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EVALUATION AND MANAGEMENT OF CHEMICAL RISK IN UNIVERSITY RESEARCH LABORATORIES USING CALCULATION ALGORITHMS

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ABSTRACT

The assessment of risk concerning the exposure to hazardous chemicals in research laboratories, is a strong element of criticality in the more general process of evaluation of occupational hazards.
The research laboratories are working realities in which a large number of chemicals, hazardous to health and safety of workers, generally in small amount and with an irregular temporal patterning of exposures, is used. This makes the performing of reliable environmental monitoring particularly difficult and sometimes impossible.
A predictive risk evaluation by the application of a mathematical calculation model plays, in such situations, considerable importance for assessing the effectiveness of preventive and protective measures adopted and to direct any necessary remediation.
In the present work three university research laboratories have been monitored and a risk assessment, by the application of the model Archi.me.de., has been performed.
This allowed to discriminate risk situations from those where the risk was under control and to test the effectiveness of taken measures. A few critical points related to the use of the chosen model have been highlighted, which may be subject to improvement to be better applied to the studied context.

KEYWORDS

chemical risk, risk assessment, research laboratory, model

1. INTRODUCTION

The risk assessment chemicals related is one of the most complex situations of workplace risks both for the variety of potentially hazardous chemicals simultaneously present in the workplace, and the multiplicity of possible actions for each hazardous substance.

In particular, in the activities of research laboratories, the situation is further complicated by the fact that workers come into contact with a large number of hazardous chemical agents, in small quantities, for short times, with irregular profiles over time of exposure.
Health effects of hazardous substances used are not always known, as not all have been classified according to the criteria expressed by REACH [1] and CLP [2]; some of them may be formed as secondary products in various chemical reactions and should still be evaluated.

The routine laboratory work is complemented by the research projects that, in a short period of time may be very different, with frequent changes of substances and preparations used and their conditions of employment.

To this must also be added the high turnover of staff (undergraduates, graduate students and research fellows), which makes difficult to reconstruct the career and the exposures to individual chemicals [3, 4].

Even in these peculiar cases, an obligation on the employer to make the chemical risk assessment, in accordance with Italian regulations on protection of health and safety in the workplace still remains. Legislative Decree 81/08 and subsequent modifications [5]) brings together and harmonizes the laws previously in force, e.g. Directive 98/24/EC [6], and sets among the general safety measures in the workplace, the assessment of any risks to health and safety, the planning of prevention and the elimination of risks and, where this is not possible, minimize them in relation to the acquired knowledge based on technical progress.

To cope with such a complex situation, as a simplified operational tools have been developed. Different valuation models which, while coming to a synthetic quantification of risk, are still unable to return to the assessor the details of the critical factors on which to address any significant corrective actions.

At international level, several algorithms have been proposed by institutions, government bodies, and private organizations [7-14]; the models BAuA-Tool, Stoffenmanager, EASE, ECETOC Target Risk Assessment are. examples In particular, the EASE model (Estimation and Assessment of Substances Exposure) [15, 16] has been developed by the UK Health & Safety Executive (HSE) specifically for the chemical workers and incorporated in EUSES (European Union System for the Evaluation of Substances), a larger computer program, adopted by the European Commission, for the quantitative calculation of overall risk, both human and environmental, of chemicals, in accordance with the provisions of the Technical Guidance Document (TDG) in Europe.

The models vary by different points of view, but so far there is no algorithmic model specifically designed for research laboratories.

This issue is part of the objective of this work, i.e. to verify the applicability of the model Archi.me.de to chemical risk assessment in some university research laboratories, responding to the principles of the Law on safety at work.
2. THE ASSESSMENT MODEL A.R.CH.I.M.E.D.E.

The model used, A.r.chi.me.d.e. vers. 3.0, arises from the EASE program and is based on the simple relationship for which the risk (R) linearly depends on the hazard (P) and exposure (E) according to the formula:

\[ R = P \times E \]  

(1)

where the risk depends on the intrinsic characteristics of the chemical agent, or on physico-chemical properties and toxicological properties, while exposure on the way in which the worker comes in contact with this hazard.

This is a model of conservative type which therefore tends to overestimate the exposure and separately evaluates the risk to health and safety.

3. ASSESSMENT OF HEALTH RISK

The P factor, expressed by the properties of danger to health and safety as shown on the classification of pure substances and preparations according to the criteria defined by European Directives 67/548/EEC and 1999/45/EC [17-18] and subsequent amendments and updates (REACH and CLP Regulations), has been deducted from the score given to the R phrases.

For each of the phrases, single or combined, a numeric value between 1 and 10 (Table 1) has been assigned. The proposed method uses for any chemical agent the highest value obtained from the labelling danger indices, the same criterion has been adopted by most of the Italians and Europeans algorithms.

The exposure may be of a type inhalation, skin or by ingestion and also for more than one route. For each route of exposure is possible to calculate individual risk according to the formulas:

\[ R_{\text{inhalation}} = P \times E_{\text{inhal}} \]  

(2)

\[ R_{\text{skin}} = P \times E_{\text{skin}} \]  

(3)

\[ R_{\text{ingestion}} = P \times E_{\text{ingestion}} \]  

(4)

When a chemical agent determines an exposure by multiple routes, the total risk (R) takes into account all the contributions by the formula:

\[ R = \sqrt{R_{\text{inhalation}}^2 + R_{\text{skin}}^2 + R_{\text{ingestion}}^2} \]  

(5)

Whereas the contribution due to ingestion in normal conditions of hygiene is negligible, the formula (5) can be simplified as follows:
The values that the coefficients can assume are

\[ 0.1 \leq R_{\text{inhalation}} \leq 100 \]

\[ 1 \leq R_{\text{skin}} \leq 100 \]

\[ 1 \leq R \leq 141 \]

It is necessary to clarify that this evaluation cannot be applied to the mutagenic and carcinogenic substances, for which it is never possible to assign a risk level “irrelevant” to health.

Product labelling can be considered a tool for evaluating the intrinsic hazard of a product. However it often happens to find substances with uncertain classification or that were formed during the production process and are not accompanied by an MSDS. In those cases will be necessary to apply their own classification, using data from the scientific literature and the classification criteria required by law.

<table>
<thead>
<tr>
<th>R PHRASES</th>
<th>RISK DESCRIPTION</th>
<th>SCORE</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>Harmful by inhalation</td>
<td>4.00</td>
</tr>
<tr>
<td>20/21</td>
<td>Harmful by inhalation and skin contact</td>
<td>4.35</td>
</tr>
<tr>
<td>20/21/22</td>
<td>Harmful by inhalation, skin contact and if swallowed</td>
<td>4.50</td>
</tr>
<tr>
<td>23</td>
<td>Toxic by inhalation</td>
<td>7.00</td>
</tr>
<tr>
<td>23/24</td>
<td>Toxic by inhalation and skin contact</td>
<td>7.75</td>
</tr>
<tr>
<td>23/24/25</td>
<td>Toxic by inhalation, skin contact and if swallowed</td>
<td>8.00</td>
</tr>
<tr>
<td>36</td>
<td>Irritating to eyes</td>
<td>2.50</td>
</tr>
<tr>
<td>37</td>
<td>Irritating to respiratory system</td>
<td>3.00</td>
</tr>
<tr>
<td>38</td>
<td>Irritating to the skin</td>
<td>2.25</td>
</tr>
</tbody>
</table>

The index of inhalation exposure- \( E_{\text{inhal}} \) – has been calculated as the product of the intensity of exposure (I) for the distance (d) according to the formula:

\[ E_{\text{inhal}} = I \times d \] (7)
The intensity of the exposure, in turn, depends on:

(i) the physical-chemical properties;

(ii) the amount of daily use (<0.1 kg from 0.1 kg and 1 kg, between 1 kg and 10 kg from 10 kg to 100 kg, > 100 kg);

(iii) the usage conditions, leading to a more or less high dispersion of the substance in the air more or less high:

a. closed system: the substance has been used and / or stored in airtight containers or reactors and transferred from one container to another through pipes watertight;

b. inclusion in the matrix: the substance has been incorporated into materials or products from which it is prevented or limited the dispersion in the environment (for example the use of materials in pellet, dispersion of solids in water);

c. controlled, non-dispersive use: it takes into account the processes in which selected groups of workers operate, expert in the process and where there are adequate control systems to control, reduce and minimize exposure;

d. use with significant dispersion: work and activities that can lead to uncontrolled exposure of employees, other workers and possibly the general population;

(iv) the type of control, taking into account the measures of prevention and protection to be provided and put in place, to prevent worker exposure to the substance, (complete containment, ventilation / local exhaust ventilation, segregation, dilution / ventilation, direct manipulation);

(v) the time of exposure (<15 min; between 15 min and 2 h, between 2 h and 4 h; between 4 h and 6 h; > 6 h).

Among the chemical-physical properties four levels in ascending order have been considered, according to the capacity of the substance to disperse in the air as a powder or steam:

a. Solid state / mists (large particle size range):

- Low availability: pellet and solid non-friable, with low dust evidence observed during use;

- Media availability: granular or crystalline solid with visible dust that quickly settling;

b. Particulate matter:

- High level of availability, fine and light dust; during use a cloud of dust that remains airborne for several minutes can form.

c. Liquids of low volatility (low vapour pressure).

d. Liquid medium and high volatility (high vapour pressure) or fine powders, gaseous state.
The 5 variables identified allow the determination of the parameter I through a matrix system with a score ranging from 1, if the intensity of the exposure is low, to 10 if the intensity is high.

The index d takes into account the distance between a source of emission and the exposed worker and takes the value 1 for a distance of 1 meter, up to 0.1 for distances longer than 10 meters (Table 2). This index allows to evaluate the exposures for workers who, though not directly in contact with the substance, remain in the same working environment and can be potentially exposed.

**Table 2. Values of the intensity indicator**

<table>
<thead>
<tr>
<th>distance (m)</th>
<th>d Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 1</td>
<td>1</td>
</tr>
<tr>
<td>Between 1 e 3</td>
<td>0.75</td>
</tr>
<tr>
<td>Between 3 e 5</td>
<td>0.50</td>
</tr>
<tr>
<td>Between 5 e 10</td>
<td>0.25</td>
</tr>
<tr>
<td>≥ 10</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Figures 1 shows the evaluation diagram for the determination of the exposure via inhalation.

The index of dermal exposure (Eskin) has been determined through a simple matrix that takes into account both the type of use (see point iii) and skin contact.
4 possible degrees of skin contact have been identified (in ascending order):

a. No contact.

b. Accidental contact: no more than one event per day. Due to occasional spillage or releases.

c. Discontinuous contact: from two to ten events per day because of the production process.

d. Extended contact: the number of events per day is higher than ten.

After granting the assumptions corresponding to the two above mentioned variables and by the help of the matrix for assessing skin, it is possible to assign the value of Eskin.

Calculated the exposure indices Einhal and Eskin and knowing the factors P of the substance, the model calculates R according to formula (1), considering the combined effects on the health and safety of workers due to exposure to multiple hazardous chemicals. The model A.r.chi.me.d.e. makes possible to highlight the cumulative effects on health through the recognition of the action of different substances on the same target organ. In this way, even small exposures of multiple substances may lead to a judgment of not inconsiderable risk to health, if all act in an unfavourable way on the same target organ.

On the bases of R value it is possible to assess the health risk from exposure to hazardous chemicals according to the diagram shown in figure 2:

<table>
<thead>
<tr>
<th>CRITERIA FOR THE ASSESSMENT OF HEALTH RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td>Irrelevant risk</td>
</tr>
<tr>
<td>for health</td>
</tr>
<tr>
<td>0,1 ≤ R &lt; 15</td>
</tr>
<tr>
<td>15 ≤ R &lt; 21</td>
</tr>
<tr>
<td>Not irrelevant risk</td>
</tr>
<tr>
<td>for health</td>
</tr>
<tr>
<td>21 ≤ R &lt; 40</td>
</tr>
<tr>
<td>40 ≤ R &lt; 80</td>
</tr>
<tr>
<td>R &gt; 80</td>
</tr>
</tbody>
</table>

If the substance is "sensitizing" or classified R42 and R43, the outcome of the assessment will lead to a not irrelevant level of risk for health and it will be necessary, if possible, replace the product.

Assessment of security risk

The model qualitatively assess the security risk due to the use of hazardous chemicals. In
fact, if you meet the following requirements, the level of security risk in the workplace will be automatically low:

(i) in the workplace it is excluded the presence of:

1. hazardous concentrations of flammable substances;
2. chemically unstable substances;
3. other combustible oxidizing or similar materials;
4. free flames, ignition sources or similar;
5. easily volatile (boiling point below 65 ° C) and flammable substances.

(ii) the workplace is classified as having a low fire risk, according to current regulations.

In all cases where the security risk will be "not low", the proposed actions will be a more thorough assessment of the risk or the replacement of the product.

4. ASSESSMENT OF CUMULATIVE EFFECTS

The model identifies substances that act on the same target organ, groups them and calculate the health risk from cumulative effects, using in the algorithm:

- The Worsening group parameters;
- Amount as the sum of all the quantities of the substances;
- As a using way the worst condition;
- As distance the smallest from the source;
- Exposure time as the longest one;
- As danger index the highest.

Although the combined effects are not always additive, the most protective scenario for workers should be considered, according to the principle of legislation for safety at work, and considering also that toxicological data do not always exist for the various combinations of substances, the adopted method can then be used for security purposes.

5. METHODOLOGY

Three research laboratories of the University of Rome La Sapienza working in different disciplinary areas of science have been monitored: they are two analytical chemistry laboratories (laboratory ICP-MS and LC-MS lab) and a laboratory of organic synthesis. As part of analytical chemistry, the two laboratories differ in the type of analytical investigations undertaken and in the equipment used.

In the laboratory called ICP-MS methods for the analysis of inorganic substances in water
and particulate material in biological matrices, by means of mass spectrometry inductively coupled plasma to (ICP-MS) have been developed and applied.

In the laboratory, referred to as LC-MS, methods for the characterization and analysis of pollutants (pesticides, pharmaceuticals and emerging contaminants) in food, environmental and biological matrices by liquid chromatography coupled to mass spectrometry (LC-MS) have been developed and applied. In this laboratory there is also a section deputed to the analysis of drugs of natural origin and synthetic.

In the laboratory of organic synthesis organic molecules biologically active, through a series of reaction step have been synthesized; at the end of each step, the obtained results and the reaction yields, for both synthetic intermediates and final products have been verified by chromatographic analysis (GC-MS or LC-MS) spectrometry or infrared Fourier transform (FTIR)

The process by which the risk assessment has been carried out is divided into six phases:

1. Preliminary fact-finding investigation;
2. verification and data collection in the laboratory using a checklist provided to individual workers;
3. inspections at the workplace and observation of processing stages;
4. Acquisition of safety data sheets for substances and preparations;
5. data processing;
6. calculation and definition of risk.

The preliminary fact-finding investigation, conducted through meetings with researchers, it was necessary for any clarification on the check list and for proper identification of homogenous risk groups among workers. After analyzing all the activities carried out in the laboratories, the hazardous properties of registered chemicals have been verified.

To calculate the inhalation and dermal index of exposure through the model Archi.me.de 3.0, information on the daily amounts used for each chemical, the operation modes, exposure time and the distance from the source have been derived from checklists completed by laboratory workers.

For the chemical agents a "controlled, non-dispersive use" has been generally considered, as the staff working in the laboratories normally has a highly specialized and professional training.

Totally around 120 hazardous chemicals have been estimated, including 10 reaction intermediates, synthesized in the laboratory of organic synthesis, to which, in the absence of an official classification, it has been decided to assign risk phrases according to the molecular
structure and of physical and chemical known properties.

In conducting the assessments, it has been necessary to adapt to a certain rigidity of the algorithm, so the use of chemical agents has been rated as a daily, while in the practice of laboratory activities, frequency of use were often minor, and not exactly quantifiable, especially if referred to the individual operator. Moreover, the quantities actually in use were generally much lower than the minimum defined by the algorithm of 0.1 Kg.

### 6. RESULTS

Table 3, 4 (a-d), and 5 show the health and safety risk level and the combinatorial effects on health, calculated with the model, for the three laboratories, taking into account, among all those analyzed, only the chemical showing critical issues. In this first phase (risk assessment) the specific prevention and protection measures adopted in the laboratories have not been taken into account.

In the laboratory LC-MS, the work process has been divided into two phases which are performed in two separate rooms: the sample preparation (prep) and the subsequent investigation, conducted by analytical instruments, in the room (instrumental). These phases are in turn distinct for the analysis section of the drugs (preparative drugs and instrumental drugs).

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>HEALTH RISK</th>
<th>SAFETY RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{INHA}$</td>
<td>$R_{SKIN}$</td>
</tr>
<tr>
<td>Hydrochloric acid, solution 30%</td>
<td>36,38</td>
<td>14,55</td>
</tr>
<tr>
<td>Fluoridric acid, solution 40%</td>
<td>49,88</td>
<td>28,50</td>
</tr>
<tr>
<td>Nitric acid 65%</td>
<td>43,88</td>
<td>17,55</td>
</tr>
<tr>
<td>Cesium ICP standard solution</td>
<td>18,00</td>
<td>24,00</td>
</tr>
<tr>
<td>Mercury ICP standard solution</td>
<td>15,75</td>
<td>21,00</td>
</tr>
<tr>
<td>Molybdenum ICP standard solution</td>
<td>15,70</td>
<td>21,00</td>
</tr>
<tr>
<td>Gold ICP standard solution</td>
<td>18,00</td>
<td>24,00</td>
</tr>
<tr>
<td>Palladium ICP standard solution</td>
<td>18,00</td>
<td>24,00</td>
</tr>
<tr>
<td>CHEMICAL</td>
<td>HEALTH RISK</td>
<td>SAFETY RISK</td>
</tr>
<tr>
<td>--------------------------------</td>
<td>-------------</td>
<td>-------------</td>
</tr>
<tr>
<td></td>
<td>$R_{INHA}$</td>
<td>$R_{SKIN}$</td>
</tr>
<tr>
<td>Hydrogen peroxide 30%</td>
<td>13.16</td>
<td>5.85</td>
</tr>
<tr>
<td>Lead ICP standard solution</td>
<td>22.50</td>
<td>30.00</td>
</tr>
<tr>
<td>Potassium ICP standard solution</td>
<td>7.43</td>
<td>23.10</td>
</tr>
<tr>
<td>Selenium ICP standard solution</td>
<td>16.31</td>
<td>21.75</td>
</tr>
<tr>
<td>Silicon ICP standard solution</td>
<td>18.00</td>
<td>24.00</td>
</tr>
<tr>
<td>Thallium ICP standard solution</td>
<td>19.69</td>
<td>26.25</td>
</tr>
<tr>
<td>Tellurium ICP standard solution</td>
<td>18.00</td>
<td>24.00</td>
</tr>
<tr>
<td>Vanadium ICP standard solution</td>
<td>19.13</td>
<td>25.50</td>
</tr>
</tbody>
</table>

(*) The security risk is "not low" because the following risk phrase that characterizes the substance: R 5 - danger of explosion if heated

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>HEALTH RISK</th>
<th>SAFETY RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{INHA}$</td>
<td>$R_{SKIN}$</td>
</tr>
<tr>
<td>Acetone</td>
<td>24.50</td>
<td>10.50</td>
</tr>
<tr>
<td>Acetonitrile</td>
<td>13.50</td>
<td>13.50</td>
</tr>
<tr>
<td>Butyl hydroxy toluene</td>
<td>7.00</td>
<td>21.00</td>
</tr>
<tr>
<td>Dichloromethane</td>
<td>21.00</td>
<td>21.00</td>
</tr>
<tr>
<td>Methanol</td>
<td>8.25</td>
<td>8.25</td>
</tr>
<tr>
<td>n-exane</td>
<td>20.70</td>
<td>20.70</td>
</tr>
<tr>
<td>Propan-2-ol</td>
<td>10.50</td>
<td>10.50</td>
</tr>
<tr>
<td>Trichloromethane</td>
<td>7.00</td>
<td>21.00</td>
</tr>
</tbody>
</table>

(*) The security risk is "not low" because the following risk phrase that characterizes the substances: R 11 – easily flammable
Table 4b. LC-MS Laboratory "instrumental" - risk evaluation

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>HEALTH RISK</th>
<th>SAFETY RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{INHA}$</td>
<td>$R_{SKIN}$</td>
</tr>
<tr>
<td>Acetonitrile</td>
<td>23.63</td>
<td>13.50</td>
</tr>
<tr>
<td>Methanol</td>
<td>43.31</td>
<td>24.75</td>
</tr>
<tr>
<td>n-exane</td>
<td>36.23</td>
<td>20.70</td>
</tr>
</tbody>
</table>

Table 4c. LC-MS Laboratory "drugs preparation" - risk evaluation

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>HEALTH RISK</th>
<th>SAFETY RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{INHA}$</td>
<td>$R_{SKIN}$</td>
</tr>
<tr>
<td>6 Monoacetilmorphine</td>
<td>9.50</td>
<td>28.50</td>
</tr>
<tr>
<td>Ethyl acetate</td>
<td>3.50</td>
<td>10.50</td>
</tr>
<tr>
<td>Acetone</td>
<td>3.50</td>
<td>10.50</td>
</tr>
<tr>
<td>Acetonitrile</td>
<td>13.50</td>
<td>13.50</td>
</tr>
<tr>
<td>Cocaine</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Deuterated amphetamine</td>
<td>9.50</td>
<td>28.50</td>
</tr>
<tr>
<td>Deuterated cocaine</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Hydrochloric acid, solution 30%</td>
<td>7.00</td>
<td>21.00</td>
</tr>
<tr>
<td>Dichloromethane</td>
<td>7.00</td>
<td>21.00</td>
</tr>
<tr>
<td>Deuterated Metamphetamine</td>
<td>8.00</td>
<td>24.00</td>
</tr>
<tr>
<td>Methanol</td>
<td>24.75</td>
<td>24.75</td>
</tr>
<tr>
<td>n-exane</td>
<td>6.90</td>
<td>20.70</td>
</tr>
<tr>
<td>Propan-2-ol</td>
<td>10.50</td>
<td>10.50</td>
</tr>
<tr>
<td>Tetrahydrocannabinol</td>
<td>6.90</td>
<td>20.70</td>
</tr>
</tbody>
</table>

(*): The security risk is "not low" because the following risk phrase which denotes substances: R 11 - highly flammable.

(**): the risk to health is "not irrelevant" because the following risk phrase which denotes substances: R 43 - may cause sensitization by skin contact.
Table 5. Organic Synthesis Laboratory risk assessment

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>HEALTH RISK</th>
<th>THRESHOLD OF RISK</th>
<th>SAFETY RISK</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R&lt;sub&gt;INL&lt;/sub&gt;</td>
<td>R&lt;sub&gt;CUTE&lt;/sub&gt;</td>
<td>R&lt;sub&gt;TOT&lt;/sub&gt;</td>
</tr>
<tr>
<td>1,4 dioxane</td>
<td>52.50</td>
<td>21.00</td>
<td>56.54</td>
</tr>
<tr>
<td>Ethyl acetate</td>
<td>26.25</td>
<td>10.50</td>
<td>28.27</td>
</tr>
<tr>
<td>Acetonitrile</td>
<td>33.75</td>
<td>13.50</td>
<td>36.35</td>
</tr>
<tr>
<td>Hydrochloric acid, solution 30%</td>
<td>14.55</td>
<td>33.95</td>
<td>36.94</td>
</tr>
<tr>
<td>Anhydrous ammonia</td>
<td>52.50</td>
<td>21.00</td>
<td>56.54</td>
</tr>
<tr>
<td>Dichloromethane</td>
<td>52.50</td>
<td>21.00</td>
<td>56.54</td>
</tr>
<tr>
<td>Dimethylsulfoxide</td>
<td>25.50</td>
<td>10.20</td>
<td>27.46</td>
</tr>
<tr>
<td>Petroleum ether</td>
<td>26.25</td>
<td>10.50</td>
<td>28.27</td>
</tr>
<tr>
<td>Diethyl ether</td>
<td>26.25</td>
<td>10.50</td>
<td>28.27</td>
</tr>
<tr>
<td>Potassium hydroxide</td>
<td>5.85</td>
<td>40.95</td>
<td>41.37</td>
</tr>
<tr>
<td>Sodium hydroxide</td>
<td>17.55</td>
<td>17.55</td>
<td>24.82</td>
</tr>
<tr>
<td>Reaction intermediate 1</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 2</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 3</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 4</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 5</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 6</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 7</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 8</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 9</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Reaction intermediate 10</td>
<td>28.50</td>
<td>28.50</td>
<td>40.31</td>
</tr>
<tr>
<td>Methyl iodide</td>
<td>7.25</td>
<td>21.75</td>
<td>22.93</td>
</tr>
<tr>
<td>Solid lithium</td>
<td>14.55</td>
<td>14.55</td>
<td>20.58</td>
</tr>
<tr>
<td>Methanol</td>
<td>61.88</td>
<td>24.75</td>
<td>66.64</td>
</tr>
<tr>
<td>n-exane</td>
<td>51.75</td>
<td>20.70</td>
<td>55.74</td>
</tr>
<tr>
<td>Pentane</td>
<td>26.25</td>
<td>10.50</td>
<td>28.27</td>
</tr>
<tr>
<td>Propan-2-ol</td>
<td>26.25</td>
<td>10.50</td>
<td>28.27</td>
</tr>
<tr>
<td>Tetrahydrofuran</td>
<td>24.75</td>
<td>9.90</td>
<td>26.60</td>
</tr>
<tr>
<td>Toluene</td>
<td>51.75</td>
<td>20.70</td>
<td>55.74</td>
</tr>
<tr>
<td>Trichloromethane</td>
<td>52.50</td>
<td>21.00</td>
<td>56.54</td>
</tr>
</tbody>
</table>

(*): the safety risk for the following substances results “not low”
- 1,4 dioxane (R 19)
- Petroleum ether (R 12)
- (Diethyl ether R 12 - R 19)
- Solid lithium (R 14/15)
- Pentane (R 12)
- Tetrahydrofuran (R 19)

For the proper management of chemical risk, for the only cases in which the index of potential risk was not irrelevant to the health and not low to security, such measures have been adopted:
- Use of adequate equipment and materials;
- Appropriate organizational and collective protection measures at source;
- Individual protection measures including personal protective equipment;

As measures of collective protection the use of fume hoods, has been provided, besides in each laboratory there were systems for emergency eyewash and fire suppression systems.

Operators, already subject to health surveillance and receiving training in information and formation, have been provided with personal protective equipment such as:
- Latex gloves, protection class 1 (breakthrough time > 10 min);
- Nitrile gloves, protective risk category III;
- Earpiece eye protection to stem.

Taking into account the specific adopted measures of prevention and protection, the new levels of health risk have been determined in order to determine whether, as implemented in the laboratories, it was sufficient to keep the risks to acceptable levels and content (management of residual risk).

With regard to the laboratory ICP-MS, as can be seen from the graph shown in Figures 3, the vapours of strong acids remain in a critical situation, where the contribution of risk inhalation maintains a high level (R > 21), while the skin risk is sufficiently managed by the personal protective equipment.

![Graph showing laboratory ICP-MS effectiveness of measures taken](image)

**Figure 3:** Laboratory ICP-MS - Effectiveness of measures taken
With regard to the laboratory LC-MS, the new levels of health risk, show an acceptable situation, except for the organic solvents used in the instrumental section and the presence of sensitizers (cocaine and cocaine deuterated) in the section drugs. The data relating to solvents are reported in Tables 6a-b.

Table 6a. Laboratory LC-MS “instrumental” – residual risk management

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>R_{INHAL}</th>
<th>R_{skin}</th>
<th>R_{TOT}</th>
<th>RISK MANAGEMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>acetonitrile</td>
<td>23.63</td>
<td>4.50</td>
<td>24.05</td>
<td>insufficient measures</td>
</tr>
<tr>
<td>methanol</td>
<td>43.31</td>
<td>8.25</td>
<td>44.09</td>
<td>insufficient measures</td>
</tr>
<tr>
<td>n- hexane</td>
<td>36.23</td>
<td>6.90</td>
<td>36.88</td>
<td>insufficient measures</td>
</tr>
</tbody>
</table>

Table 6b: Laboratory LC-MS “instrumental drugs” – residual risk management

<table>
<thead>
<tr>
<th>CHEMICAL</th>
<th>R_{INHAL}</th>
<th>R_{skin}</th>
<th>R_{TOT}</th>
<th>RISK MANAGEMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>acetonitrile</td>
<td>23.63</td>
<td>4.50</td>
<td>24.05</td>
<td>insufficient measures</td>
</tr>
<tr>
<td>methanol</td>
<td>43.31</td>
<td>8.25</td>
<td>44.09</td>
<td>insufficient measures</td>
</tr>
</tbody>
</table>

Even for the organic synthesis laboratory model shows that the most critical are found for organic solvents, for which (see figure 4) there remains a risk to health due to the high contribution inhalation.

![Organic synthesis laboratory](image)

Figure 4: Organic synthesis laboratory – Effectiveness of taken measures
As can be seen from the results the greatest risk in the three laboratories is due to the inhalation contribution, so it is necessary to strengthen the specific measures of prevention and protection, with corrective actions such as adopting collection systems at source by local exhaust ventilation, and, when possible, the change of the analytical methods for example by decreasing the quantity of solvents used and the exposure times. Beyond this should be assessed the feasibility of replacing the most hazardous substances with others of equal efficiency but less problematic.

7. ASSESSMENT OF CUMULATIVE EFFECTS

This assessment is carried out automatically because the model uses all the data already entered into the system to make individual assessments.

The calculation has been made with respect to the cumulative effects on target organs and the information was derived from INAIL (Italian Workers’ Compensation Authority) the table for occupational diseases, with the exception of ototoxic effects derived from literature.

The cumulative risk on target organs has been calculated in both the absence of specific measures (evaluation) and in the presence of these (management) in order to determine the residual risk and the effectiveness of preventive and protective measures adopted.

The results obtained show that some organs are particularly fatigued by combinatorial effects, even considering the measures put in place.

Specifically:

1. in the ICP-MS laboratory is the use of strong acids to create more problems, particularly hydrofluoric acid, for which we observe a significant risk concerning the following organs: respiratory tract, skin, eye and ocular annexes, the lymphatic system disorders, musculoskeletal and connective tissue.

2. in the LC-MS and organic synthesis laboratories, the contexts, in which the risk is not negligible, mainly involves organic solvents, therefore the target organs of the cumulative effects are: respiratory tract, skin, eye and ocular annexes, ear, central and peripheral nervous system, liver and biliary tract.

The assessment of cumulative effects integrates the assessment of health risk and is useful to the responsible physician for determining the health protocol.

8. CONCLUSIONS

The model A.r.chi.me.d.e. is simple, easy to apply and require a limited number of inputs. However, we highlight some critical points that may be object of appropriate assessment to adapt the algorithm to the examined context. The first consideration is on the unit measure, Kg, the amount used by the algorithm, which is too high when put in relation to the amounts
used in research laboratories, where the weight used order is of grams or less.

The second consideration concern the intervals of exposure which are too wide and do not allow to discriminate between phases having, for example, an exposure time of 16 minutes by phases of 2 hours. On the basis of these considerations it can be seen as the score related to the phrase R is preponderant in the calculation of P, without any attenuation for small amounts and short times of exposure. The third critical point is that the frequencies of use are not adequately considered and in this sector would be suitable to identify more precise mechanisms linking the exposure times to the frequencies with which the work phases are repeated throughout the day.

In conclusion it can be said that in chemical risk assessment for the research, the use of the model Archi.me.de is a valid help to identify those situations where the risk is definitely "irrelevant", and this is particularly useful when environmental monitoring is not possible, but it should be improved, where critical issues were identified, for a better application it at the university level.

REFERENCES


MATHEMATICAL MODELS OF DECISION-MAKING IN COUNSELING

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ABSTRACT

Some methodologies for practicing counseling are deepen in. The aim is in helping a person to assume his/her autonomous decisions and then rational actions, coherent with own identity and objectives. Moreover a mathematical formalization of the counseling procedures is considered as a dynamical decision making problem, where the awareness of alternatives and objectives and their evaluations are maieutically induced by the Counselor. Finally it is shown that fuzzy reasoning can give a useful help to the task of the Counselor, because of its flexibility and closeness to human reasoning.

KEYWORDS


1. INTRODUCTION

In Sociology, there are many well-established methodologies for practicing counseling [2], [5], [17], [18], [19], [20], [21], [25], [26]; on the other side in Mathematics and in Operational Research there are many patterns of Decision Making [6], [7], [9], [11], [13], [16], [22] in condition of certainty, randomness uncertainty [3], [4], [6], and semantic uncertainty [10], [12], [29], [30]. The aim of our paper is to establish links among these theories and to look for what possible help to practicing counseling can arise by suitable formalizations of problems in terms of Decision Making theory.

The idea is that a cooperation among experts of different fields can give new points of view to face problems of persons having unease. In particular the mathematical procedures utilized to solve Decision Making problems can be very useful for the Counselor task of showing to the Client the degree of coherence between his/her objectives and his/her actions and behaviors.

An important role is played by the de Finetti subjective probability [3], [4], [6]. Precisely, the coherence between objectives and actions is obtained with a thought-out assessment of probabilities to nature states and a rational aggregation of the issues associated to every alternative. Moreover this prevents a too pessimistic or a too optimistic behavior, that are consequences of not realistic evaluations of the possibility of occurrence of events.

Fuzzy reasoning and fuzzy algebra [1], [8], [14], [15], [24], [27], [28], [29], [30] can also give a good contribution in the counseling procedure. Fuzzy reasoning permits a quantification and a treatment of the semantic uncertainty and can help the Client to avoid extreme positions, and to consider and mediate negative and positive aspects of every situation. Fuzzy algebra [8],
permits to consider the uncertainty in all the counseling processes, and manage gradual modifications of the opinions of the Client induced by the maieutical activity of the Counselor.

2. SOME MAIN FEATURES OF COUNSELING

Any counseling intervention pursues a general aim of development of competences and resources, needed to confront and solve actual problems that a person may meet in the his/her own path of life [2], [17], [18], [19], [20], [21], [25], [26]. The intervention concerns the support for the person to develop, in complete autonomy, a knowledge of self, his/her own objectives, tools, contexts, actions, strategies to be able to learn, confront and change the problems of his/her own personal and social development, problems that are perceived like criticalities to solve.

The Counselor, in his own practices, exercises an aid process that, in a maieutic way, eases, in the person, to emerge competences and the development of resources in the criticalities of the own path of life. The aim consists in helping the person in his/her choices to assume autonomous decisions and useful and rational behaviors in order to act with effectiveness and satisfaction in own contexts perceived as problematic and difficult, with the ultimate aim to reach self-realization and the wellness of the development.

This aim is realized with different methodologies and practices. In this paper we illustrate the main features of the humanistic approach elaborated by Carl Rogers [20], [21] and Robert Carkhuff [2]. This model presents the following main features, that are fundamental in practicing counseling.

The practice of counseling by Carl Rogers, called “non directive counseling”, is centered on the person of the Client, in order that an autonomous and positive base force of the person emerges and operates.

Following Rogers, the Counselor accepts in the client the humanistic and existential dignity of a capable person, that can address him/her-self. The Counselor non directive role assists the person to implementing attitudes of reliance, authenticity, unconditional acceptation in the interpersonal communicative climate between the Counselor and the interlocutor.

The helping relation has its main aim at inducing decision making autonomy, sense of dignity and self-esteem of the person, who can experiment with a suitable climate of self-determination, assumption of responsibilities, promotion.

Non directive method is not, however, the counterpart of a professional neutrality of the Counselor, but an experience of full acceptation in the direction of the Client, to whom the Counselor practices an interesting and tolerant communication. This disposition favors the reliance of the person to change and develop him/her self toward a life more full and satisfactory, inducing in the Client the capability of self-management to recognize and get over the unease.
The empathetic technique of the Counselor consists in the attempt to understand and perceive how the client perceives and understands, to come into his/her perceptive field of meanings, where the whole situational experience finds fulfillments to realize the problem how it is lived by the Client, with just his/her words and subjective universe.

The non directive role of the Counselor presents an empathetic and non judging communication, without prejudices, that does not proceed by analysis and clinical classification of the problems, but respecting the complete initiative of the Client, toward the representation of his/her problem in the itinerary of the interview. The basis is a continuous encouragement from the Counselor to a spontaneous expression of the Client autonomy, with the goal to help him to search for his/her true self. Getting over the unease, the true transformation of the situation and personality toward him/her-self, the environment and the others, is just a client's concern: the Counselor can only help him to recover the freedom to be him/her-self.

The task of the Counselor is to take care that the interviewee recovers his/her own integrity by means of the consistent perception of his/her self. The person has a conceptual representation organized by the self, like a fluid, but coherent, dynamic system of attributes and relations that the ego assigns to the me in relation to the others. This system is self-organizing in front of the natural and social context [23]. When lived experiences in the context are harmonic and congruent with the concept of self, the person reach his/her own integrity and wellness, otherwise the person is fragmented in the unease.

The existential humanistic approach of the Counselor is developed by his dispositional attitudes, that consist in following the threads of Client's discourse, in a helping relational climate of warm reception, comprehension, listening, without pre-determined schemes.

However, an operational methodology is outlined, mainly among Rogers followers, e. g. Robert Carkhuff [2], with verbal reformulation technique of the Counselor, that aims to deepen implicit meanings of the interviewee language and behavior, that give consistency and visibility to the inner attitudes of the person.

Rogers' non directive and dispositional approach and the indirectly regulated and operational one by Carkhuff are integrated on a level of greater complexity. The non directive aid of the Counselor intends to make the Client's lived experience emerges autonomously, to help him to find integrity, wellness, and coherence of his/her self. The indirect guide of the Counselor does not neglect to reorganize the perceptions, the attitudes, the capabilities of the person under unease, to help him/her to explore his/her own lived experience of behaviors in front of the context, discover the contradictions, and find again the consistency between objectives and effective actions.

In this way of operating, Carkhuff and his continuators elaborate a helping model, addressed to the interpersonal processes management skills of the helper towards the helpee. The operational methods of the Counselor are the verbal re-formulation of Client's words to deepen the meaning of them, and the capability to pay attention, answer, personalize, involve, explore, understand, compare, communicate and analyze impressions and evaluations… The
aim of such sequential phases of the help process is to lead the person to awake to self, to have knowledge of his/her problem, to make autonomous decisions, to reorganize perceptions, behaviors and relations towards environment and persons. A progressive modification model is sketched with the aim to solve the problems of wellness in a self-regulating way.

3. A MATHEMATICAL MODELING FOR COUNSELING

Some mathematical models for Counseling can be built starting from the patterns of Decision Making and Game theories.

The first task of the Counselor is to obtain that, in a given instant $t$ of the counseling procedure, the Client is aware of a set $A_t$ of own alternatives or strategies and a set $O_t$ of own objectives and is persuaded to look for the strategies that are coherent with own objectives and to follow one of these strategies, dropping out the incoherent alternatives.

The sets $A_t$ and $O_t$ can change in the counseling process. The Counselor can obtain, maieutically, that the Client be conscious of other alternatives and objectives, realizes that some alternatives before considered are not feasible, or some objectives that seemed important in a moment of emotion, actually are not important for his/her, and so can be deleted from the list of the objectives.

A second task of the Counselor is to facilitate the Client in founding the relations between his/her own objectives and each of his/her possible behaviors, in a coherent and realistic vision of his/her wishes, tools, constraints, and possibilities; then the Client is assisted in elaborating his/her evaluation of the situation with consequent framework of expectations, perspectives of the possible alternatives, i.e. to understand how actions are tied with his/her wishes.

The Counselor, using Ars Maieutica, and establishing together with the interlocutor a relation that allows to attune his/her techniques to the emotional framework and the cognitive objectives of the Client, has to get that the client himself builds his/her answers and decides the own proper social actions.

Of course, the formalization should be understood “as needed” and anyway it should be a light, an ideal target, a far reference point, that gives the Client the logic framework of his/her thoughts, in order that he/she avoids dispersions, incoherencies between wishes and effects of his/her actions. A complete clarification of the objectives or a complete knowledge of the alternatives is neither possible, nor desirable; indeed, an excess of detail increases the complexity and loses sight of main logical thread.

The help of decision theory, in the construction of the frame of the expectations consequent to each action, is important. The person to be oriented must be aware of what he/she gets, or the frame of the possible outcomes related with any alternative and each his/her objective. Decision theory helps to collect these data in order that the subject be oriented toward the actions that are more coherent with his/her wishes.

Decisions may develop in certainty conditions or randomly. In other words an action may give
rise to just one or several possible consequences. The Counselor must help the Client to find out the possible consequences of his/her actions and set choice criteria neither too optimist, nor too pessimist. To this aim, a crucial help is given by subjective probability, based on the coherence of the opinions. In fact, subjective probability allows to assess a coherent probability distribution to the outcomes of an action.

Let \(\{a_1, a_2, \ldots, a_m\}\) be the set of alternatives, \(\{s_1, s_2, \ldots, s_r\}\) the set of nature states, i.e. a set of events pairwise disjoint and such that their union is the certain event. In the classical Decision Making problem formalization the existence of the utility matrix \(U = (u_{ih})\) is assumed, where \(u_{ih}\) is a real number that represents the utility for the Decision Maker if he choose the alternative \(a_i\) and the event \(s_h\) happens.

The pessimistic point of view leads to assume as score of \(a_i\) the minimum, respect to \(h\), of the numbers \(m_{ih}\), the optimistic one considers the maximum of such numbers. In [4], [6] it is proved that a coherent point of view leads to obtain the score of every \(a_i\) in two steps. Firstly, a subjective probability assessment \(\{p_1, p_2, \ldots, p_r\}\) to the set of events \(\{s_1, s_2, \ldots, s_r\}\) is given. After, the score of \(a_i\) is assumed to be equal to the prevision of \(a_i\) given by the formula

\[
P(a_i) = p_1 u_{i1} + p_2 u_{i2} + \ldots + p_r u_{ir}.
\]

(1)

In [10], [12] fuzzy extensions of formula (1) are considered, starting by two different points of view.

Of course, in practicing counseling, the Counselor must obtain probabilities \(p_j\) and utilities \(m_{ij}\), in a maieutical way, by the Client. The coherent synthesis of the opinions of the Client is the prevision (1).

The rational behavior is considering as scores of objectives a coherent synthesis of elements of information or opinions. Lindley [6] claims that, if the utility matrix \(U = (u_{ih})\) is given, only the prevision is a coherent synthesis. But, in general, it is very difficult to obtain the matrix \(U\). In general, the maximum result of the activity of the Counselor is to lead the Client to give a classification of pairs \((a_i, s_j)\), from the most preferable to the least desirable, i.e. a preorder relation is obtained.

In order to obtain numerical scores a useful procedure is Saaty’s AHP [13], [22]. This process is based on questions that the Counselor proposes to the Client, with the aim to get measures of the pairwise comparisons of the desirability of a set of objects (the pairs \((a_i, s_j)\) in our case).

If there are many objectives and \(O = \{o_1, o_2, \ldots, o_n\}\) is the set of objectives, then for every objective \(o_j\), there is a different utility matrix \(U^j = (u_{ih}^j)\) and then, for every alternative \(a_i\) and objective \(o_j\), the prevision of \(a_i\) with respect to \(o_j\) is the real number

\[
P^j(a_i) = p_1 u_{i1}^j + p_2 u_{i2}^j + \ldots + p_r u_{ir}^j.
\]

(2)
An important task of the Counselor is to help the Client to be aware of own objectives. Of course, every person has a very high number of objectives in her/his life, and from the ars maieutica of the Counselor the most important objectives of the Client must emerge and only they are to be considered in the mathematical Decision Making model.

Moreover, the set \(O = \{o_1, o_2, \ldots, o_n\}\) of the relevant objectives of the model must be classified by the Client. The ideal situation is that the Client finds a rational way to associate to every objective \(o_j\) a positive real number \(w_j\) that measures the importance that the Client attributes to the objective \(o_j\).

Also for the weights of the objectives a suitable procedure is given by the AHP of Saaty [13], [22]. For every pair \((o_{j1}, o_{j2})\) of objectives, the Counselor, with a set of questions, does the Client say what is the one preferred or that the objective are equally preferred. In the first case the Counselor must obtain by the client an integer number belonging to the interval \([2, 9]\) that measures to what extent the most preferred objective is more important for the Client than the last preferred.

From the responses obtained, with the AHP procedure, a vector \(w = (w_1, w_2, \ldots, w_n)\) of weights of objectives is obtained, where \(w_j\) is a positive real number expressing the importance that the Client gives to the objective \(o_j\), and the following normalization condition is satisfied:

\[
w_1 + w_2 + \ldots + w_n = 1.
\]  

Before to implement the mathematical procedure to obtain the vector \(w\) a verification of the coherence of the responses is necessary. In particular the transitivity of the preferences must be verified. On the contrary, the Counselor, with a patient procedure, must propose the questions in a different form in order to avoid incoherence.

When the previsions \(P(a_i)\) and the weights \(w_j\) are obtained, a rational measure of the opportunity of the action \(a_i\) by the Client is given by the following score

\[
s(a_i) = w_1 P^1(a_i) + w_2 P^2(a_i) + \ldots + w_n P^n(a_i).
\]  

The greater is the number \(s(a_i)\), the more agreeable is the action \(a_i\). We emphasize that this conclusion is only a coherent consequence of the opinions of the Client and the assumption of particular mathematical procedures (e.g., the consideration of formula of prevision or formula (4) to aggregate information on weights and previsions).

Obtaining scores \(s(a_i)\) is important mainly as an help to the Decision Making, without a claim to be definitive and not modifiable preference measure of the opportunity of the Client actions.
4. FUZZY REASONING FOR COUNSELING

Fuzzy reasoning and arithmetic may help. They permit a gradual procedure in the changes of points of view, gradual and dynamical attribution of the degrees of importance to the objectives, management of semantic and emotional uncertainty. Furthermore, the consequences of an action cannot be, in general, defined in a sharp way; rather it is opportune that the Client does not renounce to his/her doubts in favor of a choice that is rash, premature and of doubtful effectiveness. From this point of view, fuzzy logic and linguistic variables may be effective, in that they are expressed as imprecise numbers, but plastic and gradually modifiable numbers as far as the opinions of the Client become more clear.

The fuzzy extensions of the decision theory are able to gather the various elements and shows a clear framework, in a language that is close to the human language, of the path that goes from the awareness of the own proper expectations to the coherent action.

In particular, the utility matrix \( U = (u_{ih}) \) considered in the previous Sec. is replaced by a matrix \( U^* = (u_{ih}^*) \) where every \( u_{ih}^* \) is a fuzzy number that expresses a value of a linguistic variable [29], [30]. Moreover, the probability assessment is replaced by an assessment of fuzzy probabilities \( \{p_1^*, p_2^*, \ldots, p_r^*\} \) to the set of events \( \{s_1, s_2, \ldots, s_r\} \), where every \( p_h^* \) is a fuzzy number with support contained in the interval \([0, 1]\).

By considering the Zadeh’s extensions [28], [29], [30] of the usual addition and multiplication to fuzzy numbers, or alternative fuzzy operations [8], for every alternative \( a_i \), we can introduce the fuzzy prevision [10], [12] by means of the formula

\[
P^*(a_i) = p_1^* u_{i1}^* + p_2^* u_{i2}^* + \ldots + p_r^* u_{ir}^*.
\] (5)

If the importance of the objectives is expressed by values of a linguistic variable, then also the weights of objectives are fuzzy numbers \( w_j^* \), \( j = 1, 2, \ldots, n \). Then formula (4) is replaced by the more general

\[
s^*(a_i) = w_1^* P^*(a_i) + w_2^* P^*(a_i) + \ldots + w_n^* P^*(a_i),
\] (6)

where the addition is the extension of the usual addition with the Zadeh extension principle and the multiplication is an approximation of the Zadeh multiplication, built with the aim to preserve the shape of the class of the considered fuzzy numbers [8].

Unlike numbers \( s(a_i) \), in general the fuzzy numbers \( s^*(a_i) \) are not totally ordered. This can appear a drawback by a mathematical point of view, but, on the contrary, it is an advantage in the practice of the counseling, as the fuzzy number \( s^*(a_i) \) contains, in its core and in its support, the history of the uncertainty of the opinions of the Client and then it is a more realistic global representation and it is a measure of the opportunities of his/her choices, more coherent with his/her opinions.
5. CONCLUSIONS

From previous Secs. it seems natural the conclusion that an interaction between the maieutical ability and capability to find strategies of the Counselor and the power of the mathematical models for Decision Making can be very useful to solve unease problems e to show to the Client a clear vision of the consequence of her/his possible actions.

The illusory certainties, obtained by questionable assumptions, must be replaced by controlled uncertainties. The tools to take into account the uncertainties and control their consequences in all the counseling procedures are the probabilistic and fuzzy reasoning. In particular, before any action it is necessary to have some information about the facility of occurrence of nature states and this leads to consider the de Finetti subjective probability. More in general, the uncertainty on the assessments of these probabilities can be controlled by considering fuzzy subjective probabilities expressed by fuzzy numbers.

The utility of an action with respect to an objective is often very doubtful. Such uncertainty can be controlled by measuring the utilities with fuzzy numbers.

In the fuzzy ambit, the aggregation of utilities and subjective probabilities associated to every possible action of the Client is made with the tools of the fuzzy algebra and the fuzzy reasoning. They permit to obtain, as a final score of every action, a fuzzy number that provides not only a measure of the validity of this action, but also contains a résumé of the whole history of doubts and uncertainties on the process of evaluation.

A treatment of the aggregation of the previsions and the weights of objectives more sophisticated than the one considered in the previous Sec. takes into account also the logical relations among the objectives [12], [14], [15], [16]. A generalization of the utilities is obtained by utilizing fuzzy measures decomposable with respect to a t-conorm $\oplus$ [1], [24], [27]. From such a viewpoint a fuzzy prevision can be defined, in which the addition is replaced by the operation $\oplus$. Applications of these theories to Decision Making may be found in [12], [14], [15], [16].

However, as a final result of the application of the mathematical model, the Client obtains a vector of fuzzy (in particular crisp) numbers $s^* = (s^*(a_1), s^*(a_2), \ldots, s^*(a_m))$, where $s^*(a_i)$ measures the advisability of the action $a_i$, taking into account all the opinions and doubts expressed by the Client and the end of the maieutic work of the Counselor.

The vector $s^*$ is an important reference point for Counselor and Client, a basis for understanding the consequences of their future activities, interactions, strategies and actions.
REFERENCES


A GAME THEORETIC APPROACH FOR A REINSURANCE MARKET

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ABSTRACT

We give a survey for a reinsurance market. We present a game theoretic approach for some important concepts useful in the analysis of reinsurance as the core, the bargaining set or the Nash equilibrium. We propose some new results for a game model applied in the case of reinsurance market and then we define and give some applications with exponential and power utility functions for the market core, market core cover, stable sets of portfolios, the Reasonable and the Weber sets for a reinsurance market.

KEYWORDS

reinsurance market, optimal allocations, core.

1. INTRODUCTION

The human desire for security and fighting against unpredictable future were first steps for appearing ideas of insurance. At first, insurance means finding support in family, community or guild and it has the origins in marine commerce. Even in Chistian era there were transport covers but the oldest known contract with reinsurance characteristics was discovered much later in 1370, in Genoa.

The reinsurance business has an important and vital role in country economies because it represents a protection from the risks for the participants in the reinsurance market, i.e., for the insurance companies. In a reinsurance market there is a transfer of risks to reinsurers for reducing the fluctuations in the business performance of the insurance market, and, also, for their capital costs.

"Reinsurance is a highly international industry with a limited number of large companies. In 2002, the total reinsurance premiums of the 40 largest reinsurance groups amounted to USD 138 601 200 000, whereof USD 58 544 000 000 stemmed from EU reinsurers. In EU, Germany has a dominant position with companies as Munich Re, Hannover Re and Allianz Re, Lloyd's is the largest UK writer of reinsurance, and SCOR together with Axa Re are the largest French reinsurers"(Standard&Poor's, Global Reinsurance Highlights 2003 Edition, London/New York, 2003). Today, Standard&Poor gives a list with almost 135 professional reinsurers worldwide, as well as around 2000 direct insurers that also underwriting for the insurance companies.
The predecessor of the big group of professional reinsurance companies was the Cologne Reinsurance Company, founded in Hamburg, in 1842, after a catastrophic fire which provoked losses of 18 million marks. So, after marine insurance, the first big step in writing history of reinsurance was insurance against fire hazards.

The reinsurance market is the secondary market for insurance risks. The agents who are the direct insurers rarely trade risks with each other. For this reason, they cede percent of their initial risks to specialized professional reinsurers who have no primary business.

Reinsurance business is a prominent part of the non-life insurance market. From information given by OECD it is known that only 10% of all insurance premiums are insured and, more, 18% of non-life insurance premiums are insured, compared to 3% for life insurance premium average. The main roles of reinsurance for the cedant, i.e., the direct insurer who cede a some percent of his risks, can be summarized as follows:

- reduction of technical risks;
- permanent transfer of technical risks to the reinsurer;
- increase of homogeneity of insurance portfolio;
- reduction of volatility of technical results;
- substitute for capital/own funds;
- supply of funds for financing purposes;
- supply of service provision.

We can easily see that the underwriting policy of the cedant to a large extant is determined by its reinsurance policy. The financial strength of a cedant and its ability to fulfill obligations arising from insurance contracts vis-a-vis the policyholders depends clearly on the security of the cedant's reinsurance arrangements.

In spite of its powerful connection to direct insurance industry, there are some characteristic of reinsurance that may be important to underline before going into more specific issues. We will describe some particularities of reinsurance:

- there is no direct contractual relationship between the originally insured and the reinsurer, and the policyholders have no priority to the assets of the reinsurer to cover their claims;
- reinsurance is a business activity between professional parties;
- reinsurers largely depend on information from the direct insurers to establish claims reserves. There are furthermore significant delays in receiving claims information;
- for reinsurers the volume of nonlife operations significantly exceeds that of life operations;
• reinsurance business has higher degrees of diversity in respect of geography and combinations of insured lines than direct insurance business;

• reinsurers have significant catastrophe exposure, and special retrocession and pooling techniques to cope with those.

2. SOME MAIN DIRECTIONS

The theoretical study of insurance and reinsurance market equilibrium has been carry out under a variety of models. The one of the first results were given by Arrow [10] and Debreu [29] who used contingent space to study economic equilibrium in a simple risk exchange model with two risk averse parties. In this way, using the contingent space, these authors were able to extend certain fundamental results of economic equilibrium from an exchange of goods to an exchange of risks. They proved that competitive equilibrium exists and that both the first and the second social welfare theorems hold in an economy with uncertainty. Specifically, they showed that competitive equilibrium is Pareto optimal, and every Pareto optimal solution can be supported by a competitive equilibrium through the redistribution of endowments.

By using contingent space the economists can analyze uncertainty, it is far removed from reality of most insurance markets. In 1962 Borch presented a risk exchange model of the reinsurance market and he argued that this kind of market should contain only one price, rather than the multiplicity of prices associated with all possible states in contingent space. Because the Borch's model was over-specified, in 1971 Kihlstrom and Pauly [38] showed the form of this kind of risk loading was not consisting with the assumption's Borch that the parties in the risk exchange had quadratic utility functions. Kihlstrom and Pauly demonstrated that the single price of insurance is correlated with the prices of contingent claims, and that the competitive equilibrium of a risk exchange in price/quantity space is consistent with competitive equilibrium of a risk exchange in contingent space.

Subsequently, Arrow [11] noted the risk transfer - as opposed to risk exchange- model recognizes the reality that the parties in most traditional insurance markets are either buyers and sellers, and not suitable both to cede and to assume risk as in a risk exchange.

The reinsurance problem appears at first sight to be a problem which can be analyzed in terms of classical economic theory, if the objectives of the companies have been formulated in an operational manner by the help of Bernoulli's utility concept: it must not be maximized the expected gain, but the expected utility of the gain. "The ideas of Bernoulli reappeared in insurance literature only when the game theory of von Neumann and Morgenstein had made utility fashionable, and demonstrated that this concept must occupy a central position in any theory of decisions under risks and uncertainty"[24]. However, closer investigations show that the economic theory is only relevant part of the way. Then the problem becomes a problem of cooperation between parties who have conflicting interests, and who are free to form and break any coalitions which may serve their particular interests. Classical economic theory is powerless when it comes to analyze such problems. The only theory which at present seems to
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hold some promise of being able to sort out and explain this apparently chaotic situation, is Games Theory.

In insurance mathematics, beginning perhaps with the works of Borch [17, 18, 21, 22], later, Baton and Lemaire [15, 16] applied the Nash bargaining framework of cooperative game theory to the analysis of a reinsurance market. Kihlstrom and Roth showed in a similar analysis of a risk transfer between one insured and one insurer [39]. Game theoretic ideas have been used in the context of reinsurance by many authors but the recent papers belong to Aase [3, 8].

To succeed the implementation of the real cases from a reinsurance market in a theoretical structure who belong to game theory some elements hold:

- Participants have some benefits to share (political power, savings, or money);
- The opportunity to divide results have to results from cooperation of all participants or a sub-group of participants;
- Individuals are free to engage in negotiations, bargaining, coalition formation;
- Participants have conflicting objectives; each wants to secure the largest part of the benefits for himself.

Cooperative game theory studies those situations where participants' objectives are partially cooperative and partially conflicting. The biggest interest of the participants is to cooperate because in this way they can achieve the greatest possible total benefits. When it comes to sharing the benefits of cooperation, however, individuals have conflicting goals. Such cases are usually modeled as \( n \)-person cooperative games in characteristic function form.

3. REINSURANCE MARKET. OPTIMAL PORTFOLIOS

The economists propose two theoretical patterns to analyze reinsurance. K. Borch, in 1962, was the first one, who proposed a new method [21], for studying the reinsurance market, based on optimal risk sharing among risk-averse agents.

Later, in 1990, the second pattern to investigate a reinsurance market was taken from corporate hedging theory. This approach is based on the decision of an insurer to get reinsurance looks like the decision of any nonfinancial business to acquire reinsurance.

It was revealed that both approaches keep unexplained problems and, also, pass up the dual approach of the theory of reinsurance. The last observation was attacked in them study by Garven and Lamm-Tennant [32], when the reinsurance is both a risk management and a financing decision.

The optimal allocation problem, based on Borch theory, is the problem of allocation a risk in an optimal way to \( n \) players supplied with risk measures. K. Borch used in his theoretical approach an utility function Pareto to characterize optimal exchanges but without unique
solution [17, 18, 21]. This kind of solution may carry on substantial fixed sidepoints. These results give to many authors the idea of game theoretic approach, or additional, concepts like fairness. With this new point of view, game theory way, they reached to some specific elements in the set of Pareto optimal rules [18, 30, 13, 40, 15, 27, 3].

The reinsurance market can be analyzed as a game. The players seen as traders enter in the game with an initial allocation of risks, transform these goods and they go out with final allocation with a higher utility.

This section presents a short overview on reinsurance market and it is based on results established by Borch [17, 18, 23], Baton and Lemaire [15, 16], Aase [2, 3, 4, 6, 8].

Karl Henrik Borch is considered as the founding father of modern insurance economics. He is, in fact, more originally an actuary then in economics and finance. Since 1960 till 1974, he published the study of traditional actuarial problems as the theory of risk, optimal reinsurance or optimal insurance asset allocation. In his study he used expected utility analysis and the theory of games. Utility theory it is involved for the first time by Borch [20] for solving actuarial problems. The theory about Pareto-optimal risk exchanges and general economic equilibrium under uncertainty was developed for reinsurance markets but, later, it is shown that it could be applied also for capital markets. In this research it is proof the Borch's theorem on optimal risk exchange. The theorem shows that the exchange of risks is Pareto optimal if and only if a pool contains all risks, and each agent takes a part in this pool. All risk exchange model was developed during many research years [17, 19, 21, 25]. Also, there are some important textbooks which it can be found self-contained presentations of the Borch's model as Borch [25], Buhlmann [26], Gerber [33], among many others.

With his research in study of insurance and reinsurance markets K. Borch create a connection between actuarial science and the economic under uncertainty.

We present in the following some important theorems which belong to the basic model, in its original version:

Consider a set \( N = \{C_1, \ldots, C_n\} \) of \( n \) agents (policyholders, insurance companies, reinsurers,...) willing to improve their level of security through a risk exchange treaty. \( C_j \) has an initial wealth \( R_j \) and is subject to a risk, characterized by its distribution function \( F_j(x_j) \). Assume \( C_j \) evaluates his situation by means of utility function \( u_j(x) \) such that \( u'_j(x) > 0 \) and \( u''_j(x) \leq 0 \) for all \( x \). The expected utility of \( C_j \)'s initial situation \([R_j, F_j(x_j)]\) is

\[
U_j(x_j) = \int_0^\infty u_j(R_j - x_j)dF_j(x_j).
\]

The members of the set of agents will then try to increase their utilities by concluding a treaty,

\[
\tilde{y} = [y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)]
\]
where \( y_j(x_1, \ldots, x_n) = y_j(\bar{x}) \) is the sum \( C_j \) has to pay if the claims for the different agents respectively amount to \( x_1, \ldots, x_n \).

Since all claims have to be indemnified, the treaty has to satisfy the following admissibility condition.

**Condition**

\[
\sum_{i=1}^{n} y_j(\bar{x}) = \sum_{i=1}^{n} x_j = z
\]

where \( z \) denotes the total amount of all claims.

After the risk exchange, the expected utility of agent \( C_j \) becomes

\[
U_j(y) = \int_{\theta_N} u_j[R_j - y_j(\bar{x})]dF_N(\bar{x})
\]

where \( \theta_N \) is the positive orthant of \( E^n \), and \( F_N(\bar{x}) \) is the \( n \) - dimensional distribution function of the claims \( \bar{x} = (x_1, \ldots, x_n) \). Thus, \( U_j \) was extended from individual \( x_j \) to vector \( (y_1, \ldots, y_n) \).

A treaty \( \bar{y} \) is obviously preferred to \( \bar{y}' \) if \( U_j(\bar{y}) \geq U_j(\bar{y}') \), for all \( j \), with at least one strict inequality. \( \bar{y}' \) is then said to be dominated by \( \bar{y} \). The set of undominated treaties is called the efficient set or the pareto-optimal set.

**Condition** A treaty \( \bar{y} \) is Pareto-optimal if there is no \( \bar{y}' \) such that \( U_j(\bar{y}') \geq U_j(\bar{y}) \) for all \( j \), with at least one strict inequality.

**Theorem** A treaty \( \bar{y} \) is Pareto-optimal if and only if there exists \( n \) non-negative constants \( k_1, \ldots, k_n \), such that, almost surely,

\[
k_j u'_j[R_j - y_j(\bar{x})] = k_1 u'_1[R_1 - y_1(\bar{x})] \quad j = 1, \ldots, n
\]

Note that \( F_j(x_j) \) does not appear in equation (**). Hence, the shape of the Pareto-optimal treaties does not depend on the claim amounts distribution functions.

Later, du Mouchel proofs that there exists at least one Pareto-optimal treaty if there are mild technical conditions [31].

**Theorem** If the \( k_j \) can be chosen in such a way that the domains of the functions \( k_j u'_j(x) \) have a nonvoid intersection, then there exists a Pareto-optimal treaty.
Theorem If the \( y_j(x) \) are differentiable, a Pareto-optimal treaty depends only on the individual claim amounts \( x_j \) through their sum \( z \).

The Borch's risk exchange model of reinsurance, researched as a \( n \)-person cooperative game, was studied by Lemaire [40, 41, 42], Lemaire and Baton [15]. In them studies they approached game theoretic characteristics as the Pareto-optimal payoffs, core, bargaining set, value of the game. Also, Lemaire found some applications of the famous Borch's theorem [43].

Lemaire completed the existence theorem with a new result:

Theorem For each set \( K \) of constants \( \{k_1, \ldots, k_n\} \) satisfying the assumption of the Borch's existence theorem, there exists one, and only one, the Pareto-optimal treaty.

It was shown by Lemaire that the risk exchange model could be seen as a \( n \)-person cooperative game without transferable utilities [43]. From this point of view was natural to characterize the set of Pareto optimal treaties using ideas from game theory. Indeed, it was obvious to introduce the condition about individual rationality.

Condition

\[
U_j(y) \geq U_j(x_j), \forall j = 1, \ldots, n.
\]

When we have exponential utilities and independent claim amounts distribution Lemaire by helping the Individual Rationality Condition reduce the upon bounds on the monetary payments [42]:

\[
y_j(0) \leq P_j^I - P_j^N, \forall j = 1, \ldots, n.
\]

The last two conditions don't exclude the phenomenon like some agents could win more if they go out and constitute a subcoalition. This phenomenon is characterized in game theory as a core of a game. Let define firstly this notion.

Let \( N \) be a set with \( n \) players. Let \( S \subset N \) be a subcoalition, and \( v(S) \) be the set of Pareto-optimal treaties for \( S \); \( \bar{y} \) is said to dominate \( \bar{y}' \) with respect to coalition \( S \) if

1) \( U_j(\bar{y}) \geq U_j(\bar{y}'), \forall j \in S \)

2) \( S \) can impose \( \bar{y} : \bar{y} \in v(S) \)

or \( \bar{y} \) is said to dominate \( \bar{y}' \) with respect to \( S \). Hence, the core is said to be the set of all the nondominated treaties. With this approach the Individual Rationality Condition could be replaced with a stronger one, Collective Rationality Condition, which implies both last two
conditions.

Condition No coalition has interest in leaving the pool.

Baton and Lemaire studied the case of risk exchange market for exponential utility with the following result [15]:

**Theorem**  $\tilde{y}$ belongs to the core of the pool if and only if 

$$y_j(x_1, \ldots, x_n) = q_j z + y_j(0)$$

with

$$\sum_{j \in S} y_j(0) \leq \sum_{j \in S} \left( P_j^S - P_j^N \right) \forall S \subset N$$

defining $P_j^\emptyset = 0$.

According to Baton and Lemaire [15]: "the main disadvantage of the core is that there exists large classes of games for which it is empty. Fortunately, the next theorem shows the core of the risk exchange market always exists."

**Theorem** The core of the market is non-void.

Not only the concept about core was studied in a risk exchange market, but, game theory has produced other important solutions like:

1) **Bargaining set**

This is a solution concept introduced by Aumann and Mascheler [13], but the most used definition belong to Davis and Mascheler [28]. If the core solution consists of all allocations such that no coalition can propose an alternative set of trades, perception about bargaining set that it limits the ability of coalitions to refrain a proposed payoff allocation because of the apprehension that a second coalition can cause another counter object (set of trades).

2) **Kernel**

The kernel solution was introduced by Davis and Mascheler and is defined in the same way as the bargaining set relative to a partition $Q$ of $N$. The perception about this concept is that if two agents $i, j$ belong to the same partition, then the higher excess that $i$ can realize in a coalition without $j$ should be the same as the same excess like $j$ in the same situation with $i$.

3) **Nucleolus**
This solution concept was introduced by Schmeidler [48]. It is an imputation which is reached minimizing, firstly, the largest excess among all coalitions, and so on till the end, when remaining only one, so the kernel is unique imputation. If the core is non-empty then the nucleolus belong to the core, also, it is always in kernel and, since the kernel is contained in bargaining set.

These concepts were studied by Baton and Lemaire [16], Lucas [44], Owen [46], Maschler [45] in a exchange economy context.

If in the model proposed by Borch it has the property to be essentially, later, in 1993, A. Aase considered a stochastic process by introducing the risk. He proposed in his first approach that the preferences can depend only of the final estate of the company, so, Aase succeeded to transform a dynamic model in to a static one, or equivalent, in to a problem of equilibrium. Aase proved in his paper from 1993 that an equilibrium exists based on a theorem of existence which belong to Araujo and Monteiro [9]. Later, in 2008, Aase studied the existence and the uniqueness of the equilibrium in a reinsurance market from a new point of approaching as a finite dimensional Euclid space. He transformed the general exchange economy proposed by Arrow&Debreu [12] in a syndicate, where the negative variables were allowed because they could signify gains or losses measured in some units of account and not consumption. If it happens to a member then that person may be interpreted as a bankrupt.

In his paper from 2002, A. Aase gives a brief and oversimplified presentation for some essential elements from a market of pure exchange such that it can be seen as a game [3]:

1) Let \( N = \{1, 2, \ldots, n\} \) be a set of agents and let \( S \) be an arbitrary subset of \( N \). The characteristic function of the game \( v : 2^N \rightarrow \mathbb{R} \) gives the total payoff for the players who belong to the coalition \( S \), \( S \in 2^N \), payoff obtained by cooperating.

2) Let \( z_i \) be the payoff to player \( i \) who cooperates in this game.

3) The "collective rationality" or the fact that the players who will obtain by cooperating the maximum total payoff can be represented in this way:

\[
\sum_{i=1}^{n} z_i = v(N)
\]

4) This assumption corresponds to Pareto optimality in our reinsurance market, i.e., the optimal solution \( Y \) solver

\[
\sum_{i=1}^{n} \lambda_i Eu_i(Y_i) = Eu_{\lambda_N}(X_N)
\]

where \( \lambda_N = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( \lambda_i \in \mathbb{R}^+ \), represent the agent weights and representative
agent pricing denoted by $X_N$.

5) Another important condition $z_i \geq v(\{i\})$ represents "individual rationality" and corresponds to $\text{Eu}_i(Y_i) \geq \text{Eu}_i(X_i), \ i \in N,$ which implies that no player will participate in the game if he can obtain more alone.

6) The rationality assumption for any coalition of all players, i.e., for any $S \subseteq N$, we have

$$\sum_{i \in S} z_i \geq v(S), \ \forall S \subseteq 2^N$$

7) We can name this condition: "social stability" and it corresponds in reinsurance market to a further restriction on the investor weights $\lambda \neq 0$ such that

$$\sum_{i \in S} \lambda_i \text{Eu}_i(Y_i) \geq \text{Eu}_{\lambda,S}(X_S)$$

where

$$\text{Eu}_{\lambda,S}(X_S) = \sup_{Z_i \in S} \sum_{i \in S} \lambda_i \text{Eu}_i(Z_i) \ \text{s.t.} \ \sum_{i \in S} Z_i \leq \sum_{i \in S} X_i = X_S, \ \text{a.s.}$$

and $\lambda_S = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_s}) = (\lambda_i)_{i \in S}, \ \lambda_i \in \mathbb{R}^+, \ S = \{i_1, \ldots, i_s\}, \ S \subseteq 2^N, \ |S| = s.$

In many cases the core will be characterized by the Pareto optimal allocations according to investor weights $\lambda_i$ in some region restricted by inequalities given in the following inequalities.

The initial portfolios are denoted by $X_1, X_2, \ldots, X_n$, the "market portfolio" is $X_N = \sum_{i=1}^n X_i$ and the reinsurers have the exponential utility functions given by: $u_i(x) = 1 - a_i e^{-\frac{x}{a_i}}, x \in \mathbb{R}, \ i \in N.$

From Aase, the Pareto optimal allocations which result from coalition are [2, 3]:

$$Y_i(X_N) = \frac{a_i}{A} X_N + b_i(N), \quad \text{where} \quad b_i(N) = a_i \ln \lambda_i - a_i \frac{K}{A}, \quad A = \sum_{i=1}^n a_i$$

and for any subset $S \subseteq N$ the formulas from above becomes:

$$Y_i(X_S) = \frac{a_i}{A_S} X_S + b_i(S), \quad \text{where} \quad b_i(S) = a_i \ln \lambda_i - a_i \frac{K_S}{A_S}, \quad A_S = \sum_{i \in S} a_i$$

$$K_S = \sum_{i \in S} a_i \ln \lambda_i, \quad X_S = \sum_{i \in S} X_i$$
The values \( \lambda_i \) appear as "investor weights" are arbitrary positive constants and Aase found constraints for the values sets of these constants, or equivalently, to obtain constraints on the zero-sum side payments \( b_i \).

He used for these requirements the expected utility of the representative agents for any \( S \subseteq N : \)

\[
Eu_{\lambda_i}(X_S) = E \left[ \sum_{i \in S} \lambda_i - A_S e^{\frac{X_S}{X_S}} \right].
\]

Example In case of the reinsurance market with \( n = 3 \) agents, the market core will be characterized by the Pareto optimal allocations corresponding to investor weights \( \lambda_i \), who are solutions of the system of inequalities:

\[
\lambda_i \geq \frac{E \left[ e^{X_N} \right]}{E \left[ e^{X_N} \right]} e^{\frac{X_N}{X_N}},
\]

\[
A_Sd_{ij}E \left[ e^{-\frac{X_S}{X_S}} \right] \geq a_iE \left[ e^{\frac{X_N}{X_N}} \right] + a_jE \left[ e^{\frac{X_N}{X_N}} \right]
\]

\[
2AE \left[ e^{-\frac{X_N}{X_N}} \right] \leq \sum_{i,j \in \{1,2,3\}, i \neq j} A_Sd_{ij}E \left[ e^{\frac{X_S \cap \cap S}{X_S \cap \cap S}} \right]
\]

\[
b_1(N) + b_2(N) + b_3(N) = 0
\]

\( i,j \in \{1, 2, 3\}, \ i \neq j, S \subseteq N, \) where \( d_{ij} = e^{\left\{ \frac{b_{ij}}{n\cap \cap n} \right\}}. \)

Example In case of the reinsurance market with \( n = 3 \) agents, the market core will be characterized by the Pareto optimal allocations corresponding to investor weights \( \lambda_i \), \( i \in N, \) the solutions of the system of inequalities:

\[
\lambda_i \geq \frac{1}{E[X_N]} \left[ \frac{E[X_i^{1-a}]}{E[X_N^{1-a}]} \right] \cdot \frac{i}{a}, i \in \{1, 2, 3\}
\]

\[
\lambda_i \geq 2^{-a} \left( \sum_{i=1}^{3} \lambda_i^{\frac{1}{a}} \right)^a [a_{ij} + a_{i,k} - a_{k,j}]^a, i,j,k \in \{1, 2, 3\}, i \neq j \neq k
\]

\[
2 \left( \sum_{i=1}^{3} \lambda_i^{\frac{1}{a}} \right)^a \geq \sum_{\{i,j\} \in \{\{1,2\}, \{1,3\}, \{2,3\}\}} \left( \lambda_i^{\frac{1}{a}} + \lambda_j^{\frac{1}{a}} \right)^a a_{i,j}^{1-a}
\]

where \( a_{i,j} = \left[ \frac{E[X_i + X_j]^{1-a}}{E[X_N]^{1-a}} \right] \cdot \frac{i}{a} \) for \( \{i,j\} \in \{\{1,2\}, \{1,3\}, \{2,3\}\}. \)

The core allocations \( Y = (Y_1, Y_2, \ldots, Y_n) \) are given by
\[ Y_i(X_N) = \frac{\lambda_i^{\frac{1}{\epsilon}}}{\sum_{i=1}^{n} \lambda_i^{\frac{1}{\epsilon}}} X_N \text{ a.s. for all } i \]

and \( \lambda_i \) are defined by the inequalities remembered above.

In all studies, who are focused in finding the better solution concept of outcomes for the classical exchange model, [15, 16], [13], [14], [3, 8], it is argued that the bargaining solution is more advantageous than the core solution because in many cases just the last one, the core, is empty. Another argument, in the cases of the reinsurance market, when the core solution always exists, the bargaining set reflects better then the core solution the real behavior of the reinsurance syndicate. In 1973, Aumann, after some researches, proved that, after coalition, the core of the original game become more wide because it includes outcomes that are worse for the agents [14]; there are some situations were only outcomes that are worst for the coalition are added.

Aase considered the initial portfolio allocation of the agents denoted by \( X = (X_1, \ldots, X_n) \), or, the realizations at time zero if no reinsurance exchanges took place. The random vector \( X \) appears in his paper with the probability distribution \( F(x) = P(X_1 < x_1, \ldots, X_n < x_n) \) [8]. After take reinsurance it will be result the final portfolio, denoted, the random vector \( Y = (Y_1, \ldots, Y_n) \).

**Definition** The bargaining solution is the allocation \( Y \) which solves the problem

\[
\max_{Z_1, \ldots, Z_n} \prod_{i=1}^{n} \left[ Eu_i(Z_i) - Eu_i(X_i) \right]
\]

and satisfies

\[ \lambda_1 u'_1(Y_1) = \ldots = \lambda_n u'_n(Y_n) \]

and

\[ \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i = X_N \]

where \( u_i(\cdot), \ i \in N \) are the utility functions of the agents.

The objective in the Nash bargaining solution is to maximization the utility gains, where the utility functions are increasing and concave.

For the first time, the Nash solution in the standard reinsurance treat was analyzed by Borch in 1960 with two parties and for various utility functions [18, 19]. In the following text we
give a characterization of the Nash's bargaining solution for the affine model with zero sum side payments, often referred to as proportional reinsurance market [8].

Consider the case of the negative exponential utility functions, with the marginal utilities

\[ u_i'(x) = e^{-\frac{x}{a_i}} \]

where the \( a_i^{-1} \) is the absolute risk aversion of the agent \( i \).

It is obviously that the reinsurance contracts are affine in \( X_N \), or we can affirm that they belong to the class of proportional reinsurance. We consider \( c_i \), for all \( i \), with zero sum side payments occurred between the reinsurers, satisfying the condition:

\[ \sum_{i=1}^{n} c_i = 0. \]

The problem is to find these constants \( c_i \) if them characterize the criterion of maximization and market clearing.

In formula of maximization criterion Aase used the fact that the logarithm function is monotonically and strictly increasing, so he could reformulate it as follows:

\[ \max_{b_1,b_2,...,b_n} \mathcal{L}(c_1,c_2,...,c_n;\mu) \]

where

\[ \mathcal{L}(c_1,c_2,...,c_n;\mu) = \sum_{i=1}^{n} \ln[Eu_i(Y_i) - Eu_i(X_i)] - \mu \left( \sum_{i=1}^{n} c_i \right) \]

Theorem The bargaining solution for proportional reinsurance, or affine model with the final contracts of the form \( Y_i(X_N) = \frac{a_i}{A}X_N + c_i \) is the solution to the problem:

\[ \max_{b_1,b_2,...,b_n} \mathcal{L}(c_1,c_2,...,c_n;\mu) \]

and it has the following form:

\[ \prod_{i=1}^{n} \left( \frac{1}{\mu} + a_i \right)^{a_i} = \prod_{i=1}^{n} \left[ a_i E \left( e^{-\frac{X_i}{a_i}} \right) \right]^{a_i} = \left[ E \left( e^{-\frac{X_N}{A}} \right) \right]^{A}. \]
4. NEW RESULTS

In the last part of the paper we propose new results for a game model applied in the case of a reinsurance market and then we define and give some applications with exponential and power utility functions, for the market core, stable sets of portfolios, the market core cover, the reasonable and the Weber sets for a reinsurance market.

Definition A competitive reinsurance market is a pair denoted by \( \mathcal{RM}(u) = \langle N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N} \rangle \) consisting of the agent set \( N = \{1,2,\ldots, n\} \) interpreted as (re)insurers where the function \( u_\lambda(\cdot) : \mathbb{R} \to \mathbb{R} \) is von Neumann-Morgenstern expected utility function and \( Eu_\lambda(X_S) = 0 \).

Let \( \mathcal{RM}(N,X) = \langle \mathcal{RM}(u) | u \in \mathcal{U} \rangle \) denote the set of the all reinsurance markets where \( N = \{1, 2, \ldots, n\} \) is the set of the players, \( X = (X_1, X_2,\ldots, X_n) \) the initial random vectors, \( X_i \in L^2(\Omega, \mathcal{K}, P) \) and \( \mathcal{U} = \{ u \mid u : \mathbb{R} \to \mathbb{R} \text{ is concave and increasing} \} \).

There are some important properties proved in Ghica [34, 36, 37].

Proposition A competitive reinsurance market is monotonic, i.e.,

\[
Eu_{\lambda,S}(X_S) \leq Eu_{\lambda,T}(X_T) \text{ with } S \subseteq T \subseteq N.
\]

Theorem For all \( S, T \subseteq N \) and \( S \cap T = \emptyset \) we have

\[
Eu_{\lambda,S,T}(X_{S,T}) \geq Eu_{\lambda,S}(X_S) + Eu_{\lambda,T}(X_T).
\]

Definition A competitive reinsurance market for which

\[
Eu_{\lambda,N}(X_N) > \sum_{i=1}^{n} \lambda_i Eu_i(X_i)
\]

is said to be an essential reinsurance market.

Definition Let two reinsurance markets. Then the first reinsurance market \( \langle N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N} \rangle \) is strategically equivalent to the second reinsurance market \( \langle N, \{E\bar{u}_{\lambda,S}(X_S)\}_{S \subseteq N} \rangle \) if there exist \( k > 0 \) and a payoff \( a_i, i \in S \) such that

\[
Eu_{\lambda,S}(X_S) = kE\bar{u}_{\lambda,S}(X_S) + \sum_{i \in S} a_i, \forall S; S \subseteq N.
\]
Definition Let $\alpha, \beta \in \mathbb{R}$. A reinsurance market $\langle N, \{u_{\lambda,S}(X_S)\}_{S \subseteq N}\rangle$ is called a reinsurance market in $(\alpha, \beta)$-form if

$$Eu_{\lambda,(i)}(X_i) = \alpha, \forall i \in N \text{ and } Eu_{\lambda,N}(X_N) = \beta.$$  

Proposition Each $N$-essential reinsurance market $\langle N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N}\rangle$ is strategically equivalent to a reinsurance market $\langle N, \{E\tilde{u}_{\lambda,S}(X_S)\}_{S \subseteq N}\rangle$ in $(0, 1)$-form. The reinsurance market is unique with

$$k = \frac{1}{Eu_{\lambda,N}(X_N) - \sum_{i=1}^{n} \lambda_i Eu_i(X_i)}$$

and

$$a_i = \frac{Eu_i(X_i)}{Eu_{\lambda,N}(X_N) - \sum_{i=1}^{n} \lambda_i Eu_i(X_i)}$$

So,

$$E\tilde{u}_{\lambda,S}(X_S) = \frac{Eu_{\lambda,S}(X_S) - \sum_{a_S} \lambda_a Eu_a(X_a)}{Eu_{\lambda,N}(X_N) - \sum_{i=1}^{n} \lambda_i Eu_i(X_i)}.$$  

Definition Let $X_1, X_2, \ldots, X_n$ be the initial portfolios and the market portfolio $X_N = \sum_{i=1}^{n} X_i$. We consider $f : \mathcal{RM}(N,X) \to \mathbb{R}^n$ a map. Then we say that $f$ satisfies:

1) individual rationality: $f_i(\mathcal{RM}(u)) \geq \lambda_i Eu_i(X_i), i \in N$.

2) efficiency: $Eu_{\lambda,N}(X_N) = \sum_{i=1}^{n} f_i(\mathcal{RM}(u))$.

3) social stability: $\forall \mathcal{RM}(u_1), \mathcal{RM}(u_2) \in \mathcal{RM}(N,X)$, all additive reinsurance market $a \in \mathcal{RM}(N,X)$ and all $k > 0$, we have that

$$E(u_1)_{\lambda,N}(X_N) = kE(u_2)_{\lambda,N}(X_N) + a$$

implies the next equality

$$f(\mathcal{RM}(u_2)) = kf(\mathcal{RM}(u_1)) + a.$$

4) the dummy agent property: If

$$f_i(\mathcal{RM}(u)) = \lambda_i Eu_i(X_i), \forall \mathcal{RM}(u) \in \mathcal{RM}(N,X)$$
and for all dummy player \( i \in N \) such that
\[
Eu_{\lambda_S}(X_{S,i(j)}) = Eu_{\lambda_S}(X_S) + Eu_{\lambda_{i,j}}(X_i), \text{ for all } S \in 2^{N_{i,j}}.
\]
the anonymity property: If \( f(RM(u)^\sigma) = \sigma^*(f(RM(u))), \forall \sigma \in \Pi(N), \) where \( \Pi(N) \) is the set of all permutations on \( N \). Here \( RM(u)^\sigma \) is the reinsurance market with
\[
Eu_{\lambda_S(\sigma(S))} = Eu_{\lambda_S}(X_S) \text{ for all } S \subseteq N \text{ or } Eu_{\lambda_S}(X_S) = Eu_{\lambda_{n-1}(\sigma(S))}^{\sigma(S)}(X_{\sigma^{-1}(S)}) \text{ for all } S \subseteq N \text{ and } \sigma^*: \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is defined by } (\sigma^*(x))_{\sigma(k)} = x_k \text{ for all } x \in \mathbb{R}^n.
\]
Following we denote the set of "investor weights" \( \lambda_i, i = \overline{1,n}, \) by
\[
I^{**} = \left\{ \lambda \in \mathbb{R}^n \middle| \sum_{i=1}^{n} \lambda_i Eu_i(Y_i) \leq Eu_{\lambda_N}(X_N) \right\}
\]
and by \( I^* \) the set of efficient "investor weights" vectors in the reinsurance market \( \langle N, \{Eu_{\lambda}(X_S)\}_{S \subseteq N} \rangle, \) i.e.
\[
I^* = \left\{ \lambda \in \mathbb{R}^n \middle| \sum_{i=1}^{n} \lambda_i Eu_i(Y_i) = Eu_{\lambda_N}(X_N) \right\}
\]
Obviously we have \( I^* \subset I^{**}. \)

The "individual rationality" condition \( Eu_i(Y_i) \geq Eu_i(X_i), i \in N \) should hold in order that a weight vector \( \lambda \) has a real chance to be realized in the reinsurance market.

**Definition** A weight vector \( \lambda \in \mathbb{R}^n \) is an imputation for the reinsurance market \( \langle N, \{Eu_{\lambda}(X_S)\}_{S \subseteq N} \rangle \) if it is efficient and it has the property of individual rationality, i.e.,
1) \( \sum_{i=1}^{n} \lambda_i Eu_i(Y_i) = Eu_{\lambda_N}(X_N) \)
2) \( Eu_i(Y_i) \geq Eu_i(X_i), \forall i \in N. \)

We denote by \( I \) the set of imputations \( \lambda \). Clearly, \( I \) is empty if and only if
\[
\sum_{i=1}^{n} \lambda_i Eu_i(Y_i) > Eu_{\lambda_N}(X_N).
\]

The next theorem presents the property for an essential reinsurance market where always there are infinitely many imputations.
Moreover, $I$ is a simplex with extreme points the next vectors: $f^1, f^2, \ldots, f^n$ for $i \in N$ where $f^i = (f_1^i, f_2^i, \ldots, f_n^i)$ defined by:

$$f_j^i = \begin{cases} \lambda_i Eu_i(X_i), & \text{if } i \neq j \\ Eu_{\lambda_N}(X_N) - \sum_{k \in N \setminus \{i\}} \lambda_k Eu_k(X_k), & \text{if } i = j \end{cases}$$

Theorem Let $\langle N, \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \rangle$ be a reinsurance market. If the market $\langle N, \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \rangle$ is an essential market, then

1) $I$ is an infinite set.

2) $I$ is a convex hull of the points $f^1, f^2, \ldots, f^n$.

5. SOME GAME THEORY'S SETS STUDIED FOR A REINSURANCE MARKET

Because the set of imputations is too large for an essential reinsurance market, so, we need some criteria to single out those imputations who have the chance to appear. So, we could obtain some subsets of $I$ as solution concepts. One of this solution concept is the core of a reinsurance market.

Definition The market core denoted by $MC$ of a reinsurance market $\langle N, \{Eu_{\lambda_S}(X_S)\}_{S \subseteq N} \rangle$ is the set

$$MC = \left\{ \lambda \in \mathbb{R} \left| \sum_{i \in S} \lambda_i Eu_i(Y_i) \geq Eu_{\lambda_S}(X_S), \forall S \subseteq N \right. \right\}$$

If $MC \neq \emptyset$ then the elements of $MC$ can easily be obtained because the core is defined with the aid of a finite system of inequalities.

We introduce a set related on the market core namely: the market core cover. We can refer on it like a "market core catcher" because it contains the market core as a subset.

We denote with $M$ the upper allocation and with $m$ the lower allocation who is important for definition the new element. For each $i \in N$, the $i$-th coordinate $M_i$ of the upper vector $M$ is the marginal contribution of the reinsurer $i$ in the grand group of agents; it is also called the utopia payoff for reinsurer $i$ in the grand group of agents in the sense that if the reinsurer $i$ wants more then it is advantageous for the other agents in $N$ to throw agent $i$ out.
Definition Let \( \{N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N}\} \) be a reinsurance market. For each \( i \in N \) and for each \( S \subseteq N \) with \( i \in S \) the marginal contribution of agent \( i \) to the coalition \( S \) is

\[
M_i(S) = Eu_{\lambda,S}(X_S) - Eu_{\lambda,S\setminus\{i\}}(X_{S\setminus\{i\}})
\]

Definition Let \( S \subseteq N \), \( i \in S \). The remainder \( R(S,i) \) of reinsurer \( i \) in the group of agents \( S \) is the amount which remains for reinsurer \( i \) if group of agents \( S \) forms and all other agents in \( S \) obtain their utopia payoffs, i.e.,

\[
R(S,i) := Eu_{\lambda,S}(X_S) - \sum_{j \in S\setminus\{i\}} M_j
\]

and for each \( i \in N \), the \( i \)-th coordinate \( m_i \) of the lower vector \( m \) is then defined by

\[
m_i := \max_{S, S \ni i} R(S,i)
\]

Definition For a reinsurance market \( \{N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N}\} \) the market core cover consists of all weight vectors for which we have

\[
MCC = \{\lambda \in I | m_i \leq \lambda_i Eu_i(Y_i) \leq M_i, \forall i \in N\}
\]

That \( MCC \) is a core catcher follows from the next theorem which presents that the lower (upper) vector is a lower (upper) bound for the market core.

Theorem Let \( \{N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N}\} \) be a reinsurance market and \( \lambda \in MC \). Then \( m_i \leq \lambda_i Eu_i(Y_i) \leq M_i, \forall i \in N \).

To describe a new kind of the set we will use the same notation: \( \lambda_S = (\lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_s}) \) where we have the set \( S = \{i_1, i_2, ..., i_s\} \) and \( |S| = s \). In what it follows we present another market core catcher: the reasonable set. This set was introduced in game theory by Milnor in 1952.

Definition The reasonable set \( \mathcal{R} \) of the reinsurance market denoted by \( \{N, \{Eu_{\lambda,S}(X_S)\}_{S \subseteq N}\} \) is the set \( \mathcal{R} = \bigcap_{i \in N} \mathcal{R}_i \) where, for \( \forall i \in N \), \( \mathcal{R}_i \) has the next form:

\[
\left\{ \lambda \in \mathbb{R}^n | \lambda_i Eu_i(X_i) \leq \lambda_i Eu_i(Y_i) \leq \max_{S, S \ni i} [Eu_{\lambda,S}(X_S) - Eu_{\lambda,S\setminus\{i\}}(X_{S\setminus\{i\}})] \right\}.
\]
Remark. Obviously $MC \subset MCC \subset \mathcal{R}$.

The last market core catcher presented here correspond to the Weber set from game theory [49].

Definition. The Weber set $W$ of the reinsurance market denoted by $\langle N, \{EU_{\lambda_S}(X_S)\}_{S \subseteq N} \rangle$ is the convex hull of the $n!$ marginal vectors $m^\sigma$, corresponding to the $n!$ permutations $\sigma \in \Pi(N)$:

$$m^\sigma_{\sigma(1)} = \sup_{Z_{\sigma(1)}>X_{\sigma(1)}} \lambda_{\sigma(1)}EU_{\sigma(1)}(Z_{\sigma(1)})$$

$$m^\sigma_{\sigma(2)} = \sup_{Z_{\sigma(1)}>X_{\sigma(1)}; Z_{\sigma(2)}>X_{\sigma(2)}} EU_{\lambda_{\sigma(1),\sigma(2)}}(Z_{\sigma(1),\sigma(2)}) -$$

$$\sup_{Z_{\sigma(1)}>X_{\sigma(1)}} \lambda_{\sigma(1)}EU_{\sigma(1)}Z_{\sigma(1)}$$

$$\cdots$$

$$m^\sigma_{\sigma(k)} = \sup_{Z_{\sigma(1)}>...>Z_{\sigma(k)}} EU_{\lambda_{\sigma(1),...\sigma(k)}}(Z_{\sigma(1),...\sigma(k)}) -$$

$$\sup_{Z_{\sigma(1)}>...>Z_{\sigma(k-1)}} EU_{\lambda_{\sigma(1),...\sigma(k-1)}}(Z_{\sigma(1),...\sigma(k-1)})$$

for each $k \in N$.

The payoff vector $m^\sigma$ can be created as follows. Let the agents enter in a room one by one in the order: $\sigma(1), \ldots, \sigma(n)$ and give each agent the marginal contribution which he creates by entering.

We propose in the following a result who is necessary for proving the next theorem.

Proposition. The set $MC$ is a convex set.

The Weber set is a market core catcher as shown in the next theorem. We will prove that $MC \subset W$.

Theorem. Let $\langle N, \{EU_{\lambda_S}(X_S)\}_{S \subseteq N} \rangle$ be the reinsurance market. Then $MC \subset W$. 
6. SOME APPLICATIONS. THE EXPONENTIAL UTILITY FUNCTION

In general the core will be characterized by the Pareto optimal allocations according to investor weights \( \lambda_i \) in some region restricted by inequalities given in the following theorems.

The initial portfolios are denoted by \( X_1, X_2, \ldots, X_n \), the "market portfolio" is \( X_N = \sum_{i=1}^n X_i \) and the reinsurers have the exponential utility functions given by: 

\[
u_i(x) = 1 - a_i e^{-x}, \quad x \in \mathbb{R}, \quad i \in N.
\]

From Aase The Pareto optimal allocations who result from coalition are \([2, 3]\):

\[
Y_i(X_N) = \frac{a_i}{A} X_N + b_i(N), \quad \text{where} \quad b_i(N) = a_i \ln \lambda_i - a_i \frac{K}{A}, \quad A = \sum_{i=1}^n a_i \quad \text{and} \quad K = \sum_{i=1}^n a_i \ln \lambda_i
\]

and for any subset \( S \subseteq N \) the formulas from above becomes:

\[
Y_i(X_S) = \frac{a_i}{A_S} X_S + b_i(S),
\]

Where

\[
b_i(S) = a_i \ln \lambda_i - a_i \frac{K_S}{A_S}, \quad A_S = \sum_{i \in S} a_i, \quad K_S = \sum_{i \in S} a_i \ln \lambda_i, \quad X_S = \sum_{i \in S} X_i
\]

The values \( \lambda_i \) appear as "investor weights" are arbitrary positive constants and we want to find the constraints for the values sets of these constants, or equivalently, to deceive constraints on the zero-sum side payments \( b_i \).

We use for these requirements the expected utility of the representative agents for any \( S \subseteq N \):

\[
Eu_{\lambda_S}(X_S) = E \left[ \sum_{i \in S} \lambda_i - A_S e^{\frac{K_S}{A_S} x_S} \right]
\]

We can see that the market core is characterized by the Pareto optimal allocations corresponding to investor weights \( \lambda_i \) in some region restricted by inequalities of the above kind, in general a polyhedron from \( \text{int}(\mathbb{R}_+^n) \).

Theorem The Pareto optimal allocations who correspond to the market core cover are given by weights \( \lambda_i, \quad i \in N \), the solutions of the system of next inequalities:

\[
a_i \leq \lambda_i \leq \beta_i, \quad i \in N,
\]
where

\[
\alpha_i = e^{b_i(N)/n} \mathbb{E}[e^{\frac{b_i}{X_i}}]\left[C_N - C_{M(i)}\right]
\]

\[
\beta_i = e^{b_i(N)/n} \mathbb{E}[e^{\frac{b_i}{X_i}}] \left\{ \min_{S, S \ni i} \left( |S| - 2 \right) C_S - \sum_{j \in S \setminus \{i\}} C_{S \setminus \{j\}} \right\}
\]

\(i \in N,\ \text{where } e^{\frac{K_S}{A_S}} = B_S(X) \text{ and } A_S \cdot E[B_S(X)] = C_S.\)

Theorem The Pareto optimal allocations which correspond to a reasonable set \(R\) for a reinsurance market \(\langle N, \{E \mu_{\lambda_s}(X_S)\}_{S \subseteq N} \rangle\) are given by weights \(\lambda_i, \ i \in N,\) the solutions of the system of inequalities:

\[
\lambda_i \geq \max \left\{ \frac{E[e^{\frac{b_i}{X_i}}]}{E[e^{\frac{b_i}{X_i}}]} e^{\frac{b_i}{X_i}} \left( e^{\frac{b_i}{X_i}} - 1 \right) \max_{S, S \ni i} \left[ C_S + C_{S \setminus \{i\}} \right] \right\}
\]

\(i \in N,\ \text{where } A_S E[B_S(X)] = C_S \text{ and } e^{\frac{K_S}{A_S}} = B_S(X), \ \forall S \subseteq N.\)

Remark The Pareto optimal allocations who correspond to a Weber set \(W\) for a reinsurance market \(\langle N, \{E \mu_{\lambda_s}(X_S)\}_{S \subseteq N} \rangle\) are given by the convex hull of the \(n!\) marginal vectors \(m^\sigma\), corresponding to the \(n!\) permutations \(\sigma \in \Pi(N)\):

\[
m^\sigma_{\sigma(1)} = \lambda_{\sigma(1)} E\left(1 - a_{\sigma(1)} e^{-\frac{X_{\sigma(1)}}{a_{\sigma(1)}}}\right)
\]

\[
m^\sigma_{\sigma(2)} = E\left[ \lambda_{\sigma(1)} + \lambda_{\sigma(2)} - (a_{\sigma(1)} + a_{\sigma(2)}) e^{-\frac{a_{\sigma(1)} b_{\sigma(1)} a_{\sigma(2)} b_{\sigma(2)} (a_{\sigma(1)} + a_{\sigma(2)})}{a_{\sigma(1)} b_{\sigma(2)}}} \right] - \lambda_{\sigma(1)} E\left(1 - a_{\sigma(1)} e^{-\frac{X_{\sigma(1)}}{a_{\sigma(1)}}}\right)
\]

\[
= \lambda_{\sigma(2)} + E\left[a_{\sigma(1)} e^{-\frac{X_{\sigma(1)}}{a_{\sigma(1)}}} - (a_{\sigma(1)} + a_{\sigma(2)}) e^{-\frac{a_{\sigma(1)} b_{\sigma(1)} a_{\sigma(2)} b_{\sigma(2)} (a_{\sigma(1)} + a_{\sigma(2)})}{a_{\sigma(1)} b_{\sigma(2)}}}\right].
\]

...............

\[
m^\sigma_{\sigma(k)} = E\mu_{\lambda_{\sigma(1)},...,\sigma(k)}(X_{\sigma(1)},...,\sigma(k)) - E\mu_{\lambda_{\sigma(1)},...,\sigma(k)}(X_{\sigma(1)},...,\sigma(k-1)) =
\]
\[ \lambda_{\sigma(k)} + E \left[ A_{k-1} e^{K(\sigma(1),\ldots,\sigma(k-1))} - A_k e^{K(\sigma(1),\ldots,\sigma(k))} \right] \text{ for each } k \in \mathbb{N} \]

where we denote by \( A_{k-1} = A_{\{\sigma(1),\ldots,\sigma(k-1)\}} \) and \( A_k = A_{\{\sigma(1),\ldots,\sigma(k)\}} \).

### 7. THE POWER UTILITY FUNCTION

In this case we consider the utility function \( u_i(x) = \frac{x^{1-a_i-1}}{1-a_i} \) for \( x > 0, \ a_i \neq 1; \) for situation when \( x > 0, \ a_i \to 1 \) we have the case of the natural logarithmic utility \( u_i(x) = \ln(x) \).

This kind of power utility function only make sense in the no-bankruptcy case where \( X_i > 0 \) a.s. for all \( i \). The parameters \( a_i > 0 \) are interpreted like the relative risk aversions of the agents.

We take the case where \( a_1 = a_2 = \ldots = a_n = a \). So, the marginal utilities \( u'_i(x) = x^{-a} \) and using the Riesz representation [3] we get

\[ u'_i(Y_i(X_N)) = a_i \xi(X_N) \text{ a.s. for all } i \]

which implies a formula for the Pareto optimal allocations:

\[ Y_i = a_i^{-\frac{1}{2}} \xi^{-\frac{1}{2}}(X_N), \text{ a.s.} \]

and if we use the clearing market (\( \sum_{i=1}^{n} Y_i = X_N \) ) we get

\[ \xi(X_N) = \left( \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}} \right)^a X_N^{-a}, \text{ where } \lambda_i = \frac{1}{a_i} \]

In final, we get that the optimal sharing rules are linear:

\[ Y_i(X_N) = \frac{\lambda_i^{\frac{1}{2}}}{\sum_{i=1}^{n} \lambda_i^{\frac{1}{2}}} X_N \text{ a.s. for all } i \]

and for any subset \( S \subseteq N \) the formulas from above becomes:
\[ Y_i(X_S) = \lambda_i \frac{1}{\sum_{i \in S} \lambda_i} X_S \]

The weights \( \lambda_i \) can be determined by the condition of the budget constraints:
\[ E[Y_i \xi] = E[X_i \xi]. \]
In our case of the power utility functions we get
\[ \lambda_i = \left( \sum_{i=1}^{n} \lambda_i^a \right)^a \left[ \frac{E[X_iX_N^a]}{E[X_N^{1-a}]} \right] \text{ a.s. for all } i. \]

If we normalize such that \( E[\xi(X_N)] = 1 \) then we get that
\[ \left( \sum_{i=1}^{n} \lambda_i^a \right)^a = \frac{1}{E[X_N^{1-a}]} . \]

From the last two equations we get the formula for the weights \( \lambda_i \):
\[ \lambda_i = \frac{1}{E[X_N^{1-a}]} \left[ \frac{E[X_iX_N^a]}{E[X_N^{1-a}]} \right] . \]

With this results we can find, also, the market price:
\[ \pi(Z) = E[Z \xi] = \frac{E[ZX_N^a]}{E[X_N^{1-a}]} , \text{ for all } Z \in L^2(\Omega, K, P). \]

Notice that we have also found the characteristic function of the game, here given by the expected utility of the "representative agent" restricted to any subset \( S \subseteq N \).
\[ Eu_{\lambda_S}(X_S) = E \left[ \left( \sum_{i \in S} \lambda_i^a \right)^a X_S^{1-a} - \sum_{i \in S} \lambda_i \right] . \]

The core allocations \( Y = (Y_1, Y_2, \ldots, Y_n) \) are given by
\[ Y_i(X_N) = \lambda_i \frac{1}{\sum_{i=1}^{n} \lambda_i} X_N \text{ a.s. for all } i \]

and \( \lambda_i \) are defined by the inequalities proved above.
Theorem The Pareto optimal allocations which correspond to the market core cover are given by weights $\lambda_i$ $i \in N$, the solutions of the system of next inequalities:

$$\lambda_i \geq \frac{1}{E[X_i^{-a}]} \max_{S, S \neq i} \left[ (2 - |S|)\lambda_{a,S}E[X_S^{-a}] + \sum_{j \in S \setminus \{i\}} \lambda_{a,S \setminus \{j\}} E[X_{S \setminus \{j\}}^{-a}] \right]$$

$$\lambda_i \leq \frac{E[X_i^{-a}]}{E[X_i^{-a}]}$$

where

$$(\sum_{i \in S} \lambda_i^a)^\alpha = \lambda_{a,S}, \forall S \subseteq N.$$ 

Theorem The Pareto optimal allocations who correspond to a reasonable set $R$ for a reinsurance market $\langle N, \{E u_{\lambda,S}(X_S)\}_{S \subseteq N} \rangle$ are given by weights $\lambda_i$ $i \in N$, the solutions of the system of inequalities:

$$\lambda_i \geq \max\{R_1(X), R_2(X)\}$$

with

$$R_1,i(X) = \frac{E[X_N^{-a}]}{E[X_i^{-a}]} E[X_N^a] \text{ and }$$

$$R_2,i(X) = \frac{\max_{S, S \neq i} \left[ \lambda_{a,S}E[X_S^{-a}] - \lambda_{a,S \setminus \{j\}} E[X_{S \setminus \{j\}}^{-a}] \right]^a}{E[X_N^{-a}] E[X_N^a]}.$$ 

Remark The Pareto optimal allocations who correspond to a Weber set $W$ for a reinsurance market $\langle N, \{E u_{\lambda,S}(X_S)\}_{S \subseteq N} \rangle$ are given by the convex hull of the $n!$ marginal vectors $m^\sigma$, corresponding to the $n!$ permutations $\sigma \in \Pi(N)$:

$$m_{\sigma(1)}^a = \frac{\lambda_{a(1)}}{1-a} E[X_{\sigma(1)}^{-1} - 1]$$

$$m_{\sigma(2)}^a = E\left[ \lambda_{\sigma(1)} + \lambda_{\sigma(2)} - (a_{\sigma(1)} + a_{\sigma(2)}) e^{\frac{a_{\sigma(1)}h_{a(1),a(2)}}{e^{a_{\sigma(1)}}}X_{\sigma(1)}X_{\sigma(2)}} \right]$$

$$- \lambda_{\sigma(1)} E(1 - a_{\sigma(1)}) e^{\frac{X_{\sigma(1)}}{a_{\sigma(1)}}} =$$

$$= -\frac{\lambda_{a(2)}}{1-a} + \frac{1}{1-a} E\left[ (\lambda_{a(1)}^{1/\alpha} + \lambda_{a(2)}^{1/\alpha})^a (X_{\sigma(1)} + X_{\sigma(2)})^{1-\alpha} - \lambda_{a(1)} X_{a(1)}^{1-\alpha} \right]$$
\[ m^\alpha_{\sigma(k)} = Eu(X_{\sigma(1)},\ldots,\sigma(k)) - Eu(X_{\sigma(1)},\ldots,\sigma(k-1)) = \]
\[ \frac{-\lambda_{\sigma(1)} + \lambda_{\sigma(1)} \ldots \lambda_{\sigma(k)}}{1 - \lambda_{\sigma(k)}} \]

for each \( k \in N \),

\[ \lambda_{\sigma(1),\ldots,\sigma(k)} = \left( \sum_{i \in \{\sigma(1),\ldots,\sigma(k)\}} \lambda_i \right)^{\alpha} \]

where

8. CONCLUSIONS

An important part of the actuarial mathematics is constituted by elements of mathematical stochastic, in particular theories like: probability theory, statistical theory or risk theory. In many cases the conditions and the assumptions for the assessment of risk sharing or equilibrium market are by far less favorable. As we presented in this paper, few authors showed that original objectives of game theory was to analyze markets in many particular forms, situations when the assumptions behind the neoclassical competitive equilibrium appears unreasonable.

Despite of the fact that the most researches in life insurance mathematics are sustained by the theory of risk there are difficulties to put in the practice the results like competitive equilibrium, Pareto optimality, representative agent pricing and its implications for insurance premiums. We presented in this paper an alternative model based on game theoretic approach and we have tried to show that game theory can be applied to a number of problems in insurance, problems with unsatisfactory results in other theories who belong to general economic theory or to a stochastic theory.

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BLACK-SCHOLES MODEL AND GARCH PROCESSES
WITH RISK MANAGEMENT

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ABSTRACT

The Black-Scholes option pricing model is a mathematical description of financial market and derivative investment instrument. In BS-model volatility is a constant function where trading option is indeed, risky due to random component such as volatility. Bs-model can be used to derive the hedging strategies. The concept of non constant volatility has been introduced in GARCH processes. Time varying volatility is a natural extension of the diffusion models widely applied in the asset pricing literature (Hull-White). Jump processes are characterized by discontinuous development of the analyzed random variable. In this paper a BS-model with jump has been considered to calculate the Greeks of the model and to prove that model’s call option is a monotonous function in volatility. Finally the concepts of locally risk natural value relationship (LRNVR) and GARCH option pricing are discussed.

KEYWORDS

Option Pricing, Black-Scholes model, Jump diffusion process, LRNVR, GARCH Processes.

1. INTRODUCTION

Equally spaced increments are stationary in Brownian motion that is it can be assumed that they are independently and identically distributed, but the most of the empirical studies shows that this is generally not true. The introduction of stochastic volatility in stock prices, refine the pricing of derivative instruments. In 1973 Black and Scholes introduced that, the option is implicitly priced if the stock is traded [1]. The Black-Scholes option pricing model is a mathematical description of financial market and derivative investment instrument. The model has some contradictory assumptions, for example the model assumes that volatility of the asset remain constant while empirical studies shows that generally this is not the case and constant volatility model is not adequate for the log returns. These pricing errors resulting from unrealistic assumption of the model have serious consequences for measuring the risk of portfolios [9]. Risk measurement is becoming more and more important in the financial risk management of banks and other institutions involved in financial markets. The need to quantify risk typically arises when a financial institution has to determine the amount of capital to hold as a protection against unexpected losses. In fact, risk measurement is concerned with all types of risks encountered in finance. Market risk, the best-known type of risk, is the risk of change in the value of a financial position.

Trading options is particularly risky due to the possibly high random component. An option pricing formula was derived for the more general case when the underlying stock exchanges are generated by a mixture of both continuous and jump process [10]. Advanced strategies to
reduce and manage this risk can be derived from the Black–Scholes formula. An expensive hedging strategy, i.e. a strategy to decrease the risk associated with the sale of a call, is the so called stop–loss strategy: The bank selling the option takes an uncovered position as long as the stock price is below the exercise price, \( s_t < k \), and sets up a covered position as soon as the call is in–the–money, \( s_t > k \). The shares to be delivered in case of options exercise are bought as soon as the stock \( s_t \) trades above the exercise price \( k \), and are sold as soon as \( s_t \) falls below the exercise price \( k \). Since all stocks are sold and bought at \( k \) after time 0 and at maturity \( T \) either the stock position is zero, \( (s_T < k) \), or the stocks are sold at \( k \) to the option holder, \( (s_T > k) \), this strategy bears no costs. In order to apply the concept of risk neutral valuation (see Cox and Ross (1976)) the probability measure has to be transformed such that the price process under this new measure is a martingale. In incomplete markets, however, a multitude of such transformations exist, Harrison and Kreps (1979). In contrast to complete markets the trader cannot build up a self–financing portfolio reproducing the options payoff at maturity when the market is incomplete. Therefore hedging is no longer risk free, and option prices depend on risk preferences.

The notion of non constant volatility has been introduced by GARCH processes [3]. The study of stock price models under GARCH volatility is a new horizon in derivative investment instruments. Duan (1995) was the first to provide a solid theoretical foundation for GARCH option pricing [5]. Duan generalize the concept of risk neutral valuation and introduced the notion of Locally Risk Neutral Valuation Relationship (LRNVR), which provides a solid economic argument to choose a particular equivalent martingale measure under the GARCH model with conditionally Normal innovations. Duan model contain the Black-Scholes model as a special case of homoskedasticity and it has been proved that the discounted price processes for underlying asset is a martingale measure under \( \mathcal{Q} \). In Duan GARCH option pricing model, the conditional variance process of the underlying asset under the risk neutralized pricing probability specified by LRNVR is a nonlinear asymmetric GARCH model.

In this article section 2 is consisting of a general frame work for the famous Black–Scholes option pricing model. In section 3 using more general case of BS-formula with jump, the Greeks for the model have been obtained and it is also shown that model’s call option is monotonous function in volatility. In section 4 the notion of locally risk natural value relationship (LRNVR) and GARCH model for option pricing has been studied. Finally section 5 is consisting of conclusions.

2. PRICING EUROPEAN OPTIONS

In finance, a derivative is a financial instrument whose value depends on other underlying variables and such variables can be price of other financial instruments (underlying asset) like interest rates, volatilities and indices. A European option is a derivative which at some terminal time \( T \) has a value given by some known functions of the asset prices at time \( T \).

A European call option is a contract that gives it holder the right, but not the obligation to buy
one unit of an underlying asset for a predetermined strike price $K$ on the maturity date $T$. If $s_T$ is a price of underlying asset at maturity $T$ then value of this contract at maturity will be

$$V(s_T) = (s_T - K)^+ = s_T - K \text{ if } s_T > K \quad \text{or} \quad 0 \text{ if } s_T \leq K$$

Here in the first case the holder will exercise the option and make a portfolio $s_T - K$ by buying the stock for $K$ and selling it immediately at market price $s_T$, while in the second case the option is not exercised.

Similarly a European put option is a contract that gives its holder the right to sell a unit of asset for strike price $K$ at $T$ then payoff is

$$V(s_T) = (K - s_T)^+ = K - s_T \text{ if } s_T < K \quad \text{or} \quad 0 \text{ if } s_T \geq K$$

At time $t < T$ this contract has a value known as the derivative price. Black and Scholes used the model for Stock prices

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right)$$

Let $r$ be the risk free interest rate. The probability measure $Q$ is called an equivalent martingale measure to probability measure $P$ and we can write

$$\bar{V}_t = E_Q(S_t^T / \mathcal{F}_t)$$

$$\bar{V}_t = E_Q(e^{-rt} S_t^T / \mathcal{F}_t)$$

According to fundamental theorem of asset pricing an arbitrage free price $V_t$ of an option at time $t$ is

$$V_t = E_Q(e^{-r(T-t)} V(S_T^T) / \mathcal{F}_t)$$

And

$$Y_t = \ln \frac{S_t}{S_{t-1}} = \ln S_t - \ln S_{t-1}$$

For a European call option as described the Black-Scholes Partial Differential Equation is solved with final condition $V(s_T) = (s_T - K)^+$. Let us denote $C_{BS}(t, s)$ the pricing of European calls at time $t$ for an observed risky asset price $S_t = s$ some mathematical calculations yields the closed form solution of Black-Scholes PDE.

$T - \tau = t$ in equation we obtain the explicit solution of European call option

$$C_{BS}(t, s) = SN(d_1) - Ke^{-rT}N(d_2) \quad (2.1)$$
\[ d_1 = \frac{\log \left( \frac{S}{K} \right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \]
\[ d_2 = \frac{\log \left( \frac{S}{K} \right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \]

Where \( N(.) \) is a cumulative distribution function for a standardized Normal random variable given as
\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \]

Here
- \( S \) = Stock price
- \( K \) = Strike price
- \( r \) = Risk free interest rate
- \( T \) = Time to expiry
- \( \sigma \) = Volatility of the relative price change of the underlying stock price

The Delta for the European call option is
\[ \Delta_{call} = \frac{\partial C}{\partial S} = N(d_1) \]

The delta is the option sensitivity to small changes in the underlying price.

Similarly using Put-Call parity relation Black-Scholes formula for European Put option can be written as
\[ C - P + dK = S \]  \hspace{1cm} (2.2) \]

\( C \) and \( P \) be the prices of European call and put with a strike price \( K \) and and both defined on the same stock with price \( S \) and \( d \) is the discount factor for expiration date.

\[ P = C + dK - S \]
\[ P = SN(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S \quad , \quad d = e^{-rT} \]
\[ P = Ke^{-rT}\{1 - N(d_2)\} - S\{1 - N(d_1)\} \]
\[ P = Ke^{-rT}N(-d_2) - SN(-d_1) \]  \hspace{1cm} (2.3) \]

and \( \Delta \) for the Put option of European call is
\[ \Delta_{put} = \frac{\partial P}{\partial S} = N(d_1) - 1 \]
3. JUMP DIFFUSION PROCESS AND BLACK-SCHOLE'S OPTION PRICING FORMULA

The proportional change of stock prices is a random motion which can drive either up or down. The random motion is Gaussian distributed with standard deviation (volatility) characterizing the asset. The large volatility trigger larger volatilities that is, volatilities may cluster.

Another observation in the dynamics of prices is the arrival of a piece of abnormal information can bring about a salient jump (up or down) in the asset price. Stochastic volatility is a natural extension of the diffusion models widely applied in the asset pricing literature (Hull-White). Jump processes are characterized by discontinuous development of the analyzed random variable where unpredictable moments such as new information or an external crisis include jumps and jump diffusion process combines the characteristics of a jump process with those of diffusion process. It is distorted by irregular log-normally distributed jumps.

3.1. Definition We say that \((X_t)_{t \in \mathbb{T}}\) is \(\mathcal{F} - \text{Markov}\) processes or a Markov process w.r.t \((\mathcal{F}_t)_{t \in \mathbb{T}}\) if

(i) \(X\) is \(\mathcal{F}\)-adapted

(ii) \(\forall, t \in \mathbb{T}\), The \(\sigma\)-fields \(\mathcal{F}_t\) and \(\mathcal{B}(X_u, u \geq t)\)

are conditionally independent given \(X_t\) that is, \(E[f g / X_t] = E[f / X_t] E[g / X_t]\) Where \(f \in b\mathcal{F}_t, g \in b\mathcal{B}(X_u, u \geq t)\) and \(b\mathcal{B}\) denote family of all functions of measureable and bounded.

A Markov process is a stochastic process in which the probability distribution of \(X_{t+1}\) depends solely on \(X_t\) and not on prior states. Markov process can be divided as following.

1. Diffusion processes
2. Jump Processes
3. Jump Diffusion Processes

3.2. Definition A Markov process \((X_t)_{t \geq 0}\) is a diffusion process if following properties holds

1. Continuity

\[\forall x \text{ and } \varepsilon > 0, \int_{|x-y|>\varepsilon} P(dy, t/x, s) = 0(t-s)\]

2. Drift Coefficient

\[\int_{|x-y|\leq \varepsilon} P(dy, t/x, s) = \alpha(x-s)(s-t) + 0(t-s), s < t\]

3. Diffusion Coefficient
\[ \int_{|x-y|\leq \varepsilon} p(dy, t/x, s) = b(x - s)(s - t) + o(t - s), s < t \]

Stochastic volatility models let volatility follow a diffusion process and the occasional discontinuous jumps add more mass to the tails and help to match excess kurtosis in the empirical distribution of stock returns. We suppose that jump is a Poisson random variable.

### 3.3. Definition

A càdlàg process \((N_t)_{t \geq 0}\) is a Poisson process with parameter \(\lambda > 0\) if

- \(N_t\) take values in \(\mathbb{Z}_t\) and \(N\) is a Lévy process
- \(\lambda\) take values in \(\mathbb{R}\)
- The increment \(N_t - N_s\) has Poisson distribution of parameter \(\pi_{t-s}\)

If \(\pi_u\) is a Poisson distribution then

\[ \pi_u = \exp (-u) \sum_{n=0}^{\infty} \frac{u^n}{n!} e_n \]

In addition \((N_t)_{t \geq 0}\) is a Lévy if it is càdlàg with stationary independent increments \(N_0 = 0\) and for any \(s < t\)

\[ P \circ (N_t - N_s)^{-1} = P \circ N_{t-s}^{-1} \]

Further more if \((W_t)_{t \geq 0}\) is a continuous real process with \(W_0 = 0\) then following are equivalent

(i) \(W\) is Brownian motion

(ii) \(W\) is Lévy process

We have considered jump is Poisson random variable so we consider in time interval \(\Delta t\) the probability distribution for the number of jumps is

\[ e^{-\lambda \Delta t} \left( \frac{\lambda \Delta t}{X!} \right)^X \]

and let \(dj\) is a jump indicator function as

\[ dj = \begin{cases} 1 & \text{if jump occurs} \\ 0 & \text{if no jump occur} \end{cases} \]

\[ dj = \begin{cases} \lambda dt & \text{if jump occurs, } X = 1 \\ 1 - \lambda dt & \text{if no jump occur, } X = 0 \end{cases} \]

We say if a jump occurs then stock variable changes by amount \(\phi(S, t, \tilde{Y})\)

\[ ds = \mu(s, t) dt + \sigma(s, t) dW + \phi(S, t, \tilde{Y}) dj \]

(3.1)

Here \(\tilde{Y}\) is another random variable and we say value of

\[ \phi(S, t, \tilde{Y}) = -S \]

i.e value of \(S\) drops to zero and
\[ \phi(S, t, \bar{Y}) = S(\bar{Y} - 1) \quad \text{otherwise} \]

So change in value of a function \( f = f(s + \phi) - f(s) \) and expected change in \( f(.) \) from jump events at time \( t \) is

\[ E[f(.)] = \lambda [f(s + \phi) - f(s)] dt \]

Assuming that jump risk is not symmetric in order to use continuous modeling because introduction of jump may eliminate continuous time modeling. Using Itô lemma for jump diffusion processes, we can write

\[ df = \mu(s, t) \frac{\partial f}{\partial s} dt + \frac{\partial f}{\partial t} dt + \frac{1}{2} \sigma^2(s, t) \frac{\partial^2 f}{\partial s^2} dt + \sigma(s, t) \frac{\partial f}{\partial s} dW + [f(s + \phi) - f(s)] dj \] (3.2)

Here we can see that any stochastic process can be represented as combination of diffusion and Poisson processes.

\[ E[dW] = 0 \quad E[dj] = \lambda dt \]

Rewriting equation (3.2) using above results

\[ E[df] = \left[ \mu(s, t) \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2(s, t) \frac{\partial^2 f}{\partial s^2} \right] dt + \lambda [f(s + \phi) - f(s)] dt \] (3.3)

We know that we have set change in stock variable by occurrence of jump as \( \phi(S, t, \bar{Y}) \) and let \( \phi(S, t, \bar{Y}) = ps \).

Where \( p \), the expected gain or loss is conditional on jump so we obtain probability of jump multiplied by returns given the jump is \( \frac{p \lambda dt}{dt} \) or \( p \lambda \) so we can write

\[ \mu = \mu^* - p \lambda \] (3.4)

where \( \mu^* \) is the total rate of return on stock. Rewriting equation (3.2)

\[ \frac{ds}{s} = (\mu^* - p \lambda) dt + \sigma dW + pdj \] (3.5)

Here \( dW \) can be symmetric jump risk is not continuously hedged, mixing options and stocks to eliminate \( dW \) risk. Using Itô lemma again for jump diffusion

\[ r \left( C - \frac{\partial C}{\partial s} S \right) = -\mu^* \frac{\partial C}{\partial s} dt + \left[ \frac{\partial C}{\partial t} + (\mu^* - p \lambda)s \frac{\partial C}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} \right] dt + \lambda dt + \lambda C(S, t, \bar{Y} - C(S, t)) \] (3.6)
The portfolio is consist of $\frac{\partial C}{\partial S}$ shares and written call with $-C + \frac{\partial C}{\partial S} S$

$$
 r \left( C - \frac{\partial C}{\partial S} S \right) = \frac{\partial C}{\partial t} - p \lambda \frac{\partial C}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt + \lambda \sigma^2 C \left[ C(S, t, \bar{Y}) - C(S, t) \right]
$$

$$(r - p \lambda)S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \lambda [C(S, t, \bar{Y}) - C(S, t)] - rC = 0 \quad (3.7)
$$

Here $C(S, t, \bar{Y}) = C(0, t) = 0$

If $p = -1 \ i.e. \ if \ jump \ occurs$

From (3.7) we get

$$(r + \lambda)S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - (r + \lambda)C = 0 \quad (3.8)
$$

Equation (3.8) is similar as Black-Scholes PDE just replacing $r$ by $r + \lambda$.

Equation (2.1) of section 2 can be written as

$$
C = SN(d_1) - Ke^{-(r+\lambda)\tau}N(d_2) \quad (3.9)
$$

$$
d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r+\lambda+\frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}
$$

$$
d_2 = d_1 - \sigma \sqrt{\tau}
$$

$$(r + \lambda)S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - (r + \lambda)C = 0 \quad (3.10)
$$

Equation (3.3) is similar with Black-Scholes Partial Differential Equation just replacing $r$ by $(r + \lambda)$

Following the equation (3.10) Black-Scholes Call Option can be written as

$$
C = SN(d_1) - Ke^{-(r+\lambda)\tau}N(d_2) \quad (3.11)
$$

$$
d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r+\lambda+\frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}
$$

$$
d_2 = d_1 - \sigma \sqrt{\tau}
$$

**Theorem 3.1** The Greeks of BS-call option with jump $\lambda$ corresponding to the equation (3.11) are given by:

i. Delta=$\Delta = \frac{\partial C}{\partial S}$
ii. \( \text{Gamma} = \Gamma = \frac{\partial^2 C}{\partial S^2} \)

iii. \( \text{Theta} = \Theta = \frac{\partial C}{\partial t} \)

iv. \( \text{Rho} = \rho = \frac{\partial C}{\partial \rho} \)

v. \( \text{Vega} = \nu = \frac{\partial C}{\partial \sigma} \)

Proof.

i. \[
\Delta = \frac{\partial C}{\partial s} = \frac{\partial}{\partial s} [SN(d_1) - Ke^{-(r+\lambda)\tau}N(d_2)] \\
= N(d_1) + SN^{(1)}(d_1) \frac{\partial d_1}{\partial s} - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2) \frac{\partial d_2}{\partial s} \\
= N(d_1) + \frac{\partial d_1}{\partial s} [SN^{(1)}(d_1) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2)] \\
= N(d_1)
\]

where \[
\frac{\partial d_1}{\partial s} = \frac{\partial d_1}{\partial S} \frac{1}{S \sqrt{\tau}}
\]

\[
SN^{(1)}(d_1) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2) = 0 \quad \text{and} \quad N^{(1)}(d_1) = \frac{\partial N(d_1)}{\partial S}
\]

\[
N^{(1)}(d_2) = \frac{\partial N(d_2)}{\partial S}
\]

ii. \[
\Gamma = \frac{\partial^2 C}{\partial s^2} = \frac{\partial N}{\partial S} = \frac{N^{(1)}(d_1)}{S \sqrt{\tau}}
\]

iii. \[
\Theta = \frac{\partial C}{\partial \tau} = SN^{(1)}(d_1) \frac{\partial d_1}{\partial \tau} - (r + \lambda)Ke^{-(r+\lambda)\tau}N(d_2) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2) \frac{\partial d_2}{\partial \tau}
\]

\[
é = -(r + \lambda)Ke^{-(r+\lambda)\tau}N(d_2) + \frac{\partial d_1}{\partial \tau} [SN^{(1)}(d_1) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2)] - \frac{N^{(1)}(d_2)Ke^{-(r+\lambda)\tau}}{2\sqrt{\tau}}
\]

where \[
\frac{\partial d_1}{\partial \tau} = \frac{\partial d_1}{\partial \tau} = \frac{\sigma}{\sqrt{\tau}}
\]

\[
\Theta = -(r + \lambda)Ke^{-(r+\lambda)\tau}N(d_2) - \frac{\sigma SN^{(1)}(d_1)}{2\sqrt{\tau}}
\]

iv. \[
\rho = \frac{\partial C}{\partial \rho} = SN^{(1)}(d_1) \frac{\partial d_1}{\partial \rho} + \tau Ke^{-(r+\lambda)\tau}N(d_2) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2) \frac{\partial d_2}{\partial \rho}
\]

\[
e = \tau Ke^{-(r+\lambda)\tau}N(d_2) + \frac{\partial d_1}{\partial \rho} [SN^{(1)}(d_1) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2)]
\]

\[
= \tau Ke^{-(r+\lambda)\tau}N(d_2)
\]

v. \[
\nu = \frac{\partial C}{\partial \sigma} = SN^{(1)}(d_1) \frac{\partial d_1}{\partial \sigma} - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2) \frac{\partial d_2}{\partial \sigma}
\]

\[
= \sqrt{\tau} SN^{(1)}(d_1) + \frac{\partial d_1}{\partial \sigma} [SN^{(1)}(d_1) - Ke^{-(r+\lambda)\tau}N^{(1)}(d_2)]
\]

\[
= \sqrt{\tau} SN^{(1)}(d_1)
\]
**Theorem 3.2.** The Black-Scholes Call option is monotonous function in volatility ($\sigma$) i.e.

$$\frac{\partial C}{\partial \sigma} > 0.$$ 

The BS Formula is nonlinear in $\sigma$ and relationship between call option “C” and $\sigma$ is directly increasing. Here we want to show that BS call option is monotonous function in $\sigma$.

$$\frac{\partial C}{\partial \sigma} = S\frac{\partial N}{\partial d_1} \frac{\partial d_1}{\partial \sigma} - Ke^{-(r+\lambda)\tau} \frac{\partial N}{\partial d_2} \frac{\partial d_2}{\partial \sigma}$$  \hspace{1cm} (i)

We know

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx$$

$$\frac{\partial N}{\partial x} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r+\lambda)\tau + \sigma^2 \tau}{\sigma \sqrt{\tau}}$$ \hspace{1cm} (ii)

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + (r+\lambda)\tau - \sigma^2 \tau}{\sigma \sqrt{\tau}}$$ \hspace{1cm} (iii)

$$\frac{\partial d_1}{\partial \sigma} = \sqrt{\tau} - \frac{d_1}{\sigma}$$

$$\frac{\partial d_2}{\partial \sigma} = \sqrt{\tau} - \frac{d_2}{\sigma}$$

Substituting in (i)

$$\frac{\partial C}{\partial \sigma} = S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \left(\sqrt{\tau} - \frac{d_1}{\sigma}\right) - \frac{1}{\sqrt{2\pi}} Ke^{-(r+\lambda)\tau} \exp\left\{\left(\frac{-d_1}{2}\right)\left(\frac{-d_1}{\sigma}\right)\right\}$$

$$= S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{\tau} + \frac{d_1}{\sqrt{2\pi} \sigma} \left[-Se^{\frac{-d_1^2}{2}} + Ke^{-(r+\lambda)\tau} \exp\left\{\frac{1}{2} (d_1 - \sigma \sqrt{\tau})^2\right\}\right]$$

$$= S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{\tau} + \frac{d_1}{\sqrt{2\pi} \sigma} \left[-Se^{\frac{-d_1^2}{2}} + Ke^{-(r+\lambda)\tau} \exp\left\{\frac{1}{2} (d_1 + \sigma \sqrt{\tau})^2\right\}\right]$$

$$= S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \sqrt{\tau} + \frac{d_1}{\sqrt{2\pi} \sigma} \left[-s + K \exp\left\{-(r+\lambda)\tau + (d_1 + \sigma \sqrt{\tau})^2\right\}\right]$$ \hspace{1cm} (iv)

From (ii)

$$\frac{S}{K} = e^{\exp\left\{-(r+\lambda)\tau + (d_1 + \sigma \sqrt{\tau})^2\right\}}$$

Substituting above result in (iv)
\[
\frac{\partial c}{\partial \sigma} = \frac{s}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \sqrt{\tau} + \frac{d_1}{\sqrt{2\pi} \sigma} \left[-s + \frac{\sigma^2}{2} \right]
\]

\[
\frac{\partial c}{\partial \sigma} = \frac{s}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \sqrt{\tau} > 0
\]  

(v)

So BS call option is a monotonous function of \(\sigma\) and for every option price there must be corresponding implied volatility.

4. LRNVR AND GARCH MODELS

In order to develop a GARCH option Pricing model it is necessary to change the probability measure \(\mathbb{P}\) from \(\mathbb{P}\)-measure to risk neutral \(\mathbb{Q}\)-measure. Duan (1995) introduced the notion of LRNVR by generalizing the concept of risk neutral valuation to accommodate hetroskedasticity.

**Definition 4.1** If \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. A probability measure \(\mathbb{Q}\) is said to be local risk neutral if satisfying the following conditions.

i) \(\mathbb{Q}\) is equivalent to measure \(\mathbb{P}\)

ii) \(E^\mathbb{Q} \left[ \ln \frac{S_t}{S_0} / \mathcal{F}_{t-1} \right] = r\)

iii) \(\text{var}^\mathbb{Q} \left[ \ln \frac{S_t}{S_0} / \mathcal{F}_{t-1} \right] = \text{var}^\mathbb{P} \left[ \ln \frac{S_t}{S_0} / \mathcal{F}_{t-1} \right]\)

To avoid arbitrage opportunities an equivalent probability measure \(\mathbb{Q}\) is always introduced.

i) If \(\mathbb{Q}\) equivalent to \(\mathbb{P} \Rightarrow B \in \mathcal{F}, \mathbb{Q}(B) = 0 \Leftrightarrow \mathbb{P}(B) = 0\)

ii) The discounted prices process \(\hat{S}_t\) is a martingale measure under \(\mathbb{Q}\) w.r.t \(\mathcal{F}_t\) that is \(E^\mathbb{Q}(\hat{S}_t / \mathcal{F}_{t-1}) = S_{t-1}\)

Consider \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a completely filtered probability space, here \(\mathbb{P}\) is a probability measure, \(\mathcal{F}_t\) is a \(\sigma\)-field of \(\mathcal{F}\) giving all market information up to time \(t\) and \(T = \{t/ t = 0, 1, 2, ..., T\}\) and \(\hat{S}_t = e^{-rt} S_t\), where \(r\) is risk free interest rate and \(S_t\) is adapted to \(\mathcal{F}\).

A discrete time stock price process under GARCH volatility can be constructed as following.

As we know a stock price process can be defined as

\[
\hat{S}_t = S_0 \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right]
\]

\[
S_0 = S_{t-1} \exp \left[ \left( \mu - \frac{\sigma^2}{2} \right) (t - 1) - \sigma W_{t-1} \right]
\]
Economic and Financial Risk

Let \( S_t \) be the stock price at time \( t \), then one period ahead stock price will be

\[
\tilde{S}_t = S_t \exp \left[ (\mu - \frac{\sigma^2}{2}) t + \sigma W_t \right]
\]

\[
= S_{t-1} \exp \left[ - (\mu - \frac{\sigma^2}{2}) (t - 1) - \sigma W_{t-1} \right] \times \exp \left[ (\mu - \frac{\sigma^2}{2}) t + \sigma W_t \right]
\]

\[
= W_t - W_{t-1} \sim W_{t-t} \sim N(0,1)
\]

Let \( \epsilon_t / \mathcal{F}_{t-1} \sim N(0,1) \) under \( \mathbb{P} \) then one period ahead stock price will be

\[
\tilde{S}_t = S_{t-1} \exp \left[ (\mu - \frac{\sigma^2}{2}) + \sigma \epsilon_t \right]
\]

Let \( \xi = \frac{\mu - r}{\sigma} \)

Then

\[
\tilde{S}_t = S_{t-1} \exp \left[ r + \sigma \xi - \frac{\sigma^2}{2} + \sigma \epsilon_t \right]
\]

According to Jin-Chuan Duan a log normally distributed stock price process with stochastic volatility under measure \( \mathbb{P} \)

\[
\ln \left( \frac{S_{t-1}}{S_t} \right) = r + \sigma_t \xi - \frac{\sigma_t^2}{2} + \sigma_t \epsilon_t \sim N(r + \sigma_t \xi - \frac{\sigma_t^2}{2}, \sigma_t^2)
\]

(4.1)

Equation (4.1) means the conditional variance is allowed to change over time, while keeping unconditional variance constant.

4.1. STOCHASTIC DISCOUNT FACTOR

Suppose that we observe a vector process \( z = z_t \) and let \( I_t \) denote the information available at time \( \tau \) that is, the \( \sigma \)-field generated by \( \{ z_s \ \text{s} \leq t \} \). We are interested in the pricing of a derivative whose payoff is \( g = g(z_T) \) at time \( T \). Suppose that there exists, at time \( t < T \), a price \( c_t(Z, g, T) \) for this asset. It can be shown that, \( g \rightarrow c_t(\mathcal{Z}, g, T) \), we have the representation

\[
c(g, t, T) = E[g(z_T) \mathcal{R}_{t, T} / I_t]
\]

The variable \( \mathcal{R}_{t, T} \) is called SDF for period \([t, T]\).

\[
\mathcal{R}_{t, T} = \mathcal{R}_{t, t+1}, \mathcal{R}_{t, t+2}, \ldots, \mathcal{R}_{T-1, T}
\]

On the other hand, the one-step SDFs are constrained by

\[
B(t, t+1) = E(R_{t+1, t+1} / I_t)
\]

\[
s_t = E \left( \frac{S_{t+1} \mathcal{R}_{t, t+1}}{I_t} \right)
\]

(4.2)
Consider the model

$$\ln \left( \frac{S_t}{S_{t-1}} \right) = \mu - \frac{\sigma_t^2}{2} + \sigma_t \varepsilon_t$$

With \( \varepsilon_t = \ln \left( \frac{S_t}{S_{t-1}} \right) \)

And suppose that \( B(t, t + 1) = e^{-r + \lambda} \), a simple specification of the one-step SDF is given by the affine model, \( R_{t, t+1} = \exp \{ a_t + b_t \varepsilon_{t+1} \} \), where \( a \) and \( b \) are constants.

Now consider a general GARCH type model of the form

$$\varepsilon_t = \log \left( \frac{S_t}{S_{t-1}} \right) = \mu_t + \varepsilon_t$$

Consider \( R_{t, t+1} = \exp \{ a_t + b_t \varepsilon_{t+1} \} \)

Equation (4.2) can be written as

$$e^{-(r+\lambda)} = E(\exp \{ a_t + b_t \varepsilon_{t+1} \} / \ln)$$

(4.3)

$$1 = E\exp\{a_t + b_t \varepsilon_{t+1} + \varepsilon_{t+1} / \ln\}$$

(4.4)

After simplification

$$\alpha_t = -(r + \lambda) - \frac{b_t^2}{2}, \beta_t \varepsilon_{t+1} = (r + \lambda) - \mu_{t+1} - \frac{\sigma_{t+1}^2}{2}$$

The risk natural probability is defined through characteristic function

$$E^Q (e^{u\varepsilon_{t+1} / \ln}) = E(e^{u\varepsilon_{t+1} \frac{R_{t, t+1}}{B(t, t+1)} / \ln})$$

Where \( z_t = \mu_t + \varepsilon_t \)

$$= E(e^{u(\varepsilon_t + (r+\lambda) + \mu_{t+1} + \beta_t \varepsilon_{t+1} + \varepsilon_{t+1} / \ln})$$

Using constraint \( \alpha_t \) and \( \beta_t \varepsilon_{t+1} \)

$$= \exp(u(\mu_{t+1} + \beta_t \varepsilon_{t+1}) + u^2 \frac{\sigma_{t+1}^2}{2})$$

$$= \exp(u \left((r + \lambda) - \frac{\sigma_{t+1}^2}{2}\right) + u^2 \frac{\sigma_{t+1}^2}{2})$$

The last two qualities are obtained by using \( \alpha_t \) and \( \beta_t \) thus under the probability the law of process \( z_t \) is given by the model

$$z_t = r - \frac{\sigma_t^2}{2} + \varepsilon_t^*$$

$$\varepsilon_t^* = \sigma_t \varepsilon^* \quad \varepsilon^* \sim N(0,1)$$

The model under the risk-neutral probability is then a GARCH-type model if the variable \( \sigma_t^2 \) is a measurable function of the past of \( \varepsilon_t^* \). This generally does not hold because the relation

$$\varepsilon_t^* = \mu_t - (r + \lambda) + \frac{\sigma_t^2}{2} + \varepsilon_t$$
The past of $\varepsilon_t^*$ is included in the past of $\varepsilon_t$ but not the reverse. A GARCH(1,1) model can be written as

$$\sigma_t^2 = \omega + \alpha((r + \lambda) - \frac{\sigma_{t-1}^2}{2} - \mu_{t-1} + \varepsilon_t^*) + \beta \sigma_{t-1}^2$$

5. CONCLUSIONS

Considering the famous Black-Scholes option pricing model in this article a more general case with jump has been studied. In BS-model volatility is a constant function but empirical studies reveal the fact that volatility is time varying and trading options are risky due to these random components. It has been proved that, call option of the more general case of BS-model with jump is a monotonous function in volatility. Further the Greeks of the model are also obtained. Finally extending this model for LRNVR and GARCH model for option pricing is discussed.

REFERENCES

ON CERTAIN FUNDAMENTAL PROPERTIES of HYPERGROUPS and FUZZY HYPERGROUPS – MIMIC FUZZY HYPERGROUPS

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ABSTRACT

In this paper, properties of hypergroups pertaining to the relationship between the hypercomposition and induced hypercompositions with the empty set are presented. Analogous fuzzy hypergroup properties are also proven. Finally, the study of these properties leads to the introduction of the mimic fuzzy hypergroup (fuzzyM-hypergroup).

KEYWORDS

hypergroup, fuzzy hypergroup.

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1. INTRODUCTION

Almost all fields of science desire to and attempt to utilize mathematical models in the process of not only describing the various phenomena under study, but also in predicting, wherever and whenever possible, the results generated through the influence of various causes. Several of the mathematical models thus utilized are non-deterministic, as several phenomena incorporate numerous uncertainties. Theory of probability, stochastic processes, fuzzy set theory, soft set theory, vague set theory, hypercompositional algebra, etc., are all different ways of expressing uncertainty. The self-contained development, as well as the cross-linking of the above theories hold great attraction for mathematicians. For example, A. Maturo, in a series of articles of his, (e.g. [39, 40, 41, 42] delved into linking the theory of probability with hypercompositional algebra. On the other hand, several mathematicians worked on linking fuzzy set theory with hypercompositional algebra.

One can distinguish three approaches, which were employed in order to connect these two topics. One approach is to consider a certain hyperoperation defined through a fuzzy set (P. Corsini [4], P. Corsini - V. Leoreanu, [6], I. Cristea e.g. [7, 8, 9], I. Cristea - S. Hoskova [10], M. Stefanescu - I. Cristea [49], K. Serafimidis et al. [48] etc.). Another is to consider fuzzy hyperstructures in a similar way as Rosenfeld did for fuzzy groups [46] (Zahedi, A.
Hasankhani [12, 54, 55], B. Davvaz [11] and others). The third approach is employed in the pioneering papers by P. Corsini - I. Tofan [5] and by I. Tofan - A. C. Volf [50, 51], which introduce fuzzy hyperoperations that induce fuzzy hypergroups. This approach was further adopted by other researchers (Ath. Kehagias e.g. [15, 16, 17, 18], V. Leoreanu-Fotea e.g. [22, 23] etc.). M. K. Sen, R. Ameri and G. Chowdhury utilized this concept in defining fuzzy hypersemigroups [47].

This paper deals with the algebra of hypergroups and fuzzy hypergroups. In researching all relevant bibliography, we came to realize that there exists a certain amount of confusion in some fundamental matters. For example, in attempting to define the hypergroup, some authors view the hyperoperation on a non-void set \( H \) as a function from \( H \times H \) to the power set \( P(H) \) of \( H \), while others view it as a function from \( H \times H \) to \( P^*(H) \), i.e. to the set of all non-empty subsets of \( H \). Certain fundamental properties of hypergroups and fuzzy hypergroups, which are proven herein, afford us a clearer view of such matters.

2. ON CRISP HYPERCOMPOSITIONS

Hypercompositional algebra was born in 1934, when F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the hypergroup [24]. The hypergroup is an algebraic structure in which the result of the composition of two elements is not an element but a set of elements. To make this paper self-contained, we begin by listing some definitions from the theory of hypercompositional structures (see also [35]). A (crisp) hypercomposition or hyperoperation in a non-empty set \( H \) is a function from \( H \times H \) to the power set \( P(H) \) of \( H \). A non-void set \( H \) endowed with a hypercomposition \( “\cdot” \) is called hypergroupoid if \( ab \neq \emptyset \) for any \( a,b \) in \( H \), otherwise it is called partial hypergroupoid. Note that, if \( A, B \) are subsets of \( H \), then \( AB \) signifies the union \( \bigcup \{ab \mid (a,b) \in A \times B\} \). Since \( A \times B = \emptyset \iff A = \emptyset \lor B = \emptyset \), one can observe that if \( A = \emptyset \) or \( B = \emptyset \), then \( AB = \emptyset \) and vice versa. \( aA \) and \( Aa \) have the same meaning as \( \{a\}A \) and \( A\{a\} \) respectively. Generally, the singleton \( \{a\} \) is identified with its member \( a \).

A hypergroup is a non-void set \( H \) endowed with a hypercomposition which satisfies the following axioms:

i. \((ab)c = a(bc)\) for every \( a,b,c \in H \) (associativity) and

ii. \(aH = Ha = H\) for every \( a \in H \) (reproduction).

If only (i) is valid, then the hypercompositional structure is called semi-hypergroup, while if only (ii) is valid, then it is called quasi-hypergroup. The quasi-hypergroups in which the weak associativity is valid, i.e. \((ab)c \cap a(bc) \neq \emptyset\) for every \( a,b,c \in H \), were named \( H_v \)-groups [52].

Remark. If a non-void set \( H \) is endowed with a composition which satisfies the
associative and the reproduction axioms, then \( H \) is a group. Indeed let \( x \in H \). Per reproduction \( x \in xH \). Therefore there exists \( e \in H \) such that \( xe = x \). Next let \( y \) be an arbitrary element in \( H \). Per reproduction there exists \( z \in H \) such that \( y = zx \). Consequently \( ye = (zx)e = z(xe) = zx = y \). Hence \( e \) is a right neutral element. Now, per reproduction \( e \in xH \). Thus there exists \( x' \in H \), such that \( e = xx' \). Hence any element in \( H \) has a right inverse.

Proposition 2.1. *The result of the hypercomposition of any two elements in a hypergroup is always non-void.*

**Proof.** Let \( H \) be a hypergroup and suppose that \( ab = \emptyset \) for some \( a, b \in H \). Per reproduction, \( aH = H \) and \( bH = H \). Hence, \( H = aH = a(bH) = (ab)H = \emptyset \), which is absurd.

Proposition 2.2. *If the weak associativity is valid in a hypercompositional structure, then the result of the hypercomposition of any two elements is always non-void.*

**Proof.** Let \( H \) be a non-void set endowed with a hypercomposition satisfying the weak associativity. Suppose that \( ab = \emptyset \) for some \( a, b \in H \). Then, \( (ab)c = \emptyset \) for any \( c \in H \). Therefore, \( (ab)c \cap a(bc) = \emptyset \), which is absurd. Hence, \( ab \) is non-void.

Corollary 2.1. *The result of the hypercomposition of any two elements in a \( H \_V \)-group is always a non-void set.*

F. Marty also defined in [24] the two induced hypercompositions (right and left division) resulting from the hypercomposition of the hypergroup, i.e.:

\[
\begin{align*}
\frac{a}{b} &= \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b} = \{x \in H \mid a \in bx\}.
\end{align*}
\]

It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, \( a/b \) or \( a : b \) is used to denote the right division (as well as the division in commutative hypergroups) and \( b\backslash a \) or \( a . b \) is used to denote the left division. F. Marty's life was cut short, as he was killed during a military mission in World War II. [24, 25, 26] are the only works on hypergroups he left behind. However, several relevant papers by other authors began appearing shortly thereafter (e.g. Krasner [19, 20], Kuntzmann [21] etc). Up to the present, a vast number of papers has been produced on this subject (e.g.: see [3, 6])

In [13] and then in [14], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition \( ab \) with the hypercomposition \( ba \). One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds true:
Given a theorem, the dual statement resulting from interchanging the order of hypercomposition “.” (and, by necessity, interchanging of the left and the right divisions), is also a theorem.

This principle is used throughout this paper. The following properties are direct consequences of the hypergroup axioms and the principle of duality is used in their proofs.

Proposition 2.3. \( a/b \neq \emptyset \) and \( b\setminus a \neq \emptyset \) for all the elements \( a, b \) of a quasi-hypergroup \( H \).

Proof. Per reproduction, \( Hb = H \) for every \( b \in H \). Hence, for every \( a \in H \) there exists \( x \in H \), such that \( a \in xb \). Thus, \( x \in a/b \) and, therefore, \( a/b \neq \emptyset \). Dually, \( b \setminus a \neq \emptyset \).

Proposition 2.4. In a quasi-hypergroup \( H \), the non-empty result of the induced hypercompositions is equivalent to the reproduction axiom.

Proof. Suppose that \( x/a \neq \emptyset \) for every \( a, x \in H \). Thus, there exists \( y \in H \), such that \( x \in ya \). Therefore, \( x \in Ha \) for every \( x \in H \) and so \( H \subseteq Ha \). Next, since \( Ha \subseteq H \) for every \( a \in H \), it follows that \( H = Ha \). Per duality, \( H = aH \). Conversely now, per Proposition 2.3, the reproduction axiom implies that \( a/b \neq \emptyset \) and that \( a \setminus b \neq \emptyset \) for every \( a, b \) in \( H \).

Based on Proposition 2.4, we are now in a position to give an equivalent definition of the hypergroup.

Definition 2.1. A hypergroup is a non-void (crisp) set \( H \) endowed with a (crisp) hypercomposition, i.e. a function from \( H \times H \) to the powerset \( P(H) \) of \( H \), which satisfies the following axioms:

i. \((ab)c = a(bc)\) for every \( a, b, c \in H \) (associativity) and
ii. \( a/b \neq \emptyset \) and \( b\setminus a \neq \emptyset \) for every \( a, b \in H \).

Proposition 2.5. In a hypergroup \( H \), equalities (i) \( H = H/a = a/H \) and (ii) \( H = a\setminus H = H\setminus a \) are valid for every \( a \) in \( H \).

Proof. (i) Per Proposition 2.1, the result of hypercomposition in \( H \) is always a non-empty set. Thus, for every \( x \in H \) there exists \( y \in H \), such that \( y \in xa \), which implies that \( x \in y/a \). Hence, \( H \subseteq H/a \). Moreover, \( H/a \subseteq H \). Therefore, \( H = H/a \). Next, let \( x \in H \). Since \( H = xH \), there exists \( y \in H \) such that \( a \in xy \), which implies that \( x \in a/y \). Hence, \( H \subseteq a/H \). Moreover, \( a/H \subseteq H \). Therefore, \( H = a/H \). (ii) follows by duality.

The hypergroup is a very general structure, which was progressively enriched with additional axioms, either more or less powerful. This created a significant number of specific hypergroups. Moreover, some of these hypergroups constituted a constructive origin for the development of other new hypercompositional structures (e.g.: see [3, 6, 27, 28, 29, 32, 34, 36, 43]). Thus, W. Prenowitz enriched hypergroups with an axiom, in order to utilize them in the study of geometry [e.g.: see 44, 45]. More precisely, he introduced into the commutative hypergroup the transposition axiom

\[ a/b \cap c/d \neq \emptyset \] implies \( ad \cap bc \neq \emptyset \) for every \( a, b, c, d \in H \)
and named this new hypergroup *join space* [44]. For the sake of terminology unification, join spaces are also called *join hypergroups* [30]. It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata [34, 37]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

\[ b \backslash a \cap c / d \neq \varnothing \text{ implies } ad \cap bc \neq \varnothing \text{ for every } a, b, c, d \in H . \]

He named this particular hypergroup *transposition hypergroup* [13]. Clearly, if \( A, B, C \) and \( D \) are subsets of \( H \), then \( B \backslash A \cap C / D \neq \varnothing \) implies that \( AD \cap BC \neq \varnothing \). In [31, 38] specialized *transposition hypergroups* were studied and, in [33], the transposition axiom was introduced into \( H_V \)-groups and the *transposition \( H_V \)-group* was thus defined.

### 3. ON FUZZY HYPERCOMPOSITIONS

Zadeh, in 1965, in order to provide «a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership, rather than the presence of random variables» [53] introduced the notion of fuzzy sets. If \( H \) is a non-void crisp set, then a fuzzy subset of \( H \) is a mapping from \( H \) to the interval of real numbers \([0,1]\). If \( A \subseteq H \), then the characteristic function \( X_A \) of \( A \)

\[ X_A : H \to [0,1], \quad x \mapsto X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \]

is a fuzzy subset of \( H \). Thus, if \( A = \varnothing \), then its characteristic function is the identically zero function \( 0_H \), i.e. \( X_{\varnothing}(x) = 0_H(x) = 0 \) for every \( x \in H \). Moreover, if \( A = H \), \( H \neq \varnothing \), then the characteristic function of the entire set \( H \), is \( X_H(x) = 1_H(x) = 1 \) for every \( x \in H \). Thus, we can consider crisp sets as special case of fuzzy sets and identify every set (crisp or fuzzy) with its membership function. The collection of all fuzzy subsets of \( H \) is denoted by \( F(H) \).

A *fuzzy hypercomposition* maps the pairs of elements of the Cartesian product \( H \times H \) to fuzzy subsets of \( H \), i.e. \( \circ : H \times H \to F(H) \). Hence, if \( \circ \) is a fuzzy hyperoperation, then \( a \circ b \) is a function and the notation \( (a \circ b)(x) \) means the value of \( a \circ b \) at the element \( x \). The definition of the fuzzy hyperoperation subsumes the relevant one of crisp hyperoperation as a special case, since the later results from the former using the characteristic function.

**Definition 3.1.** [17, 18] If \( \circ : H \times H \to F(H) \) is a fuzzy hypercomposition, then for every \( a \in H, b \in F(H) \), the fuzzy sets \( a \circ B \) and \( B \circ a \) are defined respectively by

\[
(a \circ B)(z) = \vee_{y \in H} \left( [ (a \circ y)(z) \land B(y) ] \right)
\]

\[
(B \circ a)(z) = \vee_{y \in H} \left( [ (y \circ a)(z) \land B(y) ] \right).
\]

Per definition 3.1, if \( a, b, c \in H \), then:
\[(a \circ (b \circ c))(z) = \vee_{y \in H} \left[ (a \circ y)(z) \land (b \circ c)(y) \right] \]

and

\[((a \circ b) \circ c)(z) = \vee_{y \in H} \left[ (y \circ c)(z) \land (a \circ b)(y) \right] \]

Definition 3.2. [17, 18] If \( \circ : H \times H \to F(H) \) is a fuzzy hypercomposition, then, for every \( A, B \in F(H) \), the fuzzy set \( A \circ B \) is defined by

\[(A \circ B)(z) = \vee_{x,y \in H} \left[ (x \circ y)(z) \right] \land A(x) \land B(y) \].

As mentioned above, these definitions subsume the relevant ones of crisp hyperoperations as special cases. For example, if \( X_A \) is the characteristic function of the crisp set \( A \), then, per Definition 2.1:

\[(x \circ X_A)(z) = \vee_{y \in H} \left[ (x \circ y)(z) \right] \land X_A(y) = \vee_{y \in H} X_A(y).\]

Hence \( x \circ X_A \) is the characteristic function of the crisp set \( xA = \cup_{y \in H} xA \) (see also [17]).

Definition 3.3. If \( a \in H \) and \( A, B \in P(H) \), then \( a \circ B = a \circ X_B, B \circ a = X_B \circ a \) and \( A \circ B = X_A \circ X_B \).

Definition 3.4. [5, 51] If \( \circ : H \times H \to F(H) \) is a fuzzy hypercomposition, then \( H \) is called fuzzy hypergroup, if the following two axioms are valid:

i. \( (a \circ b) \circ c = a \circ (b \circ c) \) for every \( a, b, c \in H \) (associativity),

ii. \( a \circ H = H \circ a = X_H \) for every \( a \in H \) (reproduction).

If (i) is only valid, then \( H \) is called a fuzzy semi-hypergroup [47] while if (ii) is only valid, then \( H \) is called a fuzzy quasi-hypergroup.

Lemma 3.1. For every \( a, b \in H \) and \( C \in F(H) \), the following is true:

\[(a \circ b) \circ C = a \circ (b \circ C).\]

Proof. \[
\begin{align*}
((a \circ b) \circ C)(z) &= \vee_{x,y \in H} \left[ (a \circ y)(z) \land (b \circ C)(x) \right] \\
= \vee_{x,y \in H} \left[ (y \circ x)(z) \land (a \circ b)(y) \right] \land C(x) \\
= \vee_{x,y \in H} \left[ (y \circ x)(z) \right] \land (a \circ b)(y) \right] \land C(x) \\
= \vee_{x \in H} \left[ ((a \circ b)(x))(z) \right] \land C(x) \\
= \vee_{y \in H} \left[ (a \circ y)(z) \right] \land (b \circ x)(y) \right] \land C(x) \\
= \vee_{y \in H} \left[ (a \circ y)(z) \right] \land (b \circ x)(y) \right] \land C(x) \\
= \vee_{x,y \in H} \left[ (a \circ y)(z) \right] \land (b \circ x)(y) \right] \land C(x)
\end{align*}
\]
\[\forall y \in H \left[ (a \circ y)(z) \land \left( \lor_{y \in H} \left( b \circ x \right) \right)(y) \land C(x) \right] =
\]
\[\forall y \in H \left[ (a \circ y)(z) \land (b \circ C)(y) \right] = \left[ a \circ (b \circ C) \right](z).
\]

Proposition 3.1. \( a \circ b \neq 0_H \) is valid for any pair of elements \( a, b \) in a fuzzy hypergroup \( H \).

**Proof.** Suppose that \( a \circ b = 0_H \) for some \( a, b \in H \). Per reproduction, \( a \circ H = X_H \) and \( b \circ H = X_H \). Hence, \( X_H = a \circ H = a \circ (b \circ H) \). Per Lemma 3.1, the equality \( a \circ (b \circ H) = (a \circ b) \circ H \) is valid. Since \( a \circ b = 0_H \), we have: \( (a \circ b) \circ H = 0_H \circ X_H \). But \( (0_H \circ X_H)(z) = \lor_{x,y \in H} \left[ \left( (x \circ y)(z) \right) \land 0_H(x) \land X_H(y) \right] = 0_H(z) \). Therefore, \( X_H = 0_H \), which is absurd because \( H \neq \emptyset \).

The **fuzzy \( H_v \)-groups** were defined in [15]. If \( A, B \in F(H) \) and \( p \in (0,1) \), then we write \( A \Box_p B \), if there exists \( x \in H \) such that \( A(x) \land B(x) \geq p \). The fuzzy quasi-hypergroups in which the weak associativity is valid, i.e. \( (a \circ b) \circ c \Box_p a \circ (b \circ c) \) for every \( a, b, c \in H \), are named **fuzzy \( H_v \)-groups**.

Proposition 3.2. If a fuzzy hypercompositional structure \( (H, \circ) \) is endowed with the weak associativity, then \( a \circ b \neq 0_H \) is valid for every \( a, b \in H \).

**Proof.** Suppose that \( a \circ b = 0_H \) for some \( a, b \in H \). Then \( (a \circ b) \circ c = 0_H \circ c \), for any \( c \in H \). Since \( (0_H \circ c)(z) = \lor_{x \in H} \left[ \left( (y \circ c)(z) \right) \land 0_H(y) \right] = 0 \), it follows that \( 0_H \circ c = 0_H \) and, therefore, \( (a \circ b) \circ c = 0_H \). Hence, the weak associativity is not valid in \( (H, \circ) \), which contradicts our supposition.

Corollary 3.1. The result of the hypercomposition of any two elements in a fuzzy \( H_v \)-group is always a non-zero function.

### 4. THE MIMIC FUZZY HYPERGROUP

If \( H \) is a non-void set endowed with a fuzzy hypercomposition \( \circ \), then two new induced fuzzy hypercompositions “/” and “\” can be defined as follows:

\[(a \div b)(x) = (x \circ b)(a) \text{ for every } a, b, x \in H \text{ and}
\]
\[(b \div a)(x) = (b \circ x)(a) \text{ for every } a, b, x \in H .
\]

As in the case of crisp hypercompositions, the two induced fuzzy hypercompositions will be called **fuzzy right division** and **fuzzy left division** respectively (see also [2]).
Proposition 4.1. For any pair of elements \(a, b\) in a fuzzy hypergroup \(H\), \(a / b \neq 0_H\) and \(a \setminus b \neq 0_H\) is valid.

**Proof.** Per reproduction, \(H \circ b = X_H\) is valid for every \(b \in H\). Thus, equality 
\[(H \circ b)(a) = X_H(a)\] 
is true for any \(a \in H\). Since 
\[(H \circ b)(a) = \bigvee_{y \in H} \left[ (y \circ b)(a) \land X_H(y) \right], \]
it follows that there exists \(y \in H\) such that \((y \circ b)(a) = 1\) or, equivalently, \((a / b)(y) = 1\). Therefore, \(a / b \neq 0_H\). Dually, \(a \setminus b \neq 0_H\).

It becomes obvious that a statement analogous to Proposition 2.4 is not valid in the case of fuzzy hypergroups. Indeed:

**Example 4.1.** Let \(H = \{a, b\}\) and suppose that
\[
(a \circ a)(a) \leq (a \circ a)(b) \leq (a \circ b)(a) \leq (b \circ b)(a)
\]
and
\[
(a \circ a)(b) = (a \circ b)(b) = (b \circ a)(a) = (b \circ a)(b) \leq (b \circ b)(a) \leq (b \circ b)(b)
\]
Then, \((H, \circ)\) is a fuzzy semi-hypergroup. Suppose that \((x \circ y)(z) \neq 0\) for every \(x, y, z \in H\). Then, in this fuzzy semi-hypergroup, \(a / b \neq 0_H\) and \(a \setminus b \neq 0_H\) is valid for every \(a, b \in H\). Yet, if \((x \circ y)(z) \neq 1\) for any \(x, y, z \in H\), one can easily see that the reproduction is not verified.

These ideas lead to the introduction of the following definition:

**Definition 4.1.** If \(\circ : H \times H \rightarrow F(H)\) is a fuzzy hypercomposition, then \(H\) is called mimic fuzzy hypergroup (fuzzy \(M\)-hypergroup), if the following two axioms are valid:

i. \((a \circ b) \circ c = a \circ (b \circ c)\) for every \(a, b, c \in H\) (associativity),

ii. \(a / b \neq 0_H\) and \(a \setminus b \neq 0_H\) for every \(a, b \in H\).

If (ii) is only valid, then \(H\) is called mimic fuzzy quasi-hypergroup (fuzzy \(M\)-quasi-hypergroup) while, if instead of (i) the weak associativity is valid, then \(H\) is called mimic fuzzy \(H_v\)-group (fuzzy \(M_{H_v}\)-group).

**Proposition 4.2.** In a fuzzy \(M\)-hypergroup \(H\), it holds that \((H \circ a)(x) \neq 0\) and \((a \circ H)(x) \neq 0\) for every \(a, x \in H\).

**Proof.** Since \(x / a \neq 0_H\) for every \(a, x \in H\), it follows that there exists \(y \in H\) such that \((x / a)(y) \neq 0\). Hence, there exists \(y \in H\) such that \((y \circ a)(x) \neq 0\), for any \(a, x \in H\). Since
\[(H \circ a)(x) = \vee_{y \in H} \left( (y \circ a)(x) \wedge X_H (y) \right) \], it follows that \((H \circ a)(x) \neq 0\) for every \(a, x \in H\).

Per duality, \((a \circ H)(x) \neq 0\) for every \(a, x \in H\).

Proposition 4.3. In a fuzzy\(^M\)-hypergroup \(H\), it holds that \(a \circ b \neq 0_H\) for every \(a, b \in H\).

Proof. Suppose that there are \(a, b \in H\) such that \(a \circ b = 0_H\). Then \((a \circ b) \circ H = 0_H\). Per Lemma 3.1, \((a \circ b) \circ H = a \circ (b \circ H)\). Hence \(a \circ (b \circ H) = 0_H\). But
\[
\left[ a \circ (b \circ H) \right](z) = \vee_{y \in H} \left[ (a \circ y)(z) \wedge (b \circ H)(y) \right]
\]
Per Proposition 4.2, \((b \circ H)(y) \neq 0\), for every \(b, y \in H\). Therefore \((a \circ y)(z) = 0\) for every \(z, y \in H\). Thus \((a \setminus z)(y) = 0\) for every \(y \in H\), which is absurd.

It is obvious that if a fuzzy\(^M\)-hypergroup \(H\) is commutative, then \(a \circ H = H \circ a\) for any \(a \in H\). However, generally speaking, this equality is not valid. Hence, we have the following definition:

Definition 4.2. A fuzzy\(^M\)-hypergroup \(H\) will be called commutable fuzzy\(^M\)-hypergroup, if \(a \circ H = H \circ a\) for any \(a \in H\).

Example 4.2. Let \(H = \{a, b\}\) and suppose that:

\[
\begin{align*}
(a \circ a)(a) &= 0.1; \quad (a \circ a)(b) = 0.2 \\
(b \circ a)(a) &= (a \circ b)(b) = (b \circ a)(b) = 0.2; \quad (a \circ b)(a) = 0.5 \\
(b \circ b)(a) &= 0.7; \quad (b \circ b)(b) = 0.9
\end{align*}
\]

Then \((H, \circ)\) is a non-commutative fuzzy\(^M\)-hypergroup, since \((a \circ b)(a) \neq (b \circ a)(a)\).

Furthermore \((H, \circ)\) is non-commutable. Next if we define:

\[
\begin{align*}
(a \cap a)(b) &= (b \cap a)(a) = 0.1 \\
(a \cap a)(a) &= (b \cap b)(a) = (b \cap b)(b) = (a \cap b)(a) = (a \cap b)(b) = (b \cap a)(b) = 0.9
\end{align*}
\]

then \((H, \cap)\) is a non-commutative fuzzy\(^M\)-hypergroup, since \((a \circ b)(a) \neq (b \circ a)(a)\).

However \((H, \cup)\) is commutable. Moreover if we define:

\[
\begin{align*}
(a \cdot a)(a) &= (b \cdot b)(a) = (a \cdot b)(a) = (b \cdot a)(a) = 0.1 \\
(a \cdot a)(b) &= (b \cdot b)(b) = (a \cdot b)(b) = (b \cdot a)(b) = 0.2
\end{align*}
\]

then \((H, \cdot)\) is a commutative fuzzy\(^M\)-hypergroup.

As in the case of fuzzy hypergroups [1, 2, 15, 16], the transposition axiom can be
introduced in fuzzyM-hypergroups as well. Thus, we have the definition:

Definition 4.3. A fuzzyM-hypergroup \( H \) will be called transposition fuzzyM-hypergroup, if for any \( a, b, c, d \in H \) for which there exists \( p \in (0,1] \) such that \( b \Delta a \triangledown_p c / d \), there exists also \( q \in (0,1] \) such that \( a \circ d \triangledown_q b \circ c \). If \( p = q \) for every \( a, b, c, d \in H \), then \( H \) will be called \( p \)-transposition fuzzyM-hypergroup.

REFERENCES


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