dx^i = \omega + d_J \sigma.
\end{align*}
\]

Here \( P_{ij} \) is the helicoidal tensor of the mechanical system \( \Sigma_L \)

\[
P_{ij} = \frac{1}{2} \left( \frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right).
\]

Expression (30) can be rewritten as follows

\[
\dot{N}_{(ij)} = N_{(ij)} + \frac{1}{2} P_{ij}.
\]
The skew-symmetric part of the canonical nonlinear connection of the Lagrange space is given by, [4]

\[ \hat{N}_{ij} := \frac{1}{2} \left( \hat{N}_{ij} - \hat{N}_{ji} \right) = \frac{1}{4} \left( \frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j} \right). \]  

From expressions (32) and (33) we conclude that the skew-symmetric part of the evolution nonlinear connection is uniquely determined and given by

\[ N_{ij} = \frac{1}{4} \left( \frac{\partial^2 L}{\partial y^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial y^j} \right) - \frac{1}{4} \left( \frac{\partial \sigma_i}{\partial y^j} - \frac{\partial \sigma_j}{\partial y^i} \right). \]

One can conclude now that the evolution nonlinear connection of a mechanical system is uniquely determined by the two conditions of the theorem. □

**Corollary 3.2.** The evolution nonlinear connection is metric if and only if the (0,2)-type d-tensor field \( \partial \sigma_i/\partial y^j \) is skew symmetric.

**Proof.** According to first condition of the Theorem 3.1 the evolution nonlinear connection is metric if and only if the tensor \( \hat{Q} \) vanishes, which is equivalent to the fact that the (0,2)-type d-tensor field \( \partial \sigma_i/\partial y^j \) is skew symmetric. □

**Corollary 3.3.** The evolution nonlinear connection is compatible with the canonical symplectic structure of the Lagrange space if and only if the (0,2)-type d-tensor field \( \partial \sigma_i/\partial y^j \) is symmetric.

**Proof.** According to the second condition of the Theorem 3.1 the evolution nonlinear connection is compatible with the canonical symplectic structure of the Lagrange space if and only if the helicoidal tensor \( \hat{P} \) vanishes, which is equivalent to the fact that the (0,2)-type d-tensor field \( \partial \sigma_i/\partial y^j \) is symmetric. □

We remark here that the case when the (0,2)-type d-tensor field \( \partial \sigma_i/\partial y^j \) is symmetric has been discussed by Bloch in [3]. This means that, at least locally, there exists a function \( D \), which is called the dissipation function, such that \( \sigma_i = \partial D/\partial y^i \). In such a case, the covariant force field \( \sigma \) is the vertical differential of the dissipation function \( D \), which means that \( \sigma = dJ/D \).

4. **Variation of energy and Lagrangian functions**

In this section we study the variation of energy and Lagrangian functions along the horizontal curves of the evolution nonlinear connection.

For a nonconservative mechanical system \( \Sigma_L \), consider \( \dot{S} \) the evolution semispray given by expression (22) and the evolution nonlinear connection given by expression (25) \( h \) the corresponding horizontal projector. Then \( h \dot{S} \) also a semispray, its integral curves are called the horizontal curves of the evolution nonlinear connection. They are solutions of the following system of SODE:

\[ \nabla \left( \frac{dx^i}{dt} \right) = \frac{d^2 x^i}{dt^2} + N^i_j \left( \frac{dx}{dt}, \frac{dx}{dt} \right) \frac{dx^j}{dt} = 0. \]

In the previous section, expression (25) gives the variation of the energy function along the evolution curves of the nonconservative mechanical system. We study now, the variation of the energy and Lagrangian functions along the horizontal curves (35). We provide necessary and sufficient conditions for the external force field such that the Lagrangian and the energy functions are integrals for the system (35).
Proposition 4.1. The horizontal differential operator \( d_h \) of the Lagrangian \( L \) is given by:
\[
2 d_h L = d_J (S(L)) - \sigma. \tag{36}
\]
In local coordinates, formula (36) is equivalent with the following expression for the horizontal covariant derivative of the Lagrangian \( L \):
\[
2 L_{i\dot{i}} := 2 \frac{\delta L}{\delta x^i} = \frac{\partial}{\partial y^i} (S(L)) - \sigma_i. \tag{37}
\]

Proof. We first prove the following formulae regarding the Cartan 1-form \( \theta_L \) of the Lagrange space:
\[
L_S \theta_L = dL + \sigma. \tag{38}
\]
By differentiating \( \iota_S \theta_L = C(L) \) we obtain \( d_S \theta_L = dC(L) \). Using the expression of the Lie derivative \( L_S = d_S + \iota_S d \) we obtain \( L_S \theta_L = \iota_S \theta_L + dC(L) \). From the defining formulae (9) and (23) for \( \omega_L \) and \( S \) we obtain \( L_S \theta_L = dE_L + \sigma + dC(L) = dL + \sigma \). In order to prove (36) we have to show that for every \( X \in \chi(TM) \), we have that
\[
2 (d_h L)(X) := 2dL(hX) = (JX)(S(L)) - \sigma(X). \nonumber
\]

Using formula (38) we obtain
\[
0 = (L_S \theta_L - dL - \sigma)(X) = S \theta_L(X) - \theta_L[S,X] - dL(X) - \sigma(X)
= S ((JX)(L)) - J [S, X](L) - dL(X) - \sigma(X)
= (JX)(S(L)) - J [S, X](L) - dL(X) - \sigma(X)
= (JX)(S(L)) - \sigma(X) - dL(2hX).
\]
Consequently, formula (36) is true. Due to the linearity of the operators involved in formula (36) we have that formulae (36) and (37) are equivalent. \( \square \)

Corollary 4.2. The Lagrangian \( L \) is constant along the horizontal curves of the evolution nonlinear connection if and only if the external force field of the mechanical system satisfies the equation:
\[
\frac{\partial \sigma^k}{\partial y^i} y^i \frac{\partial L}{\partial y^k} = -2C \left( \dot{S}(L) \right). \tag{39}
\]

Proof. The Lagrangian \( L \) is constant along the horizontal curves of the evolution nonlinear connection if and only if \( hS(L) = 0 \). If we contract expression (37) by \( y^i \) we obtain
\[
2 (hS)(L) = 2L_{i\dot{i}} y^i = 2 \frac{\delta L}{\delta x^i} y^i = C(S(L)) - \sigma(S)
= C \left( \dot{S}(L) \right) + \frac{1}{2} \frac{\partial \sigma^k}{\partial y^i} y^i \frac{\partial L}{\partial y^k}.
\]
Therefore we can see that \( hS(L) = 0 \) if and only if the external force field satisfies equation (39). \( \square \)

Proposition 4.3. The horizontal differential operator \( d_h \) of the energy \( E_L \) is given by:
\[
d_h E_L (X) = -\omega_L \left( \dot{S}, hX \right). \tag{41}
\]
In local coordinates, this is equivalent with the following expression for the horizontal covariant derivative of the energy $E_L$:

$$E_{L|i} := \frac{\delta E_L}{\delta x^i} = 2g_{ij} \left( 2\dot{G}^{ij} - N^{ij}_k y^k \right) + \frac{1}{2} g_{ij} \frac{\partial \sigma^i}{\partial y^j} y^k.$$

**Proof.** We have that $d_h E_L (X) = (dE_L)(hX) = -\omega_L \left( \dot{S}, hX \right)$. Since $h$ is the horizontal projector for the evolution nonlinear connection, for a vector field $X = X^i \left( \partial / \partial x^i \right) + Y^i \left( \partial / \partial y^i \right)$ on $TM$ we have

$$hX = X^i \frac{\delta}{\delta x^i} = X^i \frac{\delta}{\delta x^i} + X^i \frac{1}{4} \frac{\partial \sigma^i}{\partial y^j} \frac{\partial}{\partial y^j}.$$

Using expression (13) for the symplectic structure $\omega_L$ we have

$$d_h E_L (X) = \left( -2g_{ij} \frac{\partial \sigma^i}{\partial y^j} \wedge dx^j \right) \left( \dot{S}, hX \right) = 2g_{ij} X^i \left( 2\dot{G}^{ij} - N^{ij}_k y^k \right) + \frac{1}{2} g_{ij} \frac{\partial \sigma^i}{\partial y^j} y^k X^i,$$

and therefore we have proved both formulae (41) and (42). \hfill \Box

If we contract expression (42) by $y^i$ and set the left hand side to zero we obtain the following necessary and sufficient condition such that the energy of the Lagrange space is a first integral of the systems (35).

**Corollary 4.4.** The energy $E_L$ is constant along the horizontal curves of the evolution nonlinear connection if and only if the external force field of the mechanical system satisfies the equation:

$$g_{jk} \frac{\partial \sigma^j}{\partial y^k} y^i y^j = -4g_{ij} \left( 2\dot{G}^{ij} - N^{ij}_k y^k \right) y^i.$$

5. **Finslerian mechanical systems**

Finsler geometry corresponds to the case when the Lagrangian function is second order homogeneous with respect to the velocity coordinates. Therefore, a Lagrange space $L^n = (M, L)$ reduces to a Finsler space $F^n = (M, F)$ if $L(x, y) = F^2 (x, y)$, where $F (x, y) > 0$ and $F (x, y)$ is positively homogeneous of order one with respect to $y$. Using Euler’s theorem for homogeneous functions we have $C (F^2) = (\partial F^2 / \partial y^i) y^i = 2F^2$. A first consequence of the homogeneity condition is that the energy of a Finsler space coincides with the square of the fundamental function of the space: $E_{F^2} = C (F^2) - F^2 = F^2$.

A Finslerian mechanical system is a triple $\Sigma_F = (M, F, \sigma)$, where $(M, F)$ is a Finsler space and $\sigma = \sigma^i (x, y) (\partial / \partial y^i)$ is a vertical vector field.

The evolution equations of the Finslerian mechanical system are given by Lagrange equations (20) where $L(x, y) = F^2 (x, y)$, which are equivalent with the system of second order differential equations (21). The local coefficients $\ddot{C}^i$ of the canonical semispray $\dot{S}$ of the Finsler space can be written in this case as follows:

$$2\ddot{C}^i (x, y) = \gamma^i_{jk} (x, y) \Gamma^j y^k,$$

where $\gamma^i_{jk} (x, y)$ are Christoffel symbols of the metric tensor $g_{ij} (x, y)$. [18]. Therefore, the evolution semispray $\dot{S}$ of the mechanical system has the local coefficients...
The evolution curves of the mechanical system are solutions of the following SODE:

\[
\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} - \frac{1}{2} \sigma^i \left( x, \frac{dx}{dt} \right) = 0.
\]

Local coefficients \(N^i_j\) of the evolution nonlinear connection are given by expression (25). Due to the homogeneity conditions one can express them as follows:

\[
N^i_j (x, y) = \gamma_{kj}^i (x, y) y^k - \frac{1}{4} \frac{\partial \sigma^i}{\partial y^j} (x, y).
\]

Therefore, the horizontal curves of the evolution nonlinear connection are solutions of the following SODE:

\[
\frac{d^2 x^i}{dt^2} + \gamma_{jk}^i \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} - \frac{1}{4} \frac{\partial \sigma^i}{\partial y^j} \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} = 0.
\]

Proposition 5.1. Consider a Finslerian mechanical system \(\Sigma_F = (M, F, \sigma)\), with the external force field \(\sigma\) homogeneous of order zero. Then the energy function \(F^2(x, y)\) is constant along the horizontal curves of the evolution nonlinear connection.

Proof. If the external force field \(\sigma\) is homogeneous of order zero, then by Euler's theorem we have \((\partial \sigma^i / \partial y^j) y^j = 0\) and the horizontal curves of the evolution nonlinear connection given by expression (49) coincide with the geodesics of the Finsler space. The energy function \(F^2(x, y)\) is constant along the geodesic curves of the Finsler space.

One can obtain this result by using also expressions (44) or (39). The right hand side of both expression vanishes for a Finsler space, while the left hand side vanishes due to the homogeneity of the external force field. \(\square\)

For a Finsler space \((M, F^2(x, y))\), the local coefficients \(2G^i (x, y)\) of the canonical semispray are given by expression (45) and therefore they are second order homogeneous with respect to the velocity variables. This implies that \(2G^i = N^i_k y^k\) and equation (42) can be written as follows:

\[
F^2_{yi} = \frac{1}{2} g_{jk} \frac{\partial \sigma^j}{\partial y^i} y^k = \frac{1}{2} \frac{\partial \sigma^i}{\partial y^j} y^j.
\]

Here we did use the symmetry of the Cartan tensor (27) and the zeroth homogeneity of the metric tensor \(g_{ij}\).

Corollary 5.2. For a second order homogeneous external force field \(\sigma\), the energy \(F^2\) of a Finsler space is constant along the evolution curves of the Finslerian mechanical system \((M, F, V \sigma)\) if and only if \(\sigma (F^2) = 0\).

Proof. If the external force field \(\sigma\) is second order homogeneous then equations (47) and (49) coincide and therefore the evolution curves of a Finslerian mechanical system coincide with the horizontal curves of the evolution nonlinear connection. For a Finsler space, the right hand side of both expressions (39) and (44) vanish. Using the homogeneity of \(\sigma\) and \(F^2\), we obtain that the left hand side for both
expressions (39) and (44) can be written as \( \sigma (F^2) \). Consequently, we obtain that the energy function \( F^2 \) is constant along the evolution curves of the Finslerian system if and only if \( \sigma (F^2) = 0 \). □

**Corollary 5.3.** If the helicoidal tensor of the mechanical system vanishes and the external force field is zero homogeneous then the horizontal covariant derivative of the energy function vanishes, in other words \( F_{ij}^2 = 0 \).

6. Examples

In this section we give examples of nonconservative mechanical systems which have some of the properties that we studied in the previous sections.

1. Consider the nonconservative mechanical system \( \Sigma_L = (M, L, e\mathbb{C}) \), where \( e\mathbb{C} = e y^i (\partial/\partial y^i) \) and \( e \) is a constant. We call this system a Liouville mechanical system. The evolution semispray \( S \) and the evolution nonlinear connection \( N \) have the local coefficients given by:

\[
2G^i(x, y) = 2\dot{G}^i - \frac{e}{2} y^i , \quad N_j^i = \dot{N}_j^i - \frac{e}{4} \delta_j^i .
\]

The helicoidal tensor of the system is given by expression (31). Since \( \sigma_i = e g_{ik} y^k \) we have the following expression:

\[
\frac{\partial \sigma_i}{\partial y^j} = e \left( \frac{\partial g_{ik}}{\partial y^j} y^k + g_{ij} \right) = e \left( 2C_{ijk} y^k + g_{ij} \right) .
\]

According to the above formula we have that the \((0, 2)\)-type d-tensor field \( \partial \sigma_i/\partial y^j \) is symmetric and therefore the helicoidal tensor of the system vanishes. According to Corollary 3.3, the evolution nonlinear connection is compatible with the symplectic structure of the Lagrange space.

The dynamical covariant derivative of the metric tensor \( g_{ij} \) with respect to the pair \((S, N)\) is given by the following formula:

\[
g_{ij} = \frac{e}{2} \left( 2C_{ijk} y^k + g_{ij} \right) .
\]

Consequently, the evolution nonlinear connection is metric if and only if the metric tensor is homogeneous of order \(-1\).

If the Liouville mechanical system is Finslerian, then \( C_{ijk} y^k = 0 \) and consequently \( g_{ij} = (e/2) g_{ij} \). We assume that the Liouville mechanical system is Finslerian. The system is dissipative if and only if \( 0 > g(C, e\mathbb{C}) = e g_{ij} y^i y^j = e F^2 \), which holds true if and only if \( e < 0 \).

2. Consider the Finslerian mechanical system: \( \Sigma_F = (M, F(x, y), (e/F) \mathbb{C}) \). For this system, the external force field is zero homogeneous. According to the previous section the horizontal curves of the evolution nonlinear connection coincide with the geodesic curves of the Finsler space \((M, F^2)\).

The covariant force field \( \sigma \) has the components given by

\[
\sigma_i = \frac{e}{2} g_{ij} y^j = \frac{e}{2} \frac{\partial F}{\partial y^j} ,
\]

which is equivalent to \( \sigma = d_j F \). Nonconservative mechanical systems with the external force field given by expression (54) were considered also by Djukic and Vujanovic [10]. The helicoidal tensor is the vertical differential of the covariant
force field \( \sigma \), and it vanishes since \( d^2_\sigma F = 0 \). Consequently, the evolution nonlinear connection is compatible with the symplectic structure of the Finsler space.

The horizontal covariant derivative of \( F^2 \) is given by expression (50). Using the symmetry of the tensor \( \partial \sigma_k / \partial y^l \) and the zero homogeneity of the external force field we obtain

\[
F^2_{\text{H}} = \frac{1}{2} \frac{\partial \sigma_k}{\partial y^l} y^k = \frac{1}{2} \frac{\partial \sigma_i}{\partial y^k} y^k = 0.
\]

Hence, \( F^2 \) is constant along the horizontal curves of the evolution nonlinear connection.

Since \( \sigma_i y^i = eF \) and using expression (24) along the evolution curves of the mechanical system we have

\[
\frac{d}{dt} (F^2) = eF.
\]

Therefore, the system is dissipative if and only if \((e/F) g(\mathcal{C}, \mathcal{C}) < 0\), which is equivalent to \( e < 0 \).

3. Consider the Finslerian mechanical system \( \Sigma_F = (M, F, eF) \). The external force field \( eF \) is second order homogeneous and consequently the evolution curves of the mechanical system \( \Sigma_F \) coincide with the horizontal curves of the evolution nonlinear connection. The helicoidal tensor field of this system vanishes as well and therefore the almost-symplectic form of the system coincides with the symplectic form of the Finsler space.

4. Consider the Finslerian mechanical system \( \Sigma_F = (M, F(\alpha, \beta), e\beta \mathcal{C}) \), where \( F(\alpha, \beta) = \alpha + \beta \) is the fundamental function of a Randers space, [15]. Here \( \alpha(x, y) = \sqrt{a_{ij}(x)} y^i y^j \) is the fundamental function of a Riemannian space and \( \beta(x, y) = b_i(x) y^i \). The covariant force field has the components \( \sigma_i = \beta(x, y) y_i \), where \( y_i = g_{ij}(x, y) y^j \) and \( g_{ij}(x, y) \) is the metric tensor of the Randers space. Since the external force field \( e\beta \mathcal{C} \) is second order homogeneous the evolution curves of the mechanical system \( \Sigma_F \) coincide with the horizontal curves of the evolution nonlinear connection. In this case we have

\[
\frac{\partial \sigma_i}{\partial y^i}(x, y) = e y_i b_j(x) + e \beta(x, y) g_{ij}(x, y),
\]

which is neither symmetric or skew symmetric.

5. Consider the nonconservative mechanical system \( \Sigma_L = (M, L, \sigma) \), where \( \sigma_i(x, y) = -\gamma_{ij}(x) y^j \) and \( \gamma_{ij}(x) \) is a skew symmetric tensor. This example has been used by Djukić and Vujanović [10] to find first integrals for the evolution curves that cannot be obtained using standard Noether theory. For this system the evolution nonlinear connection is metric since \( \partial \sigma_i / \partial y^j = \gamma_{ij} \) is skew-symmetric.

An alternative approach of nonconservative mechanical systems can be obtained by means of Legendre transformations by using the theory of Cartan and Hamilton spaces. In this new framework, the Hamilton equations are used instead of Euler-Lagrange equations and the Hamiltonian vector field is used instead of the canonical semispray.

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Current address: Faculty of Mathematics, “Al. Cuza” University, Iași, 700506, Romania
E-mail address: bucataru@uaic.ro, radu.miron@uaic.ro