A variant of the linear isotropic indeterminate couple stress model with symmetric local force-stress, symmetric nonlocal force-stress, symmetric couple-stresses and orthogonal boundary conditions

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Dedicated to Richard Toupin, in deep admiration of his scientific achievements.

Abstract

In this paper we venture a new look at the linear isotropic indeterminate couple stress model in the general framework of second gradient elasticity and we propose a new alternative formulation which obeys Cauchy-Boltzmann’s axiom of the symmetry of the force stress tensor. For this model we prove the existence of solutions for the equilibrium problem. Relations with other gradient elastic theories and the possibility to switch from a 4th order (gradient elastic) problem to a 2nd order micromorphic model are also discussed with the view of obtaining symmetric force-stress tensors. It is shown that the indeterminate couple stress model can be written entirely with symmetric force-stress and symmetric couple-stress. The difference of the alternative models rests in specifying traction boundary conditions of either rotational type or strain type. If rotational type boundary conditions are used in the integration by parts, the classical anti-symmetric nonlocal force stress tensor formulation is obtained. Otherwise, the difference in both formulations is only a divergence–free second order stress field such that the field equations are the same, but the traction boundary conditions are different. For these results we employ an integrability condition, connecting the infinitesimal continuum rotation and the infinitesimal continuum strain. Moreover, we provide the orthogonal boundary conditions for both models.

Key words: symmetric Cauchy stresses, generalized continua, non-polar material, microstructure, size effects, microstrain model, non-smooth solutions, gradient elasticity, strain gradient elasticity, couple stresses, polar continua, hyperstresses, Boltzmann axiomatic, dipolar gradient model, modified couple stress model, conformal invariance, micro-randomness, symmetry of couple stress tensor, consistent traction boundary conditions.

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Contents

1 Introduction 3
  1.1 General viewpoint 3
  1.2 The linear indeterminate couple stress model 4
  1.3 Our perspective 6
  1.4 Notational agreements 8

2 Preliminaries 10
  2.1 Related models in isotropic second gradient elasticity 10
  2.2 Auxiliary results 14
  2.3 Discussion of invariance properties 17
  2.4 Conformal invariance of the curvature energy and group theoretic arguments in
  favour of the modified couple stress theory 17
  2.5 The classical indeterminate couple stress model based on
  $\|\nabla [axl (skew \nabla u)]\|^2$ with skew-symmetric nonlocal force-stress 19

3 The new isotropic gradient elasticity model with symmetric nonlocal force stress
and symmetric hyperstresses 21
  3.1 Formulation of the new boundary value problem 29
  3.2 Existence and uniqueness of the solution in the
  Curl (sym $\nabla u$)-formulation 30
  3.3 Traction boundary condition in the Curl (sym $\nabla u$)-formulation
  versus the $\nabla [axl (skew \nabla u)]$-formulation 34
  3.4 Principle of virtual work in the indeterminate couple stress model 36

4 Relation to the Cosserat-micropolar and micromorphic model 38

5 Conclusion 44

6 Epilogue: Much ado about nothing 48

References 48

A.1 The traction boundary conditions in the Curl (sym $\nabla u$)-formulation and in the
$\nabla [axl (skew \nabla u)]$-formulation are different 53
A.2 From second order couple stress tensors to third order moment stress tensors and
back 56
A.3 The name of the indeterminate couple stress model 57
1 Introduction

1.1 General viewpoint

The Cosserat model is an extended continuum model which features independent degrees of rotation in addition to the standard translational degrees of particles, see [28, 83, 81, 67, 65] for a detailed exposition. The prize, which has to be paid for this extension are non-symmetric force stress tensors together with so-called couple stress tensors which then represent the response of the model due to spatially differing Cosserat rotations. The couple stress model is the Cosserat model [48] with restricted rotations, i.e. in which the Cosserat rotations coincide with the continuum rotations. As such it belongs also to a certain subclass of gradient elasticity models\(^1\), where the higher derivatives only act on the continuum rotations. This constitutes a big conceptual advantage since the interpretation of the Cosserat rotations as new physical degrees of freedom is in general a difficult task. Such a model is also called a model with “latent microstructure” [11, 12].

Let \( F = RU \) be the polar decomposition of the deformation gradient \( F = \nabla \varphi \) into rotation \( R \in \text{SO}(3) \) and positive definite symmetric right stretch tensor \( U = \sqrt{F^T F} \), where \( \varphi : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \) characterizes the deformation of the material filling the domain \( \Omega \subseteq \mathbb{R}^3 \). We write \( R = \text{polar}(F) \).

In a variational context, the energy density \( W \) to be minimized in the geometrically nonlinear constrained Cosserat model is given by

\[
W = W\left(U - \mathbb{I}, \text{polar}(F)^T \nabla \text{polar}(F)\right),
\]

which reduced form follows from left-invariance of the Lagrangian \( W \) under superposed rotations. In this paper, our objectives are much more modest. We will only be concerned with the linearized variant of (1.1), which can be written as

\[
W = W\left(\text{sym} \nabla u, \nabla [\text{axl} (\text{skew} \nabla u)]\right) = W\text{lin}(\text{sym} \nabla u) + W\text{curv}(\nabla [\text{axl} (\text{skew} \nabla u)]),
\]

where \( u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \) is the displacement and

\[
\nabla \text{axl}(\text{skew} \nabla u) = 2 \text{curl} u.
\]

The energy density (1.2) is the classical Lagrangian for the indeterminate couple stress formulation. As will be seen later, this formulation leads naturally to totally skew-symmetric nonlocal force stress contributions.

Toupin already remarked on an alternative representation of the energy (1.1) [102, Section 6] which leads, in its linearized variant given by Mindlin [73, eq. (2.4)] to a dependence on

\[
W = W\left(\text{sym} \nabla u, \text{Curl} (\text{sym} \nabla u)\right) = W\text{lin}(\text{sym} \nabla u) + W\text{curv}(\text{Curl} (\text{sym} \nabla u))
\]

due to the equivalence

\[
\nabla (\text{axl}\text{skew} \nabla u) = \frac{1}{2} \nabla \text{curl} u = (\text{Curl} (\text{sym} \nabla u))^T
\]

\(^1\)Le Roux [96] seems to give for the first time a second gradient theory in linear elasticity using a variational formulation [66, 68].
instead of (1.2). The representation $\frac{1}{2} \nabla \text{curl} \, u$ is directly derived from the original Cosserat model [15, 98]. Both authors, Toupin and Mindlin, noted that now, comparing (1.2) and (1.4) the force stress tensors and the couple stress tensors are changed while the balance of linear momentum equation remains unchanged such that these concepts are not uniquely defined (see also Truesdell and Toupin’s remark on null-tensors [105, p. 547]. However, they apparently did not realize that it is possible to use this ambiguity to obtain completely symmetric force stress tensors also in the couple stress model which is otherwise the paragon for a model having non-symmetric force stress tensors. We also need to remark that in a purely mechanical context, the observation of size-effects does not necessitate to introduce non-symmetric stress-tensors [5].

In this paper we do not discuss in detail the field of applications of such a special format of gradient elasticity model. Suffice it to say that much attention is directed to nano-scaled material in which size-effects may become important, which may make the presented model applicable at strong stress gradients in the vicinity of cracks, or more generally, in highly heterogeneous media. We must also warn the reader: the indeterminate couple stress model is, in our view, a certain singular limit of the Cosserat model with independent displacements and micro-rotation and therefore some degenerate behaviour is to be expected throughout.

1.2 The linear indeterminate couple stress model

As hinted at above, the indeterminate couple stress model is a specific gradient elastic model in which the higher order interaction is restricted to the continuum rotation skew $\nabla u$ (or equivalently, $\text{curl} \, u$). It is therefore traditionally interpreted to include interactions of rotating particles and it is possible to prescribe boundary conditions of rotational type. Superficially, this is the simplest possible generalization of linear elasticity in order to include the gradient of the local continuum rotation as a source of stress and strain energy. In this paper, we limit our analysis to linear isotropic materials and only to the second gradient

$$D^2_X u = \frac{\partial^2 u_i}{\partial X_j \partial X_k} e_i \otimes e_j \otimes e_k = \varepsilon_{ji,k} + \varepsilon_{ki,j} - \varepsilon_{jk,i} e_i \otimes e_j \otimes e_k,$$

where $\varepsilon = \text{sym} \, \nabla u$. (1.5)

In general, the strain gradient models have the great advantage of simplicity and physical transparency since there are no new independent degree of freedoms introduced which would require interpretation. Since in this model there are no additional degrees of freedom (as compared to the Cosserat or micromorphic approach) the higher derivatives introduce a "latent-microstructure" (constrained microstructure). However, this apparent simplicity has to be payed with much more complicated traction boundary conditions, as will be seen later.

We will see in Section 4, surprisingly, that the mentioned rotational interaction can equivalently be viewed as a strain type interaction in the indeterminate couple stress model. Therefore, the first interpretation of rotational interaction (which is classical) is ambiguous as long as the problem is not specified together with boundary conditions appearing as a consequence of the kind of partial integration which is performed. We may choose, contrary to our intuition, another representation of the curvature energy motivated by formal considerations of invariance properties. In this regard

\footnote{There is such a formula, which says that all second derivatives of $u$ can be obtained from linear combinations of partial derivatives of strain, i.e. $D^2 u = \text{Lin}(\nabla \text{sym} \, \nabla u)$, $u_{i,j,k} = \varepsilon_{ji,k} + \varepsilon_{ki,j} - \varepsilon_{jk,i}$, where $\varepsilon = \text{sym} \, \nabla u$.}
we highlight the fact that force stresses for a material of higher order are far from being uniquely defined: it is always possible to add a self-equilibrated force field (divergence-free tensor field) changing the constitutive stress tensor but leaving unaltered the equilibrium equations [26, 74].

Often, such kind of generalized models introduce too many additional parameters (or too many additional artificial degrees of freedom) which are neither easily interpreted, nor easily to be determined from experiments. Our discussion may also be interpreted with the background to only include those higher order terms that are really required to describe the pertinent physics. It would be desirable that higher order models should not be more complicated than is warranted by experimental observation. A permanent nuisance in this respect is the question of how to identify new material parameters which are connected to the possible non-symmetry of the total force-stress tensor having the same dimensions as the classical shear modulus $\mu$ [N/mm$^2$]. In the Cosserat model the coupling parameter is the Cosserat-couple modulus $\mu_c$ [28, 82], which, for the indeterminate couple stress model considered here, is formally $\mu_c \to \infty$.

The Cauchy-Boltzmann axiom, well known from classical elasticity, requires the symmetry of the force stress tensor and may serve us also in the realm of this higher order theory to restrict the bewildering possibilities. Already Cauchy wrote [13, p. 344-345]:

"... les composantes $A, F, E; F, B, D; E, D, C$ des pressions supportées au point $P$ par trois plans parallèles aux plans coordonnés des $yz$, des $zx$ et des $xy$, pourront être généralement considérées comme des fonctions linéaires des déplacements $\xi, \eta, \zeta$ et des leurs dérivées des divers ordres."[3]

Truesdell and Toupin [105, p. 390] write:

"Theories of elastic materials of grade 2 or higher had been proposed by several authors [Cauchy [13], St. Venant [97], Jaramillo [47]], but under the assumption that the [total-force] stress tensor is symmetric".

Indeed, Jaramillo [47] considers a second gradient elastic material and obtains the dynamic equations by Hamilton’s principle. He observes dispersion relations in wave propagation problems. For simplicity only he restricts his discussion to those second gradient formulations, which give rise to a symmetric total force-stress tensor and obtains a classification for isotropic materials [47, p. 51, Eq. (96)]. The subject was pushed forward in the late 1950’s with works of Toupin [102, 103], Grioli [37, 38], Mindlin [70] and Koiter [51], among others, see the references later in this paper. Yang et al. [106] give an erroneous motivation for a symmetric moment stress tensor, as will be shown in [75]. Neff et al. [85] considered the singular stiffening behaviour for arbitrary small samples in the Cosserat and indeterminate couple stress model and concluded that in order to avoid these non-physical singular effects one has to take a symmetric moment stress, thus providing the first rational argument in favour of symmetric moment stresses. In [85] the same model$^4$ has been derived based on a homogenization procedure and a novel invariance requirement,

$^3$Our translation: The components of the symmetric total force stress tensor $A, F, E; F, B, D; E, D, C$ can be considered in general as linear functions $\xi, \eta, \zeta$ of the displacement and their derivatives of arbitrary order.

$^4$It must be noted that the grandmaster Koiter [51, p. 17-19, 23, 41] came to reject the significant presence of couple stresses because he based his investigations on the indeterminate couple stress theory with uniformly pointwise positive definite curvature energy, which tends to maximize the influence of length scale effects in its rotational formulation. His arguments only show that this special constrained gradient theory together with it’s boundary conditions cannot be based on experimental evidence. However, the main thrust of his comments remains
called micro-randomness is introduced by Neff et al. [48] and it has been shown that the model is well-posed.

1.3 Our perspective

Our contribution is intended to clarify and delineate under what boundary conditions we may expect or use symmetric nonlocal force stresses in the indeterminate couple stress model. When trying to relax the 4.th order problem (from gradient elasticity), it also seems expedient to retain the symmetry of the force stress tensor and of the moment stress tensor. Respecting symmetry restricts the possibilities to choose among 2.nd order micromorphic models. The importance of switching to a 2.nd order problem with new independent degrees of freedom is clear from the implementational point of view with finite elements: a 2.nd order problem is much easier and more efficient. However, given the antisymmetric classical and our new symmetric formulations we may arrive at completely different 2.nd order formulations in case of mixed displacement-traction boundary conditions.

In general, the hyperstress-tensor (couple stresses, sometimes called double-stress [66]) in second gradient elasticity [102, 70, 51, 100, 16, 17] (see also the recent papers [20, 21, 19, 107, 95, 6, 18, 29, 60, 27, 94, 25]) may be defined as

\[ m_{ijk} = D D^2 u W(D^2 u). \]

Since \( D^2 u = (u_{i,jk}) \) is a third order tensor, so is \( m \). Moreover, since \( u_{i,jk} \) is symmetric in \( (jk) \) the same is usually assumed for \( m_{ijk} \). This, however, is not mandatory, see [74].

In the framework considered in this paper, the hyperstress-tensor is defined as

\[ \tilde{m} := D_{\nabla(\text{curl } u)} W_{\text{curv}}(\nabla(\text{curl } u)) \quad \text{or} \quad \hat{m} := D_{\text{Curl}(\text{sym } \nabla u)} W_{\text{curv}}(\text{Curl}(\text{sym } \nabla u)), \]

respectively, and both expressions are 2.nd order tensors\(^5\) and are also called couple stress tensors, since they act as dual objects to gradients of rotations. On the other hand, as we will see, we have two competing expressions of the nonlocal force stress tensor: a symmetric tensor \( \hat{\tau} \) versus an anti-symmetric tensor \( \tilde{\tau} \):

\[
\hat{\tau} = \mu \frac{L^2}{2} \text{sym Curl} \{2 \alpha_1 \text{dev sym Curl} (\text{sym } \nabla u) + 2 \alpha_2 \text{skew Curl} (\text{sym } \nabla u)\} \\
= \text{sym Curl} (\tilde{m}) \in \text{Sym}(3)
\]

\[
\tilde{\tau} = \mu \frac{L^2}{2} \text{anti Div} \{2 \alpha_1 \text{dev sym} \nabla [\text{axl}(\text{skew } \nabla u)] + 2 \alpha_2 \text{skew} \nabla [\text{axl}(\text{skew } \nabla u)]\} \\
= \frac{1}{2} \text{anti Div}(\tilde{m}) \in \text{so}(3),
\]

with \( \text{Div}(\hat{\tau} - \tilde{\tau}) = 0 \). Since \( [\text{Curl} (\text{sym } \nabla u)]^T = \nabla [\text{axl}(\text{skew } \nabla u)] \), it follows

\[
\hat{\tau} - \tilde{\tau} = \mu L^2 \left\{ 2 \alpha_1 (\text{sym Curl} - \frac{1}{2} \text{anti Div}).[\text{dev sym Curl} (\text{sym } \nabla u)] + 2 \alpha_2 (\text{sym Curl} + \frac{1}{2} \text{anti Div}).[\text{skew Curl} (\text{sym } \nabla u)] \right\}.
\]

valid and our symmetric formulation may compare favorable. We should also have in mind that Mindlin ceased to use these models because he could finally not see the physical relevance at this time. Truesdell and Noll also wrote [104, p. 400]: “In favour of the Grioli-Toupin theory, in which the microrotation and macrorotation coincide, we can find no experimental evidence or theoretical advantage.”

\(^5\)See the Appendix for the relation between the second order tensor \( \tilde{m} \) and the third order tensor \( m \).
The independent constitutive variable $\tilde{k} := \nabla (\text{curl } u)$ is the second gradient contribution considered by Grioli [37], Toupin [102], Mindlin [73], Koiter [51] and Sokolowski [100]. In general, neither $\tilde{m}$ nor $\hat{m}$ couple stress tensors are symmetric.

The symmetry of the force stress tensor in continuum mechanics is regularly discussed in the literature, see e.g. [52, 69, 77]. It has been suggested by McLennan [69] that a symmetric force stress tensor can always be constructed by adding divergence-free couple stresses, since only its divergence occurs in the local conservation law. However, all of the previously given expositions use anti-symmetric nonlocal force-stresses. Since there is no conclusive evidence for the real need of a non-symmetric total force-stress tensor in the purely mechanical context, we apply Ockham’s razor and discard these non-symmetric force stress formulations. Our new alternative formulation will have symmetric couple stresses and symmetric force stresses. Thus it satisfies the Cauchy-Boltzmann’s axiom. We also show that the new formulation is well-posed in statics. While conceptually very pleasing, the real merits of such a “completely symmetric” formulation have yet to be discovered.

Similarly to the classical indeterminate couple stress model which can be obtained as a constrained Cosserat model, our new Curl (sym $\nabla u$)-model can be obtained as a constrained “microtrain” model [34, 33, 79].

The question of boundary conditions in higher gradient elasticity models has been a subject of constant attention. Bleustein has formulated the conclusive answer for general gradient elastic models involving the surface divergence operator [10]. However, this set of boundary conditions obtained by Tiersten and Bleustein in [101] with respect to the special case of the indeterminate couple stress model is not unique. In a forthcoming paper [62] we discuss the form of the boundary conditions considered until now in the classical indeterminate couple stress model [73, 102, 51, 85, 93, 4, 106, 92] and we propose a new set of what we call orthogonal boundary conditions. Here, we just provide the correct answer obtained there in the form of a summarizing box.

Let us consider a boundary value problem defined by the equilibrium equation in the open set $\Omega \subset \mathbb{R}^3$

$$\Box(D^2, D^4). u = 0,$$

where $\Box(D^2, D^4)$ is a differential operator involving second and fourth order derivative in $\Omega$, and the following boundary conditions

$$u \bigg|_{\Gamma_1} = u^{\text{ext}}, \quad \Lambda(D, n). u \bigg|_{\Gamma_2} = a^{\text{ext}},$$

$$G(D, D^2, n). u \bigg|_{\partial \Omega \setminus \Gamma_1} = t^{\text{ext}}, \quad H(D, D^2, n). u \bigg|_{\partial \Omega \setminus \Gamma_2} = h^{\text{ext}},$$

where $\Gamma_1, \Gamma_2$ are open subsets of $\partial \Omega$, $G$, $\Lambda$ and $H$ are operators on the boundary, while $t^{\text{ext}}$, $u^{\text{ext}}$, $g^{\text{ext}}$, $a^{\text{ext}} \in \mathbb{R}$ are given vector functions on $\partial \Omega$.

Let us assume that from the equilibrium equation (1.6) one obtains that a solution satisfies the conservation law

$$\int_{\Omega} W(D u, D^2 u) \, dv - \int_{\partial \Omega \setminus \Gamma_1} \langle t^{\text{ext}}, u \rangle \, da - \int_{\partial \Omega \setminus \Gamma_2} \langle h^{\text{ext}}, \Lambda(D, n). u \rangle \, da$$

$$- \int_{\Gamma_1} \langle G(D, D^2, n). u, u^{\text{ext}} \rangle \, da - \int_{\Gamma_2} \langle H(D, D^2, n). u, a^{\text{ext}} \rangle \, da = 0.$$

\footnote{The open subsets $\Gamma_1, \Gamma_2$ could be chosen also equal or $\Gamma_1 = \partial \Omega \subset \Gamma_2$.}
**Definition 1.1.** [Orthogonal boundary conditions] We say that the corresponding boundary conditions (1.7) are orthogonal if variations of \( u \) on \( \partial \Omega \) do not lead to changes of \( \Lambda(D,n)u \) on \( \partial \Omega \) and vice versa.

The plan of the paper is now as follows: after a subsection fixing the notation, we outline some related models in isotropic second gradient elasticity; we prove some auxiliary results and we discuss the invariance properties of the considered energy; we recall the classical indeterminate couple stress model with skew-symmetric nonlocal force-stress (i.e. with non symmetric total force-stress tensor); we formulate the equilibrium problem for the new isotropic gradient elasticity model with symmetric nonlocal force stress (i.e. with symmetric total force-stress tensor) and we give an existence result; we discuss the difference of the classical indeterminate couple stress model; paying particular attention to the boundary virtual work principle we show that these two possible formulations are applicable for different types of traction boundary conditions; we discuss the possibility to switch from a 4.th-order problem to a 2.nd order micromorphic model. All our existence results can be extended, mutatis mutandis, to first order anisotropic behaviour [60, 23, 61], i.e. considering as total energy \( \langle C \cdot \text{sym} \nabla u, \text{sym} \nabla u \rangle + W_{\text{curv}}(D^2u) \) as long as \( C \) is a uniformly positive definite tensor. We finish with some boxes summarizing our models and findings.

### 1.4 Notational agreements

In this paper, we denote by \( \mathbb{R}^{3 \times 3} \) the set of real \( 3 \times 3 \) second order tensors, written with capital letters. For \( a,b \in \mathbb{R}^3 \) we let \( \langle a,b \rangle_{\mathbb{R}^3} \) denote the scalar product on \( \mathbb{R}^3 \) with associated vector norm \( \|a\|_{\mathbb{R}^3}^2 = \langle a,a \rangle_{\mathbb{R}^3} \). The standard Euclidean scalar product on \( \mathbb{R}^{3 \times 3} \) is given by \( \langle X,Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(XY^T) \), and thus the Frobenius tensor norm is \( \|X\|^2 = \langle X,X \rangle_{\mathbb{R}^{3 \times 3}} \). In the following we omit the index \( \mathbb{R}^3, \mathbb{R}^{3 \times 3} \). The identity tensor on \( \mathbb{R}^{3 \times 3} \) will be denoted by \( \mathbb{1} \), so that \( \text{tr}(X) = \langle X, \mathbb{1} \rangle \).

We adopt the usual abbreviations of Lie-algebra theory, i.e., \( \mathfrak{so}(3) := \{ X \in \mathbb{R}^{3 \times 3} \mid X^T = -X \} \) is the Lie-algebra of skew symmetric tensors and \( \mathfrak{sl}(3) := \{ X \in \mathbb{R}^{3 \times 3} \mid \text{tr}(X) = 0 \} \) is the Lie-algebra of traceless tensors. For all \( X \in \mathbb{R}^{3 \times 3} \) we set \( \text{sym} X = \frac{1}{2}(X + X^T) \in \text{Sym}(3) \), skew \( X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3) \) and the deviatoric part \( \text{dev} X = X - \frac{1}{3} \text{tr}(X) \mathbb{1} \in \mathfrak{sl}(3) \) and we have the **orthogonal Cartan-decomposition of the Lie-algebra** \( \mathfrak{gl}(3) \)

\[
\mathfrak{gl}(3) = \{ \mathfrak{sl}(3) \cap \text{Sym}(3) \} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \cdot \mathbb{1}, \quad X = \text{dev} \text{sym} X + \text{skew} X + \frac{1}{3} \text{tr}(X) \mathbb{1}.
\]

Throughout this paper (when we do not specify else) Latin subscripts take the values 1,2,3. Typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate. We also use the Einstein notation of the sum over repeated indices if not differently specified. Here, for

\[
\mathfrak{T} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)
\]  

(1.10)
we consider the operators $axl : so(3) \rightarrow \mathbb{R}^3$ and $\text{anti} : \mathbb{R}^3 \rightarrow so(3)$ through

$$axl(\mathbf{A}) := (a_1, a_2, a_3)^T, \quad \mathbf{A}.v = (axl \mathbf{A}) \times v, \quad (\text{anti}(v))_{ij} = -\epsilon_{ijk} v_k, \quad \forall v \in \mathbb{R}^3,$$

$$\quad (axl \mathbf{A})_k = -\frac{1}{2} \epsilon_{ijk} \mathbf{A}_{ij} = \frac{1}{2} \epsilon_{kij} \mathbf{A}_{ji}, \quad \mathbf{A}_{ij} = -\epsilon_{ijk}(axl \mathbf{A})_k =: \text{anti}(axl \mathbf{A})_{ij}, \quad (1.11)$$

where $\epsilon_{ijk}$ is the totally antisymmetric third order permutation tensor. We recall that for a third order tensor $E$ and $X \in \mathbb{R}^{3 \times 3}$, $v \in \mathbb{R}^3$ we have the contraction operations $E : X \in \mathbb{R}^3$, $E.v \in \mathbb{R}^{3 \times 3}$ and $X.v \in \mathbb{R}^3$, with the components

$$\quad (E : X)_i = E_{ijk} X_{kj}, \quad (E.v)_{ij} = E_{ijk} v_k, \quad (X.v)_i = X_{ij} v_j. \quad (1.12)$$

For multiplication of two matrices we will not use other specific notations.

We consider a body which occupies a bounded open set $\Omega$ of the three-dimensional Euclidean space $\mathbb{R}^3$ and assume that its boundary $\partial \Omega$ is a piecewise smooth surface. An elastic material fills the domain $\Omega \subseteq \mathbb{R}^3$ and we refer the motion of the body to rectangular axes $Ox_i$. By $C_0^\infty(\Omega)$ we denote the set of infinitely differentiable functions with compact support in $\Omega$. In order to realize certain boundary conditions on an open subset $\Gamma \subseteq \partial \Omega$ we make use of the space [9] of functions that vanish in a neighborhood of $\Gamma$, i.e.

$$\quad C_0^\infty(\Omega, \Gamma) := \{ u \mid \exists v \in C_0^\infty(\mathbb{R}^n \setminus \Gamma) \text{ such that } v|_\Gamma = u \}.$$

Here, $\nu^-$ is a vector tangential to the surface $\partial \Omega \setminus \Gamma$ and which is orthogonal to its boundary

$$\partial(\partial \Omega \setminus \Gamma), \quad \tau^- = n \times \nu^- \text{ is the tangent to the curve } \partial(\partial \Omega \setminus \Gamma) \text{ with respect to the orientation on } \partial \Omega \setminus \Gamma. \quad \text{Similarly, } \nu^+ \text{ is a vector tangential to the surface } \Gamma \text{ and which is orthogonal to its boundary } \partial \Gamma, \tau^+ = n \times \nu^+ \text{ is the tangent to the curve } \partial \Gamma \text{ with respect to the orientation on } \Gamma. \quad \text{In the following, given any vector field } a \text{ defined on the boundary } \partial \Omega \text{ we will also set}
$$

$$\quad \| (a, \nu) \| := (a^+, \nu^+) + (a^-, \nu^-) = (a^+, \nu) - (a^-, \nu) = (|a|^+ - |a|^-, \nu), \quad (1.13)$$
which defines a measure of the jump of $a$ through the line $\partial \Gamma$, where $\nu := \nu^+ = -\nu^-$ and 

$$[\cdot]^- := \lim_{x \to \partial \Omega \setminus \Gamma} [\cdot], \quad [\cdot]^+ := \lim_{x \to \Gamma} [\cdot].$$

Since we have assumed $\partial \Omega$ is a smooth surface, there are no singularities of the boundary. The jump $[\cdot]$ arises only as consequence of possible discontinuities which follows from the prescribed boundary conditions on $\Gamma$ and $\partial \Omega \setminus \Gamma$.

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on $\Omega$ with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3 \times 3}$, respectively will be denoted by $L^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces [1, 36, 58]

$$\begin{align*}
    H^1(\Omega) &= \{ u \in L^2(\Omega) \mid \text{grad } u \in L^2(\Omega) \}, \\
    H(\text{curl}; \Omega) &= \{ v \in L^2(\Omega) \mid \text{curl } v \in L^2(\Omega) \}, \\
    H(\text{div}; \Omega) &= \{ v \in L^2(\Omega) \mid \text{div } v \in L^2(\Omega) \}, \\
    H_0(\text{curl}; \Omega) &= \{ v \in H(\text{curl}; \Omega) \mid [v] = 0 \text{ on } \partial \Omega \}, \\
    H_0(\text{div}; \Omega) &= \{ v \in H(\text{div}; \Omega) \mid [v] = 0 \text{ on } \partial \Omega \},
\end{align*}$$

of functions $u$ or vector fields $v$, respectively. Furthermore, we introduce their closed subspaces $H^1_0(\Omega)$, $H_0(\text{curl}; \Omega)$ as completion under the respective graph norms of the scalar valued space $C^\infty(\Omega)$. We also consider the spaces

$$H^1(\Omega; \Gamma), \quad H^1(\text{div}; \Omega; \Gamma), \quad H^1(\text{curl}; \Omega; \Gamma)$$

as completion under the respective graph norms of the scalar-valued space of the scalar-values space $C^\infty(\Omega, \Gamma)$. Therefore, these spaces generalize the homogeneous Dirichlet boundary conditions:

$$u \big|_{\Gamma} = 0, \quad \langle u, n \rangle_{\Gamma} = 0 \quad \text{and} \quad u \times n \big|_{\Gamma} = 0,$$

respectively. For vector fields $v$ with components in $H^1(\Omega)$, i.e. $v = (v_1, v_2, v_3)^T$, $v_3 \in H^1(\Omega)$, we define $\nabla v = \left( (\nabla v_1)^T, (\nabla v_2)^T, (\nabla v_3)^T \right)^T$, while for tensor fields $P$ with rows in $H(\text{curl}; \Omega)$, respectively $H(\text{div}; \Omega)$, i.e. $P = (P_1^T, P_2^T, P_3^T)$, $P_i \in H(\text{curl}; \Omega)$ respectively $P_i \in H(\text{div}; \Omega)$ we define $\text{Curl } P = ((\text{curl } P_1)^T, (\text{curl } P_2)^T, (\text{curl } P_3)^T)^T$, $\text{Div } P = (\text{div } P_1, \text{div } P_2, \text{div } P_3)^T$. The corresponding Sobolev-spaces will be denoted by

$$H^1(\Omega), \quad H^1(\text{Div}; \Omega), \quad H^1(\text{Curl}; \Omega), \quad H^1_0(\Omega; \Gamma), \quad H^1_0(\text{Div}; \Omega; \Gamma), \quad H^1_0(\text{Curl}; \Omega; \Gamma).$$

2 Preliminaries

2.1 Related models in isotropic second gradient elasticity

One aim of this paper is to propose a new representation of the curvature energy $W_{\text{curv}}(D^2u)$ and to prove that the corresponding minimization problem

$$I(u) = \int_{\Omega} \left[ \mu \|\text{sym } \nabla u\|^2 + \frac{\lambda}{2} [\text{tr} (\text{sym } \nabla u)]^2 + W_{\text{curv}}(D^2u) \right] dV \quad \rightarrow \quad \min \text{ w.r.t. } u, \quad (2.1)$$
admit unique minimizers under some appropriate boundary condition. Here $\lambda, \mu$ are the usual Lamé constitutive coefficients of isotropic linear elasticity, which is fundamental to small deformation gradient elasticity. If the curvature energy has the form $W_{\text{curv}}(D^2 u) = W_{\text{curv}}((D \text{ sym } \nabla u)$, the model is called a strain gradient model. We define the third order hyperstress as $D_{D^2 u} W_{\text{curv}}(D^2 u)$.

In the following we outline some curvature energies already proposed in different isotropic second gradient elastic models:

- Mindlin [70, 71, 73] considered energies (gradient elastic) based on the tensors

$$
\eta_{ijk} = u_{k,ij}, \quad \text{i.e. } \eta = \nabla(\nabla u), \\
\tilde{\eta}_{ijk} = \frac{1}{2} (u_{k,ji} + u_{j,ki}) = \varepsilon_{k,ij}, \quad \text{i.e. } \tilde{\eta} = \nabla(\text{sym } \nabla u), \\
\tilde{k}_{ij} = \frac{1}{2} \varepsilon_{jik} u_{k,li}, \quad \text{i.e. } \tilde{k} = \frac{1}{2} \nabla(\text{curl } u), \\
\eta^S_{ijk} = \frac{1}{3} (u_{k,ij} + u_{i,jk} + u_{j,ki}).
$$

The most general isotropic curvature energy defined in terms of $D^2 u$ has 5 material constants, while the anisotropic representation is much more complicated and still subject of ongoing research [72].

Mindlin and Eshel [72] have also proposed the following three alternative forms:

$$
W_{\text{curv}}(D^2 u) = \mu L_c^2 \left[ a_1^{(1)} \eta_{kii}^2 + a_2^{(1)} \eta_{ijjk} + a_3^{(1)} \eta_{j,jk} + a_4^{(1)} \eta_{j,jk}' + a_5^{(1)} \eta_{j,jk}'' \right] \quad \text{(I)} \\
= \mu L_c^2 \left[ a_1^{(2)} \eta_{kij}^2 + a_2^{(2)} \eta_{jijk} + a_3^{(2)} \eta_{ijk}'' + a_4^{(2)} \eta_{ijk} + a_5^{(2)} \eta_{ijk}'' \right] \quad \text{(II)} \\
= \mu L_c^2 \left[ a_1^{(3)} \eta_{ij}^2 + a_2^{(3)} \eta_{ji}^2 + a_3^{(3)} \eta_{ij}'' + a_4^{(3)} \eta_{ij} + a_5^{(3)} \eta_{ij}'' \right] \quad \text{(III)}
$$

which are frequently cited in the literature, where $L_c$ is the smallest characteristic length in the body and $a_1^{(j)}, a_2^{(j)}, a_3^{(j)}, a_4^{(j)}, a_5^{(j)}$ are dimensionless weighting parameters.

- a simple curvature energy is considered by Lam [53, 88]

$$
W_{\text{curv}}(D^2 u) = \mu L_c^2 \left[ a_0 \| \nabla \text{div } u \|_2^2 + a_1 \tilde{\eta}_{ijk} \tilde{\eta}_{ijk} + a_2 \| \text{sym } \nabla(\text{curl } u) \|_2^2 \right] \quad \text{(2.2)}
$$

$$
= \mu L_c^2 \left[ a_0 \| \nabla \text{tr}(\text{sym } \nabla u) \|_2^2 + a_1 \tilde{\eta}_{ijk} \tilde{\eta}_{ijk} + 4 a_2 \| \text{sym } \text{Curl}(\text{sym } \nabla u) \|_2^2 \right],
$$

where $\tilde{\eta}_{ijk}$ is called the deviatoric stretch gradient and which is defined (see e.g. [85]) by

$$
\tilde{\eta}_{ijk} = \eta_{ijk}^S - \eta_{ijk}^{(0)}, \quad \eta_{ijk}^{(0)} = \frac{1}{5} (\delta_{ij} \eta_{mkk}^S + \delta_{jk} \eta_{mmi}^S + \delta_{ki} \eta_{mmj}^S).
$$

- another simplified strain gradient elasticity model is proposed in [3, 56, 57] based on the curvature energy

$$
W_{\text{curv}}(D^2 u) = \mu L_c^2 \left[ a_0 \| \nabla \text{tr}(\text{sym } \nabla u) \|_2^2 + a_1 \| \nabla(\text{sym } \nabla u) \|_2^2 \right] \quad \text{(2.3)}
$$

$$
= \mu L_c^2 \left[ a_0 \| \nabla \text{div } u \|_2^2 + a_1 \| \nabla(\text{sym } \nabla u) \|_2^2 \right],
$$

which already leads to symmetric nonlocal force-stresses, see Section 3.
• in the same line, using also the second order curvature tensor $\tilde{k} = \frac{1}{2} \nabla \text{curl} u$, in [108, 50] the following energy is considered

$$
W_{\text{curv}}(D^2 u) = \mu L_c^2 \left[ a_0 \| \nabla \text{div} u \|^2 + a_1 \| \nabla \text{curl} u \|^2 \right]
$$

$$
= \mu L_c^2 \left[ a_0 \| \nabla \text{tr} (\text{sym} \nabla u) \|^2 + a_1 \| \nabla [\text{axl} (\text{skew} \nabla u)] \|^2 \right]
$$

$$
= \mu L_c^2 \left[ a_0 \| \nabla \text{tr} (\nabla u) \|^2 + a_1 \| \nabla (\text{skew} \nabla u) \|^2 \right].
$$

(2.4)

Let us remark that $\text{tr}(\tilde{k}) = \text{tr}(\nabla \text{axl} \text{skew} \nabla u) = \text{div}(\text{curl} u) = 0$.

• the indeterminate couple stress model (Grioli-Koiter-Mindlin-Toupin model) [37, 2, 51, 73, 103, 100, 39] in which the higher derivatives (apparently) appear only through derivatives of the infinitesimal continuum rotation $\text{curl} u$. Hence, the curvature energy has the equivalent forms

$$
W_{\text{curv}}(D^2 u) = \mu L_c^2 \left[ \frac{\alpha_1}{4} \| \text{sym} \nabla (\text{curl} u) \|^2 + \frac{\alpha_2}{4} \| \text{skew} \nabla (\text{curl} u) \|^2 \right]
$$

$$
= \mu L_c^2 \left[ \alpha_1 \| \text{sym} \nabla [\text{axl} (\text{skew} \nabla u)] \|^2 + \alpha_2 \| \text{skew} \nabla [\text{axl} (\text{skew} \nabla u)] \|^2 \right]
$$

$$
= \mu L_c^2 \left[ \frac{\alpha_1}{4} \| \text{dev} \text{sym} \nabla (\text{curl} u) \|^2 + \frac{\alpha_2}{4} \| \text{skew} \nabla (\text{curl} u) \|^2 \right]
$$

$$
= \mu L_c^2 \left[ \alpha_1 \| \text{sym} \text{Curl} (\text{sym} \nabla u) \|^2 + \alpha_2 \| \text{skew} \text{Curl} (\text{sym} \nabla u) \|^2 \right]
$$

$$
= \mu L_c^2 \left[ \alpha_1 \| \text{dev} \text{sym} \text{Curl} (\text{sym} \nabla u) \|^2 + \alpha_2 \| \text{skew} \text{Curl} (\text{sym} \nabla u) \|^2 \right]
$$

$$
= \frac{\mu L_c^2}{8} (\alpha_1 + \alpha_2) \left[ \| \nabla \text{curl} u \|^2 + \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \langle \nabla \text{curl} u, (\nabla \text{curl} u)^T \rangle \right].
$$

(2.5)

Note carefully that $\text{tr} [\text{sym} \text{Curl} (\text{sym} \nabla u)] = \text{tr}[\text{sym} \nabla [\text{axl} (\text{skew} \nabla u)]] = 0$. Therefore, we are entitled to use the deviatoric-representation, which is useful when regarding the model in the larger context of micromorphic models. Here, we have used the master identity to be established in Corollary 2.2

$$
\nabla [\text{axl} (\text{skew} \nabla u)] = \left[ \text{Curl} (\text{sym} \nabla u) \right]^T,
$$

which allows us easily to switch from considerations on the level of strain gradients to the level of rotational gradients and vice versa.

We also used the identities

$$
2 \text{axl} (\text{skew} \nabla u) = \text{curl} u, \quad \text{sym} \nabla (\text{curl} u) = 2 \text{sym} \text{Curl} (\text{sym} \nabla u),
$$

$$
\text{skew} \nabla (\text{curl} u) = -2 \text{skew} \text{Curl} (\text{sym} \nabla u), \quad \text{tr} [\text{Curl} (\text{sym} \nabla u)] = 0.
$$

Although this energy admits the equivalent forms (2.5) and (2.5), the equations and the boundary value problem of the indeterminate couple stress model is usually formulated only using the form (2.5) of the energy. Hence, we may reformulate the main aim of the present paper: to formulate the boundary value problem for the indeterminate couple stress...
model using the alternative form \((2.5)_5\) of the energy of the Grioli-Koiter-Mindlin-Toupin model. We also remark that the spherical part is \(\text{tr}(\nabla (\text{curl} \ u)) = \text{div}(\text{curl} \ u) = 0\). Note that pointwise uniform positivity is often assumed when deriving analytical solutions for simple boundary value problems because it allows to invert the couple stress-curvature relation. We will see subsequently, that pointwise positive definiteness is not necessary for well-posedness.

- In this setting, \textbf{Grioli} [37, 39] (see also Fleck [30, 31, 32]) initially considered only the choice \(\alpha_1 = \alpha_2\). In fact, the energy originally proposed by Grioli [37] is

\[
W_{\text{curv}}(D^2 u) = \mu L_c^2 \frac{\alpha_1}{4} \left[ \| \nabla (\text{curl} \ u) \|^2 + \eta \, \text{tr}(\nabla (\text{curl} \ u)^2) \right] = \mu L_c^2 \frac{\alpha_1}{4} \left[ \| \text{dev sym} \nabla (\text{axl}(\text{skew} \nabla u)) \|^2 + \| \text{skew} \nabla (\text{axl}(\text{skew} \nabla u)) \|^2 \right.
\]

\[
+ \eta \langle \nabla (\text{axl}(\text{skew} \nabla u)), (\nabla (\text{axl}(\text{skew} \nabla u))^T) \rangle \]  

\[
= \mu L_c^2 \frac{\alpha_1}{4} \left[ \| \text{dev sym} \nabla (\text{curl} \ u) \|^2 + \| \text{skew} \nabla (\text{curl} \ u) \|^2 \right.
\]

\[
+ \eta \langle \nabla (\text{curl} \ u), (\nabla (\text{curl} \ u))^T \rangle \]  

\[
= \mu L_c^2 \frac{\alpha_1}{4} \left[ (1 + \eta) \| \text{dev sym} \nabla (\text{curl} \ u) \|^2 + (1 - \eta) \| \text{skew} \nabla (\text{curl} \ u) \|^2 \right].
\]  

Mindlin [73, p. 425] (with \(\eta = 0\)) explained the relations between Toupin’s constitutive equations [102] and Grioli’s [37] constitutive equations and concluded that the obtained equations in the linearized theory are identical, since the extra constitutive parameter \(\eta\) of Grioli’s model does not explicitly appear in the equations of motion but enters only the boundary conditions, since \(\nabla \text{axl}(\text{skew} \nabla u) = [\text{Curl} (\text{sym} \nabla \nabla)]^T\), \(\text{Div Curl} (\cdot) = 0\), and

\[
\text{Div} \{\text{anti Div} \nabla \text{axl}(\text{skew} \nabla u)^T\} = \text{Div} \{\text{anti Div} [\text{Curl} (\text{sym} \nabla \nabla)]\} = \text{Div} \{\text{anti}(0)\} = 0.
\]

The same extra constitutive coefficient appears in Mindlin and Eshel’s (III) and Grioli’s version (2.6).

- \textbf{the modified - symmetric couple stress model - the conformal model}. On the other hand, in the conformal case [85, 84] one may consider that \(\alpha_2 = 0\), which makes the second order couple stress tensor \(\tilde{m}\) symmetric and trace free [17]. This conformal curvature case has been considered by Neff in [85], the curvature energy having the form

\[
W_{\text{curv}}(D^2 u) = \mu L_c^2 \frac{\alpha_1}{4} \| \text{sym} \nabla (\text{curl} \ u) \|^2 = \mu L_c^2 \frac{\alpha_1}{4} \| \text{dev sym} \text{Curl} (\text{sym} \nabla u) \|^2.
\]  

Indeed, there are two major reasons uncovered in [85] for using the modified couple stress model. First, in order to avoid singular stiffening behaviour for smaller and smaller samples in bending [83] one has to take \(\alpha_2 = 0\). Second, based on a homogenization procedure invoking an intuitively appealing natural “micro-randomness” assumption (a strong statement of microstructural isotropy) requires conformal invariance, which is again equivalent to \(\alpha_2 = 0\). Such a model is still well-posed [48], leading to existence and uniqueness results with only...
one additional material length scale parameter, while it is not pointwise uniformly positive definite.

- the skew-symmetric couple stress model. Hadjesfandiari and Dargush strongly advocate [42, 43, 44] the opposite extreme case, \( \alpha_1 = 0 \) and \( \alpha_2 > 0 \), i.e. they propose the curvature energy

\[
W_{\text{curv}}(D^2u) = \mu L_c^2 \alpha_2 \frac{\alpha_2}{4} \| \text{skew} \nabla (\text{curl} u) \|^2 = \mu L_c^2 \alpha_2 \frac{\alpha_2}{2} \| \text{axl skew} \nabla (\text{curl} u) \|^2 \\
= \mu L_c^2 \alpha_2 \frac{\alpha_2}{8} \| \text{curl} (\text{curl} u) \|^2 = \mu L_c^2 \alpha_2 \| \text{skew Curl} (\text{sym} \nabla u) \|^2.
\]

In that model the nonlocal force stresses and the couple stresses are both assumed to be skew-symmetric. Their reasoning, based in fact on an incomplete understanding of boundary conditions (see [62]) is critically discussed and generally refuted in [87], while mathematically it is also well-posed.

2.2 Auxiliary results

Further on, we consider a simply connected domain \( \Omega \subseteq \mathbb{R}^{3\times 3} \). The starting point is given by the well-known Nye’s formula [90, 86]

\[
- \text{Curl} \ A = (\nabla \text{axl} A)^T - \text{tr}[(\nabla \text{axl} A)^T] \mathbb{1},
\]

(2.9)

\[\pi := \nabla(\text{axl} A) = -(\text{Curl} \ A)^T + \frac{1}{2} \text{tr}[(\text{Curl} \ A)^T] \mathbb{1} = \alpha^T - \frac{1}{2} \text{tr}[\alpha] \mathbb{1} \quad \text{“Nye’s curvature tensor”},\]

for all skew-symmetric matrices \( A \in \mathfrak{so}(3) \), where \( \alpha := - \text{Curl} \ A \) is the micro-dislocation density tensor.

**Proposition 2.1.** Let \( p : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^{3\times 3} \) be given. The formula

\[
\nabla[\text{axl skew} \ p] = [\text{Curl} (\text{sym} \ p)]^T
\]

(2.10)

holds true if and only if there is \( u \in C^2(\Omega) \) such that \( p = \nabla u \).

**Proof.** Let us first prove that

\[
\nabla[\text{axl}(\text{skew} \nabla u)] = [\text{Curl} (\text{sym} \nabla u)]^T, \quad \text{for all} \quad u \in C^2(\Omega).
\]

(2.11)

On the one hand, using Nye’s formula for \( A = \text{skew} \nabla u \), we obtain

\[
- \text{Curl} \ (\text{skew} \nabla u) = (\nabla[\text{axl}(\text{skew} \nabla u)])^T - \text{tr}[(\nabla[\text{axl}(\text{skew} \nabla u)])^T] \mathbb{1},
\]

(2.12)

which implies

\[
- \text{Curl} \ (\text{skew} \nabla u) = (\nabla[\text{axl}(\text{skew} \nabla u)])^T - \frac{1}{2} \text{tr}[(\nabla(\text{curl} u))^T] \mathbb{1} \\
= (\nabla[\text{axl}(\text{skew} \nabla u)])^T - \frac{1}{2} \text{div} (\text{curl} u) \mathbb{1} = (\nabla[\text{axl}(\text{skew} \nabla u)])^T.
\]

(2.13)
On the other hand, \( \text{Curl} (\nabla u) = 0 \), \( \nabla u = \text{sym} \nabla u + \text{skew} \nabla u \). Thus, we deduce

\[
\text{Curl} (\text{sym} \nabla u) = \text{Curl} (\nabla u - \text{skew} \nabla u) = \text{Curl} (\nabla u) - \text{Curl} (\text{skew} \nabla u) \tag{2.14}
\]

\[
\overset{(2.13)}{=} \text{Curl} (\nabla u) + (\nabla \text{axl} (\text{skew} \nabla u))^T - \frac{1}{2} \text{div} (\text{curl} u) 1 = (\nabla \text{axl} (\text{skew} \nabla u))^T. \tag{2.13}
\]

This establishes the first part of the claim.

Now, we prove that \( \nabla \text{axl} (\text{skew} p) = [\text{Curl} (\text{sym} p)]^T \) implies that there is a function \( u \in C^2(\Omega) \) such that \( p = \nabla u \). Using again Nye’s formula, we obtain

\[
(\nabla \text{axl} (\text{skew} p))^T (\text{Nye}) = - (\text{Curl} (\text{skew} p)) + \text{tr} [\nabla \text{axl} (\text{skew} p)]. \tag{2.15}
\]

Hence, our new hypothesis is \( \text{Curl} (\text{sym} p) = (\nabla \text{axl} (\text{skew} p))^T \), which implies

\[
\text{Curl} (\text{sym} p) = - (\text{Curl} \text{ skew} p) + \text{tr} [\nabla \text{axl} (\text{skew} p)] 1. \tag{2.15}
\]

Hence, we obtain

\[
\text{Curl} (\text{sym} p + \text{skew} p) = \text{tr} [\nabla \text{axl} (\text{skew} p)] 1 \quad \Leftrightarrow \quad \text{Curl} (p) = \text{tr} [\nabla \text{axl} (\text{skew} p)] 1, \tag{2.16}
\]

or, in the equivalent form

\[
\text{Curl} (p) = \text{div} (\text{axl} \text{ skew} p) 1. \tag{2.17}
\]

We have obtained the formula

\[
\text{tr} [\text{Curl} p] = 3 \text{div} (\text{axl} \text{ skew} p). \tag{2.18}
\]

Let us also remark that considering a matrix \( B \in \mathbb{R}^{3\times3} \), we have

\[
\text{tr} [\text{Curl} (\text{skew} B)] = 2(b_{1,1} + b_{2,2} + b_{3,3}) = 2 \text{div} b, \quad b = \text{axl} (\text{skew} B). \tag{2.19}
\]

Therefore, from (2.19) we also have obtained

\[
\text{tr} [\text{Curl} (\text{skew} p)] = 2 \text{div} [\text{axl} (\text{skew} p)]. \tag{2.20}
\]

Moreover, for a matrix \( B \in \mathbb{R}^{3\times3} \), we have that

\[
\text{tr} [\text{Curl} B] = (B_{1,3,2} - B_{3,1,2}) + (B_{2,1,3} - B_{2,1,3}) + (B_{3,2,1} - B_{2,3,1}).
\]

We deduce \( \text{tr} [\text{Curl} S] = 0 \) for all \( S \in \text{Sym}(3) \). Hence,

\[
\text{tr} [\text{Curl} (\text{sym} p)] = 0 \quad \forall p \in \mathbb{R}^{3\times3}. \tag{2.21}
\]

The relations (2.20) and (2.21) lead to \( \text{tr} [\text{Curl} p] = 2 \text{div} [\text{axl} (\text{skew} p)] \), and together with (2.18) to

\[
\text{div} [\text{axl} (\text{skew} p)] = 0. \tag{2.22}
\]

Using (2.17), we obtain \( \text{Curl} p = 0 \). Since \( \Omega \) is an open domain in \( \mathbb{R}^3 \), it follows that there is a vector \( u \), such that \( p = \nabla u \) and the proof is complete. \( \Box \)
Corollary 2.2. For $u \in C^2(\Omega)$ the following formula holds true
\[
\nabla [axl (\text{skew} \nabla u)] = [\text{Curl} \ (\text{sym} \nabla u)]^T. \tag{2.23}
\]

Corollary 2.3. For $u \in C^2(\Omega)$ the following formula holds true
\[
[\nabla \text{curl} u]^T = \text{Curl} \ [(\nabla u)^T]. \tag{2.24}
\]

Therefore, $(\nabla u)^T \in H(\text{Curl}; \Omega)$ is equivalent to $\text{curl} u \in H^1(\Omega)$.

As consequence of the above remark, it follows that if $\text{curl} u \in H^1(\Omega)$, then $(\nabla u)^T \cdot \tau \in L^2(\partial \Omega)$ for any tangential direction $\tau$ at the boundary and, since $\langle (\nabla u)^T \cdot \tau, n \rangle = \langle \tau, (\nabla u) \cdot n \rangle$, it results that $(1 - n \otimes n) (\nabla u) \cdot n \in L^2(\partial \Omega)$, in the sense of trace.

Let us also recall the Saint-Venant compatibility condition

Proposition 2.4. (see e.g. [14]) Let a symmetric tensor field $\hat{\varepsilon} : \Omega \subseteq \mathbb{R}^3 \to \text{Sym}(3)$ be given. Then,
\[
\text{inc}(\varepsilon) := \text{Curl} \ [(\text{Curl} \hat{\varepsilon})^T] = 0 \iff \text{there is } u \in C^2(\Omega) \text{ such that } \hat{\varepsilon} = \text{sym}(\nabla u). \tag{2.25}
\]

We note that
\[
\text{inc}(p) := \text{Curl} \ [(\text{Curl} \text{sym} p)^T] \in \text{Sym}(3) \tag{2.26}
\]
while $\text{Curl} \ [(\text{Curl} \text{skew} p)^T] \in \mathfrak{so}(3) \quad \forall \ p \in \mathbb{R}^{3 \times 3}$.

We also remark that a direct consequence of Proposition 2.1 is the following first order compatibility condition

Proposition 2.5. Let $p : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$ be given. Then,
\[
\text{INC}(p) := [\text{Curl} \ (\text{sym} p)]^T - \nabla [axl \text{skew} p] = 0 \iff \text{there is } u \in C^2(\Omega) \text{ such that } p = \nabla u. \tag{2.27}
\]

We observe that
\[
\text{INC}(p) \in \mathbb{R}^{3 \times 3} \quad \forall \ p \in \mathbb{R}^{3 \times 3}.
\]
We recall the well known first order compatibility condition

Proposition 2.6. Let $p : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$ be given. Then,
\[
\text{Curl} \ p = 0 \iff \text{there is } u \in C^2(\Omega) \text{ such that } p = \nabla u. \tag{2.28}
\]

Hence, we have the following equivalence

Corollary 2.7. Let $p : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$ be given. Then,
\[
\text{Curl} \ p = 0 \iff \text{INC}(p) = 0. \tag{2.29}
\]

We finally remark that
\[
\text{Curl} \ (	ext{INC}(p)) = \text{Curl} \ [(\text{Curl} \text{sym} p)^T] = \text{inc}(\text{sym} p). \tag{2.30}
\]
2.3 Discussion of invariance properties

The difference between the $\nabla [\text{axl} (\text{skew} \nabla u)]$ formulation and the $\text{Curl} (\text{sym} \nabla u)$ formulation can be seen when considering the results under superposed incompatible tensor fields:

**Remark 2.8.** The quantity $[\text{Curl} (\text{sym} \nabla u)]^T$ is invariant under locally adding a skew-symmetric non-constant tensor field $W(x) \in \mathfrak{s}o(3)$, i.e.

$$[\text{Curl} (\text{sym} (\nabla u + W(x)))]^T = [\text{Curl} (\text{sym} \nabla u)]^T.$$  \hfill (2.31)

However, since $\nabla [\text{axl} \text{skew} p] \neq [\text{Curl} (\text{sym} p)]^T$ for general incompatible $p \in \mathbb{R}^{3\times3}$, $p \neq \nabla u$, the quantity $\nabla [\text{axl} (\text{skew} \nabla u)]$ is not invariant under locally adding $W(x) \in \mathfrak{s}o(3)$, i.e.

$$\nabla [\text{axl} (\text{skew} (\nabla u + W(x)))] \neq \nabla [\text{axl} (\text{skew} \nabla u)].$$  \hfill (2.32)

**Remark 2.9.** The term $\nabla [\text{axl} (\text{skew} \nabla u)]$ is invariant under locally adding a symmetric, non-constant tensor field $S(x) \in \text{Sym}(3)$, i.e.

$$\nabla [\text{axl} (\text{skew} (\nabla u + S(x)))] = \nabla [\text{axl} (\text{skew} \nabla u)].$$  \hfill (2.33)

Let us recall the Lie-group decomposition $GL^+(3)$ and the corresponding Lie-algebra decomposition:

$$GL^+(3) = \{\text{SL}(3)/\text{SO}(3)\} \cdot \text{SO}(3) \cdot (\mathbb{R}^+ \cdot 1) \quad \text{Lie-group decomposition},$$

$$T^*_3 GL^+(3) = \mathbb{R}^{3\times3} = \mathfrak{gl}(3) = \{\mathfrak{sl}(3) \cap \text{Sym}(3)\} \oplus \mathfrak{s}o(3) \oplus \mathbb{R} \cdot 1 \quad \text{Lie-algebra decomposition}. \hfill (2.34)$$

The space $\text{Sym}(3)$ is not a Lie-algebra, it is only a vector space and it does not have a group structure: the set $\text{GL}(3)/\text{SO}(3) = \text{P} \text{Sym}(3)$ is not a group, neither is the set $\mathfrak{sl}(3) \cap \text{Sym}(3)$ a Lie-algebra. Hence, the invariance requirement in (2.31), i.e., locally adding $W(x) \in \mathfrak{s}o(3)$ is much more plausible than assuming (2.33) since it yields $\mathfrak{s}o(3)$-Lie invariance.

2.4 Conformal invariance of the curvature energy and group theoretic arguments in favour of the modified couple stress theory

An infinitesimal conformal mapping \cite{82, 85} preserves (to first order) angles and shapes of infinitesimal figures. The included inhomogeneity is therefore only a global feature of the mapping (see Figure 2). There is locally no shear-type deformation. Therefore it seems natural to require that the second gradient model should not ascribe energy to such deformation modes.

A map $\phi_c : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is infinitesimal conformal if and only if its Jacobian satisfies pointwise $\nabla \phi_c(x) \in \mathbb{R} \cdot 1 + \mathfrak{s}o(3)$, where $\mathbb{R} \cdot 1 + \mathfrak{s}o(3)$ is the conformal Lie-algebra. This implies \cite{82, 85, 83} the representation (see Figure 2)

$$\phi_c(x) = \frac{1}{2} (2 \langle \text{axl} \mathbf{W}, x \rangle x - \text{axl} \mathbf{W} \|x\|^2) + [\hat{p} \mathbf{1} + \hat{A}] \cdot x + \hat{b}, \hfill (2.35)$$

$$\text{axl} (\nabla u + S(x)) = \text{axl} (\nabla u).$$
where $\mathbb{W}, \hat{A} \in \mathfrak{so}(3)$, $\hat{b} \in \mathbb{R}^3$, $\hat{p} \in \mathbb{R}$ are arbitrary given constants. For the infinitesimal conformal mapping $\phi_c$ we note

\[
\nabla \phi_c(x) = [(\text{axl} \mathbb{W}, x) + \hat{p}] \mathbb{1} + \text{anti}(\mathbb{W}, x) + \hat{A}, \quad \text{skew} \nabla \phi_c(x) = \text{anti}(\mathbb{W}, x) + \hat{A}, \\
\text{div} \phi_c(x) = \text{tr}[\nabla \phi_c(x)] = 3 [(\text{axl} \mathbb{W}, x) + \hat{p}], \quad \text{sym} \nabla \phi_c(x) = [(\text{axl} \mathbb{W}, x) + \hat{p}] \mathbb{1}, \\
\text{dev} \text{sym} \nabla \phi_c(x) = 0, \quad \nabla \text{curl} \phi_c(x) = 2 \mathbb{W} \in \mathfrak{so}(3), \\
\text{sym} \nabla \text{curl} \phi_c(x) = 0, \quad \text{skew} \nabla \text{curl} \phi_c(x) = 2 \mathbb{W}.
\]

(2.36)

These relations are easily established. By \textbf{conformal invariance} of the curvature energy term we mean that the curvature energy vanishes on infinitesimal conformal mappings. This is equivalent to

\[
W_{\text{curv}}(D^2 \phi_c) = 0 \quad \text{for all conformal maps} \quad \phi_c,
\]

(2.37)

or in terms of the second order couple stress tensor $\tilde{m} := D \nabla \text{curl} \nabla \text{curl} u$,

\[
\tilde{m}(D^2 \phi_c) = 0 \quad \text{for all conformal maps} \quad \phi_c.
\]

(2.38)

The classical linear elastic energy still ascribes energy to such a deformation mode, but only related to the bulk modulus, i.e.,

\[
W_{\text{lin}}(\nabla \phi_c) = \mu \| \text{dev} \text{sym} \nabla \phi_c \|^2 + \frac{2\mu + 3\lambda}{2} \| \text{tr}(\nabla \phi_c) \|^2 = \frac{2\mu + 3\lambda}{2} \| \text{tr}(\nabla \phi_c) \|^2.
\]

(2.39)

In case of a classical infinitesimal perfect plasticity formulation with von Mises deviatoric flow rule, conformal mappings are precisely those inhomogeneous mappings, that do not lead to plastic flow \cite{78}, since the deviatoric stresses remain zero: \text{dev sym} $\nabla \phi_c = 0$.

In that perspective

\textit{\textbf{conformal mappings are ideally elastic transformations and should not lead to moment stresses.}}

Using the formulas (2.36), it can be easily remarked that $\| \nabla [\text{dev sym} \nabla u] \|^2$, $\| \text{dev sym} \nabla u \|^2$, $\| \text{dev sym} \nabla (\text{curl} u) \|^2$, $\| \text{sym} \text{Curl} (\text{sym} \nabla u) \|^2 = \frac{1}{4} \| \text{sym} \nabla (\text{curl} u) \|^2$ are conformally invariant. Let us note that, using Lemma 2.1, we have

\[
\| \text{sym} \text{Curl} (\text{sym} \nabla u) \|^2 = \| \text{sym} \nabla [\text{axl} (\text{skew} \nabla u)] \|^2 = \frac{1}{4} \| \text{sym} \nabla (\text{curl} u) \|^2, \\
\| \text{skew} [\text{Curl} (\text{sym} \nabla u)] \|^2 = \| \text{skew} [\text{axl} (\text{skew} \nabla u)] \|^2 = \frac{1}{4} \| \text{skew} \nabla (\text{curl} u) \|^2.
\]

(2.40)

(2.41)

Hence $\| \text{sym} \text{Curl} (\text{sym} \nabla u) \|^2 = \frac{1}{4} \| \text{sym} \nabla (\text{curl} u) \|^2$ is also conformally invariant (use (2.36)$_6$), while

\[
\| \text{skew} [\text{Curl} (\text{sym} \nabla \phi_c)] \|^2 = \frac{1}{4} \| \text{skew} \nabla (\text{curl} \phi_c) \|^2 = \| \mathbb{W} \|^2,
\]

(2.42)
and therefore $\|\text{Curl}(\text{sym } \nabla u)\|^2$ is not conformally invariant, nor is $\|\nabla(\text{axl}(\text{skew } \nabla u))\|^2$ conformally invariant. Nor is $\|\nabla \text{tr}(\text{sym } \nabla u)\|^2 = \|\nabla \text{div } u\|^2$ conformally invariant.

The underlying additional invariance property of the modified couple stress theory is precisely conformal invariance. In the modified couple stress model, these deformations are free of size-effects, while e.g. the Hadjesfandiari and Dargush choice would describe size-effects. In other words, the generated couple stress tensor $\tilde{m}$ in the modified couple stress model is zero for this inhomogeneous deformation mode, while in the Hadjesfandiari and Dargush choice $\tilde{m}$ is constant and skew-symmetric.

2.5 The classical indeterminate couple stress model based on $\|\nabla[\text{axl}(\text{skew } \nabla u)]\|^2$ with skew-symmetric nonlocal force-stress

We are now re-deriving the classical equations based on the $\nabla[\text{axl}(\text{skew } \nabla u)]$-formulation of the indeterminate couple stress model. This part does not contain new results, see, e.g., [62] for further details, but is included for setting the stage of our new modelling approach.

Taking free variations $\delta u \in C^\infty(\Omega)$ in the energy $W(e, k) = W_{\text{lin}}(e) + W_{\text{curv}}(k)$, but using the following equivalent curvature energy based on $\tilde{k} = \nabla[\text{axl}(\text{skew } \nabla u)] = \frac{1}{2} \nabla(\text{curl } u)$:

$$W_{\text{curv}}(k) = \mu L_c^2 \left[ \alpha_1 \| \text{dev } \text{sym } \nabla[\text{axl}(\text{skew } \nabla u)] \|^2 + \alpha_2 \| \text{skew } \nabla[\text{axl}(\text{skew } \nabla u)] \|^2 \right]$$

$$= \mu L_c^2 \left[ \alpha_1 \| \text{sym } \text{Curl } (\text{sym } \nabla u) \|^2 + \alpha_2 \| \text{skew } \text{Curl } (\text{sym } \nabla u) \|^2 \right],$$

we obtain the virtual work principle taking free variations $\delta u \in C^\infty(\Omega)$ in the energy (2.43)

$$\frac{d}{dt} \int_\Omega W(\nabla u + t \delta u) dv \bigg|_{t=0} = \int_\Omega \left[ 2\mu \langle \text{sym } \nabla u, \text{sym } \nabla \delta u \rangle + \lambda \text{tr}(\nabla u) \text{tr}(\nabla \delta u) \right]$$

This observation is a further development in understanding why the Hadjesfandiari and Dargush [41, 42, 45] choice is rather meaningless, while mathematically not forbidden [87].
+ \mu L_c^2 [2 \alpha_1 \text{dev sym} \nabla [\text{axl} \text{skew} \nabla u], \text{dev sym} \nabla [\text{axl} \text{skew} \nabla \delta u])]
+ 2 \alpha_2 \text{skew} \nabla [\text{axl} \text{skew} \nabla [\text{axl} \text{skew} \nabla \delta u)]) + \langle f, \delta u \rangle \right] dv = 0.

The classical divergence theorem leads to

\int_\Omega (\text{Div} (\sigma - \tilde{\tau}) + f, \delta u) dv - \int_{\partial \Omega} \left\langle (\sigma - \tilde{\tau}), n, \delta u \right\rangle dv - \int_{\partial \Omega} \langle \tilde{m}, n, \text{axl} \text{skew} \nabla \delta u \rangle da = 0, \quad (2.45)

for all virtual displacements \delta u \in C^\infty(\Omega), where \(n\) is the unit outward normal vector at the surface \(\partial \Omega\), \(\sigma\) is the symmetric local force-stress tensor

\begin{equation}
\sigma = 2 \mu \text{sym} \nabla u + \lambda \text{tr}(\nabla u) \mathbb{1} \in \text{Sym}(3) \tag{2.46}
\end{equation}

and \(\tilde{\tau}\) represents the nonlocal force-stress tensor (which here is automatically skew-symmetric)

\begin{equation}
\tilde{\tau} = \mu L_c^2 \left[ \alpha_1 \text{antidiv} \left( \text{dev sym} \nabla [\text{axl} \text{skew} \nabla u] \right) + \alpha_2 \text{antidiv} \left( \text{skew} \nabla [\text{axl} \text{skew} \nabla u] \right) \right] \tag{2.47}
\end{equation}

\begin{equation}
= \mu L_c^2 \left[ \frac{\alpha_1}{2} \text{antidiv} \left( \text{dev sym} \nabla (\text{curl} u) \right) + \frac{\alpha_2}{2} \text{skew} \nabla (\text{curl} u) \right]
= \frac{1}{2} \text{antidiv} \tilde{m} \in \mathfrak{so}(3),
\end{equation}

where

\begin{equation}
\tilde{m} = \mu L_c^2 \left[ \alpha_1 \text{sym} \nabla (\text{curl} u) + \alpha_2 \text{skew} \nabla (\text{curl} u) \right]
= \mu L_c^2 \left[ \alpha_1 \text{dev sym} \nabla (\text{curl} u) + \alpha_2 \text{skew} \nabla (\text{curl} u) \right]
= \mu L_c^2 \left[ 2 \alpha_1 \text{dev sym} \nabla [\text{axl} \text{skew} \nabla u] + 2 \alpha_2 \text{skew} \nabla [\text{axl} \text{skew} \nabla u] \right], \quad (2.48)
\end{equation}

is the hyperstress tensor (couple stress tensor) which may or may not be symmetric, depending on the material parameters.

The non-symmetry of force stress is a constitutive assumption. Thus, if the test function \(\delta u \in C^\infty(\Omega)\) also satisfies \(\text{axl} \text{skew} \nabla \delta u = 0\) on \(\Gamma\) (equivalently \(\text{curl} \delta u = 0\)), then we obtain the equilibrium equation

\begin{equation}
\text{Div} \left\{ \begin{array}{l}
2 \mu \text{sym} \nabla u + \lambda \text{tr}(\nabla u) \mathbb{1} \in \text{Sym}(3) \\
\begin{array}{l}
- \mu L_c^2 \left[ \alpha_1 \text{dev sym} \nabla [\text{axl} \text{skew} \nabla u] + \alpha_2 \text{skew} \nabla [\text{axl} \text{skew} \nabla u] \right]
\end{array} \end{array} \right\} + f = 0, \quad (2.49)
\end{equation}

or equivalently

\begin{equation}
\text{Div} \tilde{\sigma}_{\text{total}} + f = 0, \quad (2.50)
\end{equation}

where

\begin{equation}
\tilde{\sigma}_{\text{total}} = \sigma - \tilde{\tau} \notin \text{Sym}(3). \quad (2.51)
\end{equation}

The complete consistent boundary conditions for this formulation is presented for the first time in [62] and recapitulated in Figure 9 and Figure 12.
3 The new isotropic gradient elasticity model with symmetric nonlocal force stress and symmetric hyperstresses

As independent constitutive variables for our novel gradient elastic model we choose now

\[ \varepsilon = \text{sym} \nabla u, \quad \hat{k} = \text{Curl} (\text{sym} \nabla u) = \text{Curl} \varepsilon. \]  

(3.1)

We use again the orthogonal Lie-algebra decomposition of \( \mathbb{R}^{3 \times 3} \)

\[ \text{Curl} (\text{sym} \nabla u) = \text{dev} \text{sym} (\text{Curl} (\text{sym} \nabla u)) + \text{skew} \text{Curl} (\text{sym} \nabla u). \]  

(3.2)

The term \( \frac{1}{7} \text{tr} (\text{Curl} (\text{sym} \nabla u)) \mathbb{1} \) is missing since \( \text{tr} (\text{Curl} (\text{sym} \nabla u)) = 0 \) anyway (already \( \text{tr} (\text{Curl} S) = 0 \) for \( S \in \text{Sym}(3) \)). The model is derived from the free energy

\[ W(e, \hat{k}) = W_{\text{lin}}(e) + W_{\text{curv}}(\hat{k}), \]

with

\[ W_{\text{lin}}(\varepsilon) = \mu \| \text{sym} \nabla u \|^2 + \lambda \| \text{dev} \text{sym} \nabla u \|^2 + \kappa \| \text{tr} (\text{sym} \nabla u) \|^2, \]  

\[ W_{\text{curv}}(\hat{k}) = \mu L_c^2 \left[ 2 \alpha_1 \| \text{dev} \text{sym} \text{Curl} (\text{sym} \nabla u) \|^2 + 2 \alpha_2 \| \text{skew} \text{Curl} (\text{sym} \nabla u) \|^2 \right], \]

where \( \alpha_1, \alpha_2 \) are non-negative constitutive curvature coefficients and \( \kappa = \frac{2\mu + 3\lambda}{3} \) is the infinitesimal bulk modulus, while \( \mu \) is the classical shear modulus.

The hyperstress-tensor (moment stress tensor, couple stress tensor)

\[ \hat{m} := D \hat{k} \] \[ = \mu L_c^2 \left[ 2 \alpha_1 \text{dev} \text{sym} \text{Curl} (\text{sym} \nabla u) + 2 \alpha_2 \text{skew} \text{Curl} (\text{sym} \nabla u) \right] \]

is symmetric in the conformal case \( \alpha_2 = 0 \), while the nonlocal force stress tensor is always symmetric, see eq. (3.11).

Due to isotropy, the curvature energy \( W_{\text{curv}}(k) \) involves in principle only 2 additional constitutive constants. Taking free variations \( \delta u \in C^\infty(\Omega) \) in the energy (3.3), we obtain the virtual work principle

\[ \frac{d}{dt} \int_\Omega W(\nabla u + t \nabla \delta u) dv \bigg|_{t=0} = \int_\Omega \left[ 2 \mu \langle \text{sym} \nabla u, \text{sym} \nabla \delta u \rangle + \lambda (\nabla u) \text{tr} (\nabla \delta u) \\
+ \mu L_c^2 \left[ 2 \alpha_1 \langle \text{dev} \text{sym} \text{Curl} (\text{sym} \nabla u), \text{dev} \text{sym} \text{Curl} (\text{sym} \nabla \delta u) \rangle \\
+ 2 \alpha_2 \langle \text{skew} \text{Curl} (\text{sym} \nabla u), \text{skew} \text{Curl} (\text{sym} \nabla \delta u) \rangle \right] + \langle f, \delta u \rangle \right] dv = 0, \]

where \( f \) is the body force per unit volume. We have the formulas

\[ \text{div} (\varphi_i Q_i) = \langle Q_i, \nabla \varphi_i \rangle + \varphi_i \text{div} Q_i \quad \text{not summed}, \]

\[ \text{div} (R_i \times S_i) = \langle S_i, \text{curl} R_i \rangle - \langle R_i, \text{curl} S_i \rangle \quad \text{not summed}, \]

(3.5)
for all $C^1$-functions $\varphi_i : \Omega \to \mathbb{R}$ and $Q_i, P_i, S_i : \Omega \to \mathbb{R}^3$, where $\varphi_i$ are the components of the vector $\varphi$ and $Q_i, P_i, S_i$ are the rows of the matrix $Q, P$ and $S$, respectively, where $\times$ denotes the vector product. If we take in (3.5) $R_i = [\text{dev sym Curl} (\text{sym } \nabla u)]_i, S_i = (\text{sym } \nabla \delta u)_i$, we get

$$\sum_{i=1}^{3} \text{div} \left( [\text{dev sym Curl} (\text{sym } \nabla u)]_i \times (\text{sym } \nabla \delta u)_i \right) = \sum_{i=1}^{3} \langle (\text{sym } \nabla \delta u)_i, \text{curl } [\text{dev sym Curl} (\text{sym } \nabla u)]_i \rangle - \sum_{i=1}^{3} \langle [\text{dev sym Curl} (\text{sym } \nabla u)]_i, \text{curl } (\text{sym } \nabla \delta u)_i \rangle. \tag{3.6}$$

Hence, we obtain

$$\sum_{i=1}^{3} \text{div} \left( [\text{dev sym Curl} (\text{sym } \nabla u)]_i \times (\text{sym } \nabla \delta u)_i \right) = \langle (\text{sym } \nabla \delta u), \text{Curl } [\text{dev sym Curl} (\text{sym } \nabla u)] \rangle - \langle [\text{dev sym Curl} (\text{sym } \nabla u)], \text{Curl } (\text{sym } \nabla \delta u) \rangle. \tag{3.7}$$

Doing a similar calculus, but choosing $R_i = [\text{skew Curl} (\text{sym } \nabla u)]_i, S_i = (\text{sym } \nabla \delta u)_i$, we obtain

$$\sum_{i=1}^{3} \text{div} \left( [\text{skew Curl} (\text{sym } \nabla u)]_i \times (\text{sym } \nabla \delta u)_i \right) = \langle (\text{sym } \nabla \delta u), \text{Curl } [\text{skew Curl} (\text{sym } \nabla u)] \rangle - \langle [\text{skew Curl} (\text{sym } \nabla u)], \text{Curl } (\text{sym } \nabla \delta u) \rangle. \tag{3.8}$$

The above formulas lead, for all variations $\delta u \in C^\infty(\Omega)$, to

$$\int_\Omega \left[ \alpha_1 \langle \text{dev sym Curl} (\text{sym } \nabla u), \text{Curl } (\text{sym } \nabla \delta u) \rangle + \alpha_2 \langle \text{skew Curl} (\text{sym } \nabla u), \text{Curl } (\text{sym } \nabla \delta u) \rangle \right] \text{dv}$$

$$= \int_\Omega \left[ \alpha_1 \langle \text{sym Curl } [\text{dev sym Curl} (\text{sym } \nabla u)], \nabla \delta u \rangle + \alpha_2 \langle \text{sym Curl } [\text{skew Curl} (\text{sym } \nabla u)], \nabla \delta u \rangle \rangle \right.$$

$$- \sum_{i=1}^{3} \text{div} \left[ \alpha_1 \langle [\text{dev sym Curl} (\text{sym } \nabla u)]_i, \text{sym } \nabla \delta u \rangle \rangle \right.$$

$$- \sum_{i=1}^{3} \text{div} \left[ \alpha_2 \langle [\text{skew Curl} (\text{sym } \nabla u)]_i, \text{sym } \nabla \delta u \rangle \rangle \right.$$

$$\left. + \alpha_2 \langle [\text{skew Curl} (\text{sym } \nabla u)]_i, \text{sym } \nabla \delta u \rangle \rangle \right] \text{dv}. \tag{3.9}$$

Therefore, using the divergence theorem and a special format of the partial integration which is
suggested by the matrix Curl-operator, it follows that

\[ \int_{\Omega} \left[ \alpha_1 \langle \text{dev sym Curl} \left( \text{sym} \nabla u \right), \text{Curl} \left( \text{sym} \nabla \delta u \right) \rangle + \alpha_2 \langle \text{skew Curl} \left( \text{sym} \nabla u \right), \text{Curl} \left( \text{sym} \nabla \delta u \right) \rangle \right] dv \\
= \int_{\Omega} \left[ \alpha_1 \langle \text{Curl} \left[ \text{dev sym Curl} \left( \text{sym} \nabla u \right) \right], \nabla \delta u \rangle + \alpha_2 \langle \text{Curl} \left[ \text{skew Curl} \left( \text{sym} \nabla u \right) \right], \nabla \delta u \rangle \right] dv \\
- \int_{\partial \Omega} \left[ \sum_{i=1}^{3} \left( \alpha_1 \langle \text{dev sym Curl} \left( \text{sym} \nabla u \right) \rangle_i \times \langle \text{sym} \nabla \delta u \rangle_i, n \rangle \right)
+ \alpha_2 \langle \text{skew Curl} \left( \text{sym} \nabla u \right) \rangle_i \times \langle \text{sym} \nabla \delta u \rangle_i, n \rangle \right] da \\
= - \int_{\Omega} \left[ \alpha_1 \langle \text{Div} \left[ \text{dev sym Curl} \left( \text{sym} \nabla u \right) \right], \delta u \rangle \right] dv \\
+ \int_{\partial \Omega} \left[ \alpha_1 \langle \text{Curl} \left[ \text{sym} \nabla u \right], \delta u \rangle \right] n, \delta u \right] da \\
- \int_{\partial \Omega} \left[ \sum_{i=1}^{3} \left( \alpha_1 \langle \text{dev sym Curl} \left( \text{sym} \nabla u \right) \rangle_i \right.
+ \alpha_2 \langle \text{skew Curl} \left( \text{sym} \nabla u \right) \rangle_i \times \langle \text{sym} \nabla \delta u \rangle_i \right] n, \delta u \right] da,
\]

where \( n \) is the unit outward normal vector at the surface \( \partial \Omega \). Hence, the relation (3.4) leads to

\[ \int_{\Omega} \text{Div} \left\{ 2 \mu \text{sym} \nabla u + \lambda \text{tr} \left( \nabla u \right) \mathbb{I} + \mu L^2_c \left[ 2 \alpha_1 \text{Curl} \left[ \text{dev sym Curl} \left( \text{sym} \nabla u \right) \right] + 2 \alpha_2 \text{sym Curl} \left[ \text{skew Curl} \left( \text{sym} \nabla u \right) \right] \right], \delta u \right\} dv \\
+ \int_{\partial \Omega} \left( \left\{ 2 \mu \langle \text{sym} \nabla u + \lambda \text{tr} \left( \nabla u \right) \mathbb{I}, n, \delta u \rangle \right\} da \\
- \int_{\partial \Omega} \left( 2 \mu L^2_c \left[ \alpha_1 \text{Curl} \left[ \text{dev sym Curl} \left( \text{sym} \nabla u \right) \right] + \alpha_2 \text{sym Curl} \left[ \text{skew Curl} \left( \text{sym} \nabla u \right) \right] \right]. n, \delta u \right] da \right) \right) \\
+ \int_{\partial \Omega} \left( \sum_{i=1}^{3} 2 \mu L^2_c \left[ \alpha_1 \langle \text{dev sym Curl} \left( \text{sym} \nabla u \right) \rangle_i \right.
+ \alpha_2 \langle \text{skew Curl} \left( \text{sym} \nabla u \right) \rangle_i \times n \rangle da = 0, \right) \]

for all variations \( \delta u \in C^\infty(\Omega). \)

\( ^8 \)This is an extra constitutive assumption since it is finally the form of the partial integration that determines, on the one hand, which force-stress tensor is generated and, on the other hand, which boundary condition is obtained. It is only the Curl-operator that seems to suggest this choice - but it remains a choice!
We can write the above variational formulation, for all variations $\delta u \in C^\infty(\Omega)$, in the following form

$$
\int_\Omega \langle \text{Div } (\sigma + \hat{\tau}) + f, \delta u \rangle \, dv - \int_{\partial \Omega} \langle (\sigma + \hat{\tau}).n, \delta u \rangle \, da - \int_{\partial \Omega} \sum_{i=1}^3 (\hat{m}_i \times n, (\text{sym} \nabla \delta u)_i) \, da = 0, 
$$

(3.10)

where

$$\sigma = 2 \mu \text{sym} \nabla u + \lambda \text{tr}(\nabla u) \mathbf{I} \in \text{Sym}(3),$$

(local force-stress)

$$\hat{\tau} = \mu L^2 \text{sym}[2 \alpha_1 \text{Curl } (\text{sym} \nabla u)] + 2 \alpha_2 \text{Curl } (\text{sym} \nabla u)]$$

(3.11)

$$= \text{sym} \text{Curl } (\hat{m}) \in \text{Sym}(3),$$

(non-local force stress)

$$\hat{m} = \mu L^2 (2 \alpha_1 \text{dev sym}[\text{Curl } (\text{sym} \nabla u)] + 2 \alpha_2 \text{skew}[\text{Curl } (\text{sym} \nabla u)]) \in \mathfrak{gl}(3).$$

We call $\sigma$ the local force stress tensor, $\hat{\tau}$ the non-local force stress tensor and $\hat{m} = D_k \text{W}_{\text{curv}}(k)$ the hyperstress tensor (couple stress tensor).

Thus, if the test function $\delta u \in C^\infty_0(\Omega)$ also satisfies $$(\text{sym} \nabla \delta u)_i \times n = 0$$ (or equivalently $(\text{sym} \nabla \delta u).\tau = 0$ for all tangential vectors $\tau$ at $\Gamma$), then we obtain the equilibrium equation

$$\text{Div } \{ 2 \mu \text{sym} \nabla u + \lambda \text{tr}(\nabla u) \mathbf{I} + \mu L^2 \text{sym} \text{Curl } [2 \alpha_1 \text{dev sym} \text{Curl } (\text{sym} \nabla u)] + 2 \alpha_2 \text{skew} \text{Curl } (\text{sym} \nabla u) \} + f = 0.$$

(3.12)

The first impulse is to prescribe on $\Gamma \subseteq \partial \Omega$ the following geometric boundary conditions

$$u = \hat{u}^{\text{ext}}$$

on $\Gamma$, (3.13)

$$(\mathbf{1} - n \otimes n) (\text{sym} \nabla u)_i \times n = (\mathbf{1} - n \otimes n) \hat{a}^{\text{ext}}_i, \quad i = 1, 2, 3$$

on $\Gamma$,

where $\hat{u}^{\text{ext}}, \hat{a}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ are prescribed functions (i.e. $3+2+2+2=9$ boundary conditions), and the following traction boundary conditions on $\partial \Omega \setminus \Gamma$,

$$(\sigma + \hat{\tau}).n = \hat{g}^{\text{ext}},$$

on $\partial \Omega \setminus \Gamma$,

$$(\mathbf{1} - n \otimes n) \hat{m}_i \times n = (\mathbf{1} - n \otimes n) \hat{h}^{\text{ext}}_i, \quad i = 1, 2, 3$$

on $\partial \Omega \setminus \Gamma$,

where $\hat{g}^{\text{ext}}, \hat{h}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3$ are prescribed functions (i.e. $3+2+2+2=9$ boundary conditions).

However, we need to separate normal and tangential derivatives of the test function $\delta u$ in (3.10) which is standard in general strain gradient elasticity, since tangential derivatives of $\delta u$ are not independent of $\delta u$. Let us define the matrix

$$\hat{M} := \begin{pmatrix} \hat{m}_1 \times n_{\text{-------}} \\ \hat{m}_2 \times n_{\text{-------}} \\ \hat{m}_3 \times n_{\text{-------}} \end{pmatrix}, \quad \text{where} \quad \hat{m} := \begin{pmatrix} \hat{m}_{1\text{-------}} \\ \hat{m}_{2\text{-------}} \\ \hat{m}_{3\text{-------}} \end{pmatrix}. \quad (3.15)$$

$^9 \langle a \times b, c \rangle = -\langle a, c \times b \rangle$. 

24
{\delta u \text{ and } \nabla \delta u \times n} \text{ cannot orthogonally be prescribed}

{\delta u \text{ and } \nabla \delta u \cdot \tau} \text{ cannot orthogonally be prescribed}

{\delta u \text{ and } (1 - n \otimes n) \nabla \delta u} \text{ cannot orthogonally be prescribed}

{\delta u \text{ and } \nabla \delta u \cdot n} \text{ cannot be orthogonally prescribed (3 bc)}

{\delta u \text{ and } (1 - n \otimes n) \nabla \delta u \cdot n} \text{ can be orthogonally prescribed (2 bc)}

{\delta u \text{ and } \langle \text{curl} \delta u, n \rangle} \text{ cannot orthogonally be prescribed}

{\delta u \text{ and } \langle \text{curl} \delta u, \tau \rangle} \text{ can be orthogonally prescribed (2 bc)}

{\delta u \text{ and } (1 - n \otimes n) \text{curl} \delta u} \text{ can be orthogonally prescribed (2 bc)}

Figure 3: The possible independent geometrically boundary conditions. The joining bracket means that the conditions are equivalent. Here \( n \) is the unit outward normal vector at the surface \( \partial \Omega \), while \( \tau \) is a tangent vector at the boundary \( \partial \Omega \).

With the help of this matrix \( \hat{M} \), we may write

\[ \sum_{i=1}^{3} \langle \hat{m}_i \times n, (\text{sym } \nabla \delta u)_i \rangle = \langle \hat{M}, \text{sym } \nabla \delta u \rangle = \langle \text{sym } \hat{M}, \nabla \delta u \rangle. \]  

(3.16)

At this point, it must be considered that the tangential trace of the gradient of virtual displacement can be integrated by parts once again and that the surface divergence theorem can be applied to this tangential part of \( \nabla \delta u \). Before doing so, one needs to introduce (see also [23, 99, 24] for details) two second order tensors \( T \) and \( Q \) which are the two projectors on the tangent plane and on the normal to the considered surface, respectively. As it is well known from differential geometry, such projectors actually allow to split a given vector or tensor field in one part projected on the plane tangent to the considered surface and one projected on the normal to such surface (see e.g. [23]). Let \( \{\tau, \nu\} \) be an orthonormal local basis of the tangent plane to the considered surface at point \( P \) and let \( n \) be the unit normal vector at the same point. We can introduce the quoted projectors as

\[ T = \tau \otimes \tau + \nu \otimes \nu = 1 - n \otimes n \quad (T_{ij} = \delta_{ij} - n_i n_j), \quad Q = n \otimes n \quad (Q_{ij} = n_i n_j). \]  

(3.17)

In our abbreviations, for a surface \( S \subset \mathbb{R}^3 \), the surface divergence theorem means [40, p. 58, ex. 7]

\[ \int_S \langle T, \nabla (T \cdot v) \rangle \, da = \oint_{\partial S} \langle v, \nu \rangle \, ds, \]  

(3.18)

for any field \( v \in \mathbb{R}^3 \) and \( \nu = \tau \times n \). We explicitly remark that if \( S \) coincides with the boundary \( \partial \Omega \) of the considered body and \( \Gamma \) is a open subset of \( \partial \Gamma \), then the surface divergence theorem (3.18) implies (see Fig. 1 and eq. (1.13))

\[ \int_{\partial \Omega} \langle T, \nabla (T \cdot v) \rangle \, da = \int_{\partial \Gamma} \langle \nabla \cdot v, n \rangle \, ds. \]  

(3.19)
Regarding the boundary conditions, similar as in [72], we obtain

\[
\sum_{i=1}^{3} \langle \hat{m}_i \times n, (\text{sym} \nabla \delta u)_i \rangle = \langle \text{sym} \hat{M}, \nabla \delta u \rangle = \langle \text{sym} \hat{M}, \nabla \delta u (T + Q) \rangle
\]

\[
= \langle (\text{sym} \hat{M}), (\nabla \delta u) T \rangle + \langle (\text{sym} \hat{M}), (\nabla \delta u) Q \rangle = \langle (\text{sym} \hat{M}) T, \nabla \delta u \rangle + \langle (\text{sym} \hat{M}), (\nabla \delta u) Q \rangle
\]

\[
= \langle (\text{sym} \hat{M}) T T, \nabla \delta u \rangle + \langle (\text{sym} \hat{M}), (\nabla \delta u) Q \rangle = \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle (\text{sym} \hat{M}), (\nabla \delta u) n \otimes n \rangle.
\]

The last term on the right hand side may be rewritten in the form

\[
\langle (\text{sym} \hat{M}), (\nabla \delta u) Q \rangle = \frac{1}{2} \langle \hat{M}, (\nabla \delta u) Q + Q(\nabla \delta u)^T \rangle = \frac{1}{2} \langle \hat{M} Q, \nabla \delta u \rangle + \frac{1}{2} \langle \hat{M}, Q(\nabla \delta u)^T \rangle
\]

\[
= \frac{1}{2} \langle \hat{M} n \otimes n, \nabla \delta u \rangle + \frac{1}{2} \langle \hat{M}, n \otimes n(\nabla \delta u)^T \rangle.
\]

Thus, we deduce

\[
\sum_{i=1}^{3} \langle \hat{m}_i \times n, (\text{sym} \nabla \delta u)_i \rangle = \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle (\text{sym} \hat{M}), (\nabla \delta u) Q \rangle
\]

\[
= \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle (\text{sym} \hat{M}), (\nabla \delta u) n \otimes n \rangle
\]

\[
= \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle n \otimes \{ (\text{sym} \hat{M})(\nabla \delta u).n \}, 1 \rangle
\]

\[
= \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle n, (\text{sym} \hat{M})(\nabla \delta u).n \rangle
\]

\[
= \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle (\text{sym} \hat{M}).n, (\nabla \delta u).n \rangle
\]

\[
= \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle + \langle (\text{sym} \hat{M}).n, \frac{\partial \delta u}{\partial n} \rangle.
\]

We can therefore recognize in the last term of this formula that the normal derivative

\[
\frac{\partial}{\partial n} \delta u = (\nabla \delta u).n = (\delta u_i, n_h)_i
\]

of the test function field \( \delta u \) (the virtual displacement) appears. We deduce by gathering the results in (3.20)-(3.27) that the last integral on the right hand side is given by

\[
\int_{\partial \Omega} \sum_{i=1}^{3} \langle \hat{m}_i \times n, (\text{sym} \nabla \delta u)_i \rangle da
\]

\[
= \int_{\partial \Omega} \langle T, T(\text{sym} \hat{M}) \nabla \delta u \rangle da + \int_{\partial \Omega} \langle (\text{sym} \hat{M}).n, (\nabla \delta u).n \rangle da.
\]
Since
\[
\{\nabla [T(\text{sym } \bar{M}) \delta u]\}_{ik} = \{T(\text{sym } \bar{M})\}_{i,k} = \{(T(\text{sym } \bar{M}))_{ij} (\delta u)_{jk}\}_{ik}
\]
\[
= (T(\text{sym } \bar{M}))_{ij,k} (\delta u)_{j} + (T(\text{sym } \bar{M}))_{ij} (\delta u)_{j,k}\] (3.24)
\[
= (T(\text{sym } \bar{M}))_{ij,k} (\delta u)_{j} + (T(\text{sym } \bar{M}))_{ij} (\nabla \delta u)_{jk}
\]
\[
= (T(\text{sym } \bar{M}))_{ij,k} (\delta u)_{j} + \{T(\text{sym } \bar{M}) \nabla \delta u\}_{ik},
\]
we obtain
\[
\int_{\partial \Omega} \sum_{i=1}^{3} \langle \hat{m}_i \times n, \text{sym } \nabla \delta u_i \rangle da = \int_{\partial \Omega} \langle T, \nabla [T(\text{sym } \bar{M}) \cdot \delta u] \rangle da - \int_{\partial \Omega} T_{ik} (T(\text{sym } \bar{M}))_{ij,k} (\delta u)_{j} da
\]
\[
+ \int_{\partial \Omega} \langle \text{sym } \bar{M} \cdot n, (\nabla \delta u) \cdot n \rangle da.
\] (3.25)

In order to write in a compact form the above relation, let us remark that
\[
(T(\text{sym } \bar{M}))_{ij} = T_{il}(\text{sym } \bar{M})_{ij} = T_{il}(\text{sym } \bar{M})_{jl} = T_{il}(\text{sym } \bar{M})_{jl} T_{li} = \{(\text{sym } \bar{M}) T\}_{ji},
\]
and further that
\[
(T(\text{sym } \bar{M}))_{ij,k} T_{ik} = \{(\text{sym } \bar{M}) T\}_{ji,k} T_{ik} = (\nabla [(\text{sym } \bar{M}) T] : T)_{j}.
\] (3.26)

We obtain
\[
T_{ik} (T(\text{sym } \bar{M}))_{ij,k} (\delta u)_{j} = (\nabla [(\text{sym } \bar{M}) T] : T)_{j} (\delta u)_{j} = (\nabla [(\text{sym } \bar{M}) T] : T, \delta u).
\] (3.27)

We deduce by gathering the results in (3.20)-(3.27) that the last integral on the right hand side is given by
\[
\int_{\partial \Omega} \sum_{i=1}^{3} \langle \hat{m}_i \times n, \text{sym } \nabla \delta u_i \rangle da
\]
\[
= \int_{\partial \Omega} \langle T, \nabla [T(\text{sym } \bar{M}) \cdot \delta u] \rangle da - \int_{\partial \Omega} \langle \nabla [(\text{sym } \bar{M}) T] : T, \delta u \rangle da + \int_{\partial \Omega} \langle \text{sym } \bar{M} \cdot n, (\nabla \delta u) \cdot n \rangle da
\]
\[
= - \int_{\partial \Omega} \langle \nabla [(\text{sym } \bar{M}) T] : T, \delta u \rangle da + \int_{\partial \Omega} \langle (\text{sym } \bar{M}) \cdot n, (\nabla \delta u) \cdot n \rangle da + \oint_{\partial \Omega} \langle \| (\text{sym } \bar{M}) \cdot n \|, \delta u \rangle ds.
\]

In the above computation $\nabla [(\text{sym } \bar{M}) T]$ is not a matrix, but a third order tensor and $\nabla [(\text{sym } \bar{M}) T] : T \in \mathbb{R}^{3}$ is a contraction operation, i.e.
\[
\{\nabla [(\text{sym } \bar{M}) T] : T\}_{i} = \{(\text{anti}[(\text{sym } \bar{M}) T])_{ij,k} T_{jk}\}.
\]

27
Therefore, the variational formulation (3.10) can be rewritten as

\[
\int_\Omega \langle \text{Div} (\sigma + \hat{\tau}) + f, \delta u \rangle \, dv - \int_{\partial \Omega} \langle (\sigma + \hat{\tau}).n - \nabla [(\text{sym } \hat{M})^T : T] , \delta u \rangle \, da
\]

\(= \int_\Omega \langle (\sigma + \hat{\tau}).n - \nabla [(\text{sym } \hat{M})^T : T] , \delta u \rangle \, dv - \int_{\partial \Omega} \langle (\text{sym } \hat{M}).n, (\nabla \delta u).n \rangle \, da - \oint_{\partial \Gamma} \langle J^T (\text{sym } \hat{M}).\nu, \delta u \rangle \, ds = 0,
\]

for all variations \(\delta u \in C^\infty(\Omega)\), where we have used that for the regular surface \(\partial \Omega\) it holds \(\nu^+ = -\nu^- = \nu\). Moreover, we also obtain

\[
\hat{M}.n = \begin{bmatrix} \hat{m}_1 \times n \\ \hat{m}_2 \times n \\ \hat{m}_3 \times n \end{bmatrix}.n = \begin{bmatrix} \langle \hat{m}_1 \times n, n \rangle \\ \langle \hat{m}_2 \times n, n \rangle \\ \langle \hat{m}_3 \times n, n \rangle \end{bmatrix}.n = 0.
\]

Hence

\[
(\text{sym } \hat{M}).n = \frac{1}{2} \hat{M}^T.n.
\]

On the other hand, we deduce

\[
\hat{M}^T.n = T \hat{M}^T.n + Q \hat{M}^T.n = (\mathbb{1} - n \otimes n) \hat{M}^T.n + n \otimes n \hat{M}^T.n
\]

\[
= (\mathbb{1} - n \otimes n) \hat{M}^T.n + [n \otimes n \hat{M}^T].n = (\mathbb{1} - n \otimes n) \hat{M}^T.n + n \otimes [\hat{M}.n].n
\]

\[
= (\mathbb{1} - n \otimes n) \hat{M}^T.n + n \otimes \begin{bmatrix} \langle \hat{m}_1 \times n, n \rangle \\ \langle \hat{m}_2 \times n, n \rangle \\ \langle \hat{m}_3 \times n, n \rangle \end{bmatrix}.n
\]

\[
= (\mathbb{1} - n \otimes n) \hat{M}^T.n + n \otimes \begin{bmatrix} \langle \hat{m}_1 \times n, n \rangle \\ \langle \hat{m}_2 \times n, n \rangle \\ \langle \hat{m}_3 \times n, n \rangle \end{bmatrix}.n
\]

In view of (3.30), we see

\[
(\text{sym } \hat{M}).n = (\mathbb{1} - n \otimes n) (\text{sym } \hat{M}).n.
\]
Therefore, finally we get from (3.10)

\[
\int_{\Omega} \left< \text{Div} (\sigma + \hat{\tau}) + f, \delta u \right> dv - \int_{\partial \Omega} \left< (\sigma + \hat{\tau}).n - \nabla [(\text{sym} \hat{M}) T] : T, \delta u \right> da
\]

\[
- \int_{\partial \Omega} \left< (\mathbb{1} - n \otimes n)(\text{sym} \hat{M}).n, (\mathbb{1} - n \otimes n)(\nabla \delta u).n \right> da - \int_{\partial \Gamma} \left< \left\| (\text{sym} \hat{M}).n \right\|, \delta u \right> ds = 0,
\]

\[
\delta u - \text{independent first order variation}
\]

\[
\text{normal variation of gradient}
\]

for all variations \( \delta u \in C^\infty(\Omega) \). An equivalent form, replacing simply \( \hat{\tau} = \text{sym Curl} (\hat{m}) \), is

\[
\int_{\Omega} \left< \text{Div} (\sigma + \text{sym Curl}(\hat{m})) + f, \delta u \right> dv - \int_{\partial \Omega} \left< (\sigma + \text{sym Curl}(\hat{m})).n - \nabla [(\text{sym} \hat{M}) T] : T, \delta u \right> da
\]

\[
- \int_{\partial \Omega} \left< (\mathbb{1} - n \otimes n)(\text{sym} \hat{M}).n, (\mathbb{1} - n \otimes n)(\nabla \delta u).n \right> da - \int_{\partial \Gamma} \left< \left\| (\text{sym} \hat{M}).n \right\|, \delta u \right> ds = 0.
\]

\[
\delta u - \text{independent second order}
\]

\[
\text{normal variation of the gradient}
\]

\[
\text{“edge line forces”}
\]

\[3.1 \text{ Formulation of the new boundary value problem}\]

\[3.1.1 \text{ Equilibrium equation}\]

In terms of the symmetric force-stress tensor \( \sigma \) and of the nonlocal force-stress tensor \( \hat{\tau} \) which is also here symmetric, while the hyperstress \( \hat{m} \in \mathfrak{gl}(3) \) is symmetric only for \( \alpha_2 = 0 \), the equilibrium equations may now be written in the format\(^{10}\)

\[
\text{Div} \hat{\sigma}_{\text{total}} + f = 0,
\]

\[ (3.34) \]

where the symmetric total force stress\(^{11}\) is given by \( \hat{\sigma}_{\text{total}} = \sigma + \hat{\tau} \in \text{Sym}(3) \).

\[3.1.2 \text{ Geometric (essential) boundary conditions}\]

To the above equilibrium equation, we adjoin on \( \Gamma \subseteq \partial \Omega \) the following boundary conditions

\[
u(x) = \hat{\nu}^{\text{ext}}(x) \quad \text{on } \Gamma, \quad (3 \text{ bc}) \]

\[
[\mathbb{1} - n \otimes n](\nabla \nu).n](x) = [(\mathbb{1} - n \otimes n)\hat{\nu}^{\text{ext}}](x) \quad \text{on } \Gamma, \quad (2 \text{ bc})
\]

where \( \hat{\nu}^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}^3 \) are prescribed functions (i.e. 3+2=5 boundary conditions).

\(^{10}\)Here, infinitesimal frame-indifference amounts to \( W(\nabla u) = W(\nabla u + \bar{W}), \forall \bar{W} \in \mathfrak{so}(3) \), which is obviously satisfied.

\(^{11}\)Vidoli et al. call this tensor the “effective stress tensor” [22].
3.1.3 Traction boundary conditions

Corresponding to the geometric boundary conditions, which are now orthogonal, we have to prescribe the following traction boundary conditions

\[
\begin{align*}
\{(\sigma + \hat{\tau}).n - \nabla[(\text{sym } \hat{M})(\mathbb{I} - n \otimes n)] : (\mathbb{I} - n \otimes n)\} \bigg|_{\partial \Omega \setminus \Gamma} &= \hat{\tau}^\text{ext}, \\
[(\mathbb{I} - n \otimes n)(\text{sym } \hat{M}).n] \bigg|_{\partial \Omega \setminus \Gamma} &= (\mathbb{I} - n \otimes n)\hat{g}^\text{ext}, \\
[(\text{sym } \hat{M}).\nu] \bigg|_{\partial \Gamma} &= \hat{\pi}^\text{ext},
\end{align*}
\]

(3 bc) (2 bc) (3.36)

\[
\frac{J}{(\text{sym } \hat{M}).\nu} \bigg|_{\partial \Gamma} = \hat{\pi}^\text{ext},
\]

(3 bc)

where \(\hat{\tau}^\text{ext}, \hat{g}^\text{ext} : \mathbb{R}^3 \to \mathbb{R}^3\) are prescribed functions on \(\partial \Omega \setminus \Gamma\) (i.e. 3+2=5 boundary conditions), while \(\hat{\pi}^\text{ext} : \mathbb{R}^3 \to \mathbb{R}^3\) is prescribed on \(\partial \Gamma = \partial(\partial \Omega \setminus \Gamma)\) and leads to 3 boundary conditions on \(\partial \Gamma\).

Remark 3.1. If \(\Gamma = \partial \Omega\) then the solution in the \(\text{Curl (sym } \nabla u)\)-formulation and in the \(\nabla[\text{axl}(\text{skew } \nabla u)]\)-formulation are the same, since the Euler-Lagrange equations are the same and the geometric boundary conditions are the same. Differences appear only if \(\Gamma \neq \partial \Omega\) due to different specifications of traction boundary conditions.

3.2 Existence and uniqueness of the solution in the \(\text{Curl (sym } \nabla u)\)-formulation

In the linear couple stress theory with constrained rotations, Hlaváček and Hlaváček [46, Remark 2, p. 426] recognized the couple stress model already in the form (1.4) but did not give an existence result. There are many existence and uniqueness results for the indeterminate couple stress model in its classical anti-symmetric formulation. Recently, optimal results have been obtained in [48, 49].

In this section we establish an existence theorem for the solution of the boundary value problem (P) defined by (3.34), (3.35) and (3.36), where \(\hat{t} = 0, \hat{g} = 0, \hat{h} = 0, \hat{u}^0 = 0\) and \((\mathbb{I} - n \otimes n)(\nabla \hat{u}^0).n = 0\) for simplicity only.

\[
\begin{align*}
\text{Lemma 3.2.} \text{ Let } u \in H^1_0(\Omega; \Gamma) \text{ be such that } \text{sym } \text{Curl (sym } \nabla u) \in L^2(\Omega). \text{ Then, } \text{Curl (sym } \nabla u) \in L^2(\Omega) \text{ and there is a positive constant } c^+ \text{ such that }
\int_\Omega \left[ \| \text{sym } \nabla u \|^2 + \| \text{sym } \text{Curl (sym } \nabla u) \|^2 \right] dv \geq c^+ \int_\Omega \left[ \| \text{sym } \nabla u \|^2 + \| \text{Curl (sym } \nabla u) \|^2 \right] dv. \quad (3.37)
\end{align*}
\]

\[
\text{Proof.} \text{ For } u \in H^1_0(\Omega; \Gamma), \text{ the first Korn’s inequality implies that there is a positive constant } c^+ \text{ such that }
\int_\Omega \| \nabla u \|^2 dv = \int_\Omega \left( \frac{1}{2} \| \text{sym } \nabla u \|^2 + \frac{1}{2} \| \text{skew } \nabla u \|^2 \right) dv
\geq \frac{1}{2} \int_\Omega \| \nabla u \|^2 dv + \frac{c^+}{2} \int_\Omega \| \nabla u \|^2 dv. \quad (3.38)
\]

On the other hand the orthogonality of sym and skew implies

\[
\int_\Omega \| \nabla u \|^2 dv \geq \int_\Omega \| \text{skew } \nabla u \|^2 dv. \quad (3.39)
\]
Therefore, there is another positive constant $c^+$ such that
\[
\int_{\Omega} \left[ \| \text{sym} \, \nabla u \|^2 + \| \text{sym} \, \text{Curl} (\text{sym} \, \nabla u) \|^2 \right] dv \\
= \int_{\Omega} \left[ \frac{1}{2} \| \nabla \nabla u \|^2 + \frac{1}{2} \| \nabla \nabla u \|^2 + \| \text{sym} \, \text{axl} (\text{skew} \, \nabla u) \|^2 \right] dv \\
\geq c^+ \int_{\Omega} \left[ \| \nabla \nabla u \|^2 + \| \text{skew} \, \nabla u \|^2 + \| \text{sym} \, \text{axl} (\text{skew} \, \nabla u) \|^2 \right] dv \tag{3.40}
\]
\[
\geq c^+ \int_{\Omega} \left[ \| \nabla \nabla u \|^2 + \| \text{axl} \, \text{skew} \, \nabla u \|^2 + \| \text{sym} \, \text{axl} (\text{skew} \, \nabla u) \|^2 \right] dv.
\]
Moreover, since $\text{axl} \, \text{skew} \, \nabla u \in L^2(\Omega)$, the second Korn’s inequality\(^\sqrt{12}\) (without boundary conditions and applied to $\text{axl}(\text{skew} \, \nabla u)$) implies the existence of a positive constant $c^+$ such that
\[
\int_{\Omega} \left[ \| \text{axl} \, \text{skew} \, \nabla u \|^2 + \| \text{sym} \, \text{axl} (\text{skew} \, \nabla u) \|^2 \right] dv \\
\geq c^+ \int_{\Omega} \left[ \| \text{axl} \, \text{skew} \, \nabla u \|^2 + \| \text{axl} \, \text{skew} \, \nabla u \|^2 \right] dv \tag{3.41}
\]
Thus, there are positive constants $c^+, c^+_1$ such that
\[
\int_{\Omega} \left[ \| \nabla \nabla u \|^2 + \| \text{sym} \, \text{Curl} (\text{sym} \, \nabla u) \|^2 \right] dv \\
\geq c^+_1 \int_{\Omega} \left[ \| \nabla \nabla u \|^2 + \| \text{axl} \, \text{skew} \, \nabla u \|^2 + \| \nabla \text{axl} \, \text{skew} \, \nabla u \|^2 \right] dv \\
= c^+_1 \int_{\Omega} \left[ \| \nabla \nabla u \|^2 + \| \text{axl} \, \text{skew} \, \nabla u \|^2 + \| \text{Curl} (\text{sym} \, \nabla u) \|^2 \right] dv \\
\geq c^+ \int_{\Omega} \left[ \| \text{sym} \, \nabla u \|^2 + \| \text{Curl} (\text{sym} \, \nabla u) \|^2 \right] dv. \tag{3.42}
\]
The proof is complete. \qed

Let us consider null boundary conditions for simplicity. Hence, in the following we study the existence of the solution in the space
\[
\mathcal{X}_0 = \{ u \in H^1_0(\Omega; \Gamma) \mid \text{sym} \, \nabla u \in H(\text{Curl} ; \Omega) \}. \tag{3.43}
\]
On $\mathcal{X}_0$ we define the norm
\[
\| u \|_{\mathcal{X}_0} = \left( \| \nabla u \|_{L^2(\Omega)}^2 + \| \text{Curl} (\text{sym} \, \nabla u) \|_{L^2(\Omega)}^2 \right)^\frac{1}{2}, \tag{3.44}
\]
\(^\sqrt{12}\)Since $\text{curl} \, u$ is divergence free we also have the following Maxwell type inequality [8, 9]:
\[
\| \nabla \text{curl} \, u \|_{L^2(\Omega)}^2 \leq c_M (\| \text{curl} \, \text{curl} \, u \|_{L^2(\Omega)}^2), \quad \text{for } u \in \{ u \in H^1_0(\Omega) \mid \text{curl} \, u \in H(\text{curl} ; \Omega) \}.
\]
31
and the bilinear form
\[
(u, v) = \int_{\Omega} \left[ 2\mu \langle \text{sym} \nabla u, \text{sym} \nabla v \rangle + \lambda \text{tr}(\nabla u) \text{tr}(\nabla v) \\
+ \mu L_c^2 \left[ 2\alpha_1 \langle \text{dev sym Curl}(\text{sym} \nabla u), \text{dev sym Curl}(\text{sym} \nabla v) \rangle \\
+ 2\alpha_2 \langle \text{skew Curl}(\text{sym} \nabla u), \text{skew Curl}(\text{sym} \nabla v) \rangle \right] \right] dv, 
\]
where \( u, v \in X_0 \). Let us define the linear operator \( l : X_0 \to \mathbb{R} \), describing the influence of external loads,
\[ l(v) = \int_{\Omega} \langle f, v \rangle dv \]
for all \( \tilde{w} \in X_0 \). We say that \( w \) is a weak solution of the problem \((P)\) if and only if
\[ (u, v) = l(v) \quad \text{for all} \quad v \in X_0. \]
A classical solution \( u \in X_0 \) of the problem \((P)\) is also a weak solution.

**Theorem 3.3.** Assume that
i) the constitutive coefficients satisfy \( \mu > 0, \quad 3\lambda + 2\mu > 0, \quad \alpha_1 > 0, \quad \alpha_2 \geq 0; \)
ii) the loads satisfy the regularity condition \( f \in L^2(\Omega) \).

Then there exists one and only one solution of the problem \((3.46)\).

**Proof.** Let us first consider the case \( \alpha_2 > 0 \). The Cauchy-Schwarz inequality, the inequalities \((a \pm b)^2 \leq 2(a^2 + b^2)\) and the assumption upon the constitutive coefficients lead to
\[
(u, v) \leq C \left[ \int_{\Omega} \left( \| \text{sym} \nabla u \|^2 + \| \text{Curl}(\text{sym} \nabla u) \|^2 \right) dv \right]^{1/2} \int_{\Omega} \left( \| \text{sym} \nabla v \|^2 + \| \text{Curl}(\text{sym} \nabla v) \|^2 \right) dv
\]
\[
\leq C \| w \|_{X_0} \| \tilde{w} \|_{X_0}, \]
which means that \((\cdot, \cdot)\) is bounded. On the other hand, we have
\[
(u, u) = \int_{\Omega} \left[ 2\mu \| \text{sym} \nabla u \|^2 + \lambda \| \text{tr}(\nabla u) \|^2 \\
+ \mu L_c^2 \left[ 2\alpha_1 \| \text{dev sym Curl}(\text{sym} \nabla u) \|^2 + 2\alpha_2 \| \text{skew Curl}(\text{sym} \nabla u) \|^2 \right] \right] dv,
\]
for all \( u \in X_0 \). Moreover, as a consequence of the properties i) of the constitutive coefficients we have that there exists the positive constant \( c \)
\[
(u, u) \geq c \int_{\Omega} \left( \| \text{sym} \nabla u \|^2 + \| \text{Curl}(\text{sym} \nabla u) \|^2 \right) dv. \quad \text{(3.48)}
\]
From linearized elasticity we have the first Korn’s inequality [76], that is
\[
\| \nabla u \|_{L^2(\Omega)} \leq C \| \text{sym} \nabla u \|_{L^2(\Omega)}, \quad \text{(3.49)}
\]
for all functions $u \in H^1_0(\Omega; \Gamma)$ with some constant $C > 0$, for bounding the deformation of an elastic medium in terms of the symmetric strains. Hence, using the Korn’s inequality (3.49), it results that there is a positive constant $C$ such that
\[
(u, u) \geq c \int_\Omega \left( \| \nabla u \|^2 + \| \text{Curl (sym} \nabla u) \|^2 \right) dv = c \| u \|^2_{X_0}.
\] (3.50)

Therefore our bilinear form $(\cdot, \cdot)$ is coercive. The Cauchy-Schwarz inequality and the Poincaré-inequality imply that the linear operator $l(\cdot)$ is bounded. By the Lax-Milgram theorem it follows that (3.46) has one and only one solution. The proof is complete in the case $\alpha_2 > 0$.

Now, we consider the case $\alpha_2 = 0$. Using Lemma 3.2 it follows that the bilinear form $(\cdot, \cdot)$ is also coercive for $\alpha_2 = 0$. Using similar estimates as above the existence follows also in this case and the proof is complete.

**Remark 3.4.** The Lax-Milgram theorem used in the proof of the previous theorem also offers a continuous dependence result on the load $f$. Moreover, the weak solution $u$ minimizes on $X_0$ the energy functional
\[
I(u) = \int_\Omega \left[ 2 \mu \| \text{sym} \nabla u \|^2 + \lambda \| \text{tr}(\nabla u) \|^2 \\
+ \mu L_c^2 \| 2 \alpha_1 \| \text{dev sym} \text{Curl (sym} \nabla u) \|^2 + 2 \alpha_2 \| \text{skew} \text{Curl (sym} \nabla u) \|^2 \right] dv - \int_\Omega \langle f, u \rangle dv.
\]

Let us consider $v \in C_0^\infty(\Omega; \Gamma)$ and $u$ a solution of problem (3.34)–(3.36). Then we obtain
\[
(u, v) = \int_\Omega \langle f, v \rangle dv + \int_{\partial \Omega} \langle (\sigma + \text{sym} \text{Curl (}\hat{m})) \cdot n - \nabla[(\text{sym} \hat{M}) : T, v] \rangle da \tag{3.51}
+ \int_{\partial \Omega} \langle (\mathbf{1} - n \otimes n)(\text{sym} \hat{M}) \cdot n, (\mathbf{1} - n \otimes n)(\nabla v) \cdot n \rangle da + \int_{\partial \Omega} \langle [(\text{sym} \hat{M}) \cdot \nu], v \rangle ds
= \int_\Omega \langle f, v \rangle dv + \int_{\partial \Omega} \langle (\sigma + \text{sym} \text{Curl (}\hat{m})) \cdot n, v \rangle da - \int_{\partial \Omega} \sum_{i=1}^3 \langle \hat{m}_i \times n, (\text{sym} \nabla v)_i \rangle da.
\]

Therefore, the corresponding existence results assures that there exists the weak solution $u$ minimizing on $C_0^\infty(\Omega; \Gamma)$ the energy functional
\[
I(u) = \int_\Omega \left[ 2 \mu \| \text{sym} \nabla u \|^2 + \lambda \| \text{tr}(\nabla u) \|^2 \\
+ \mu L_c^2 \| 2 \alpha_1 \| \text{dev sym} \text{Curl (sym} \nabla u) \|^2 + 2 \alpha_2 \| \text{skew} \text{Curl (sym} \nabla u) \|^2 \right] dv - \int_\Omega \langle f, u \rangle dv
- \int_{\partial \Omega} \langle (\sigma + \text{sym} \text{Curl (}\hat{m})) \cdot n - \nabla[(\text{sym} \hat{M}) : T, u] \rangle da
- \int_{\partial \Omega} \langle (\mathbf{1} - n \otimes n)(\text{sym} \hat{M}) \cdot n, (\mathbf{1} - n \otimes n)(\nabla u) \cdot n \rangle da - \int_{\partial \Omega} \langle [(\text{sym} \hat{M}) \cdot \nu], u \rangle ds \tag{3.52}
= \int_\Omega \left[ 2 \mu \| \text{sym} \nabla u \|^2 + \lambda \| \text{tr}(\nabla u) \|^2 \right] dv.
\]
\[ + \mu L^2 \left[ 2 \alpha_1 \| \text{dev sym} \text{Curl} (\text{sym} \nabla u) \|^2 + 2 \alpha_2 \| \text{skew} \text{Curl} (\text{sym} \nabla u) \|^2 \right] \, dv - \int_\Omega (f, u) \, dv \]
\[ - \int_{\partial \Omega} \langle (\sigma + \text{sym} \text{Curl} (\hat{m})). n, u \rangle \, da + \int_{\partial \Omega} \sum_{i=1}^{3} \langle \hat{m}_i \times n, (\text{sym} \nabla v)_i \rangle \, da. \]

3.3 Traction boundary condition in the \( \text{Curl} (\text{sym} \nabla u) \)-formulation versus the \( \nabla [\text{axl} (\text{skew} \nabla u)] \)-formulation

In this section we compare the possible traction boundary conditions in the \( \nabla [\text{axl} (\text{skew} \nabla u)] \)-formulation and the \( \text{Curl} (\text{sym} \nabla u) \)-formulation. The conclusion is summarized in Figure 12 and Figure 3. Prescribing \( \delta u \) and \( (\mathbb{1} - n \otimes n). \text{curl} u \) on the boundary means that we have prescribed independent geometrical boundary conditions, this is also the argumentation of Mindlin and Tiersten [73], Koiter [51], Sokolowski [100], etc. However, the prescribed traction conditions remain not independent, in the sense that \( \tilde{g} \) leads to a further energetic conjugate, besides \( \tilde{t} \), of \( u \). From this reason we claim that, in order to prescribe independent geometric boundary conditions and their corresponding completely independent energetic conjugate (traction boundary conditions), we have to prescribe \( u \) and \( (\mathbb{1} - n \otimes n) \nabla u.n \). In other words, we prescribe

\[ \int_{\partial \Omega} \langle \tilde{t}, u \rangle \, da + \int_{\partial \Omega} \langle \tilde{g}, (\mathbb{1} - n \otimes n) \nabla u.n \rangle \, da, \] (3.53)

in which now \( u \) and \( (\mathbb{1} - n \otimes n) \nabla u.n \) are orthogonal in the sense of Definition 1.1 and \( \tilde{g} \) does not produce work against \( u \), see [62] for further detailed explanations. This type of orthogonal boundary conditions are also correctly considered already by Bleustein [10], but for the full strain gradient elasticity case only. In order to have a complete overview on the subject, in Table 1 we also summarize the equivalent form of the equilibrium equations.
Table 1. Euler-Lagrange equations in various formulations

<table>
<thead>
<tr>
<th>Euler-Lagrange equations in direct tensor format</th>
<th>Euler-Lagrange equations in indices</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Euler-Lagrange equations for Curl (sym ( \nabla u ))</strong></td>
<td><strong>Euler-Lagrange equations for Curl (sym ( \nabla u ))</strong></td>
</tr>
<tr>
<td>( \text{Div}(\sigma + \tilde{\tau}) + f = 0 )</td>
<td>( (\sigma_{ij} + \tilde{\tau}_{ij})_j + f_i = 0 )</td>
</tr>
<tr>
<td>( \sigma = D_{\text{sym}} \nabla u W_{\text{lin}}(\text{sym} \nabla u) \in \text{Sym}(3) )</td>
<td>( \sigma_{ij} = D^2_{\text{sym}} \left( u_{i,j} + u_{j,i} \right) W_{\text{lin}} \left( \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \right) \in \text{Sym}(3) )</td>
</tr>
<tr>
<td>( \tilde{\tau} = \text{sym}(\nabla \tilde{m}) \in \text{Sym}(3) )</td>
<td>( \tilde{\tau}<em>{ij} = \frac{1}{2} \left( \epsilon</em>{i,jk} \tilde{m}<em>{k,l} + \epsilon</em>{j,kl} \tilde{m}_{i,l} \right) \in \text{Sym}(3) )</td>
</tr>
<tr>
<td>( \tilde{m} = D_{\text{Curl}}(\text{sym} \nabla u) W_{\text{Curv}}(\text{Curl} \text{sym} \nabla u) ), second order</td>
<td>( \tilde{m}<em>{ij} = D^2</em>{\text{Curl}}(\epsilon_{ijk} u_{k,l} + u_{k,ij}) W_{\text{Curv}} \left( \frac{1}{2} \epsilon_{ijl} u_{l,ij} \right) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Euler-Lagrange equations for ( \nabla [\text{axl} \text{ skew} \nabla u] )</strong></th>
<th><strong>Euler-Lagrange equations for ( \nabla [\text{axl} \text{ skew} \nabla u] )</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Div}(\sigma - \tilde{\tau}) + f = 0 )</td>
<td>( (\sigma_{ij} - \tilde{\tau}_{ij})_j + f_i = 0 )</td>
</tr>
<tr>
<td>( \sigma = D_{\text{sym}} \nabla u W_{\text{lin}}(\text{sym} \nabla u) \in \text{Sym}(3) )</td>
<td>( \sigma_{ij} = D^2_{\text{sym}} \left( u_{i,j} + u_{j,i} \right) W_{\text{lin}} \left( \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \right) \in \text{Sym}(3) )</td>
</tr>
<tr>
<td>( \tilde{\tau} = \text{Div} \tilde{m} = \frac{1}{2} \text{anti}(\text{Div} \tilde{m}) \in \mathfrak{so}(3) )</td>
<td>( \tilde{\tau}<em>{ij} = \frac{1}{2} \epsilon</em>{ijk} \tilde{m}_{k,l} \in \mathfrak{so}(3) )</td>
</tr>
<tr>
<td>( \tilde{m} = D_{\nabla \text{axl} \text{ skew} \nabla u} W_{\text{Curv}}(\nabla \text{axl} \text{ skew} \nabla u) ), second order</td>
<td>( \tilde{m}<em>{ij} = D</em>{\nabla \text{axl} \text{ skew} \nabla u} W_{\text{Curv}} \left( \epsilon_{ijk} u_{k,j} \right) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Euler-Lagrange equations for ( \nabla [\text{axl} \text{ skew} \nabla u] )</strong> (3rd order)</th>
<th><strong>Euler-Lagrange equations for ( \nabla [\text{axl} \text{ skew} \nabla u] )</strong> (3rd order)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Div}(\sigma - \tilde{\tau}) + f = 0 )</td>
<td>( (\sigma_{ij} - \tilde{\tau}_{ij,k})_j + f_i = 0 )</td>
</tr>
<tr>
<td>( \sigma = D_{\text{sym}} \nabla u W_{\text{lin}}(\text{sym} \nabla u) \in \text{Sym}(3) )</td>
<td>( \sigma_{ij} = D^2_{\text{sym}} \left( u_{i,j} + u_{j,i} \right) W_{\text{lin}} \left( \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \right) \in \text{Sym}(3) )</td>
</tr>
<tr>
<td>( \tilde{\tau} = \text{Div} \tilde{m} = \frac{1}{2} \text{anti}(\text{Div} \tilde{m}) \in \mathfrak{so}(3) )</td>
<td>( \tilde{\tau}<em>{ij} = \frac{1}{2} \epsilon</em>{ijk} \tilde{m}_{k,l} \in \mathfrak{so}(3) )</td>
</tr>
<tr>
<td>( \tilde{m} = D_{\nabla \text{axl} \text{ skew} \nabla u} W_{\text{Curv}}(\nabla \text{axl} \text{ skew} \nabla u) ), third order</td>
<td>( \tilde{m}<em>{ijk} = D</em>{\nabla \text{axl} \text{ skew} \nabla u} \tilde{W}<em>{\text{Curv}} \left( \epsilon</em>{ijk} u_{k,j} \right) )</td>
</tr>
</tbody>
</table>

We outline that there exists a relation between the allowed traction boundary conditions in the \( \text{Curl} \text{sym} \nabla u \)-formulation and those from the \( \nabla \text{axl} \text{ skew} \nabla u \)-formulation which we take from [62]

\[
- \int_{\partial \Omega} \langle (\sigma - \tilde{\tau}) . n, \delta u \rangle \, da - \int_{\partial \Omega} \langle \tilde{m}, n, \text{axl} \text{ skew} \nabla u \rangle \, da = - \int_{\partial \Omega} \langle (\sigma + \tilde{\tau}) . n, \delta u \rangle \, da - \int_{\partial \Omega} \sum_{i=1}^{3} \langle \tilde{m}_i \times n, \text{sym} \nabla \delta u_i \rangle \, da.
\] (3.54)
or after splitting up with use of the surface divergence theorem on both sides

\[- \int_{\partial \Omega} \langle (\sigma + \text{sym Curl} (\hat{m})) \cdot n - \nabla[\text{sym} (\hat{M})] (\mathbb{1} - n \otimes n), \delta u \rangle \, da \quad (1) \]

\[- \int_{\partial \Omega} \langle (\mathbb{1} - n \otimes n) \text{sym} (\hat{M}) \cdot n, \nabla \delta u \cdot n \rangle \, da - \oint_{\partial \Gamma} \langle \text{sym} (\hat{M}) \cdot \nu, \delta u \rangle \, ds \quad (2) \]

\[- \int_{\partial \Omega} \langle (\sigma - \frac{1}{2} \text{anti Div} [\hat{m}]) \cdot n - \frac{1}{2} \nabla [(\text{anti} (\hat{m} \cdot n)) (\mathbb{1} - n \otimes n)], (\mathbb{1} - n \otimes n), \delta u \rangle \, da \quad (1') \]

\[- \int_{\partial \Omega} \langle \frac{1}{2} (\mathbb{1} - n \otimes n) \text{anti} (\hat{m} \cdot n), \nabla \delta u \cdot n \rangle \, da - \int_{\partial \Gamma} \langle \frac{1}{2} [\text{anti} (\hat{m} \cdot n)] \cdot \nu, \delta u \rangle \, ds, \quad (2') \]

\[- \int_{\partial \Gamma} \langle \text{axl} (G) \cdot \nu, \text{axl} (\hat{A}) \rangle \, da, \quad (3') \]

\[\text{for all variations } \delta u \in C^\infty (\Omega). \text{ Naively, we might expect that the quantities involved have to be equal term by term, i.e. } (1) = (1'), (2) = (2'), (3) = (3') \text{ or } (a) = (a'), (b) = (b'), (c) = (c'). \text{ However, this is not true, see Appendix A.1.}\]

### 3.4 Principle of virtual work in the indeterminate couple stress model

#### 3.4.1 Principle of virtual work in Cosserat theory

Let us first recall that in the Cosserat theory with independent fields of displacement and micro-rotations the internal energy has the form \( W(\nabla u, \hat{A}, \nabla \hat{A}) \) in which \( u : \Omega \to \mathbb{R}^3 \) is the displacement and \( \hat{A} : \Omega \to \mathfrak{so}(3) \) is the infinitesimal microrotation. The virtual work principle of the Cosserat theory is given by

\[ p^{\text{int}} = p^{\text{ext}}, \quad (3.56) \]

where

\[ p^{\text{int}} = \frac{d}{dt} \int_{\Omega} W((\nabla u + t \nabla \delta u), \hat{A} + t \delta \hat{A}, \nabla (\hat{A} + t \delta \hat{A})) \bigg|_{t=0}, \]

\[ p^{\text{ext}} = \int_{\Omega} \langle \text{force}, u \rangle \, dv + \int_{\Omega} \langle \text{axl} (\mathfrak{so}(3)), \text{axl} (\hat{A}) \rangle \, dv \]

\[ + \int_{\partial \Omega} \langle \text{tractions}, u \rangle \, da + \int_{\partial \Omega} \langle \text{axl} (\mathfrak{gl}), \text{axl} (\hat{A}) \rangle \, da, \quad (3.57) \]
with \( f : \partial \Omega \setminus \Gamma \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), \( t : \partial \Omega \setminus \Gamma \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), \( \mathcal{M} : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \), \( \mathcal{G} : \overline{\Omega} \subseteq \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \). From this virtual work principle, one obtains the equilibrium equations

\[
\text{Div } \sigma + f = 0, \quad (3.58)
\]

\[
\text{Div } m + \text{axl}(\text{skew } \sigma) + \text{axl}(\mathcal{M}) = 0,
\]

where \( \sigma = D_{\text{sym}} \nabla u W(\text{sym } \nabla u, A) \) and \( m = D_{\nabla A} W(\nabla u, \overline{A}, \nabla \overline{A}) \), and the boundary conditions

\[
\sigma.n = t, \quad m.n = \text{axl}(\mathcal{G}). \quad (3.59)
\]

In order to obtain these equilibrium equations and the form of the boundary conditions, we have used the fact that \( u \) and \( \overline{A} \) are automatically orthogonal being independent constitutive variables.

### 3.4.2 Virtual work principle in the Curl (sym(\nabla u))-formulation of the indeterminate couple stress model

Using again the fact that \( u \) and \( 2 \text{curl } u = \text{axl}(\text{skew } \nabla u) \sim \overline{A} \) are not independent constitutive variables, and the identity

\[
\nabla(\text{axl}(\text{skew } \nabla u)) = [\text{Curl } \text{sym}(\nabla u)]^T
\]

we consider the energy \( W(\text{sym}(\nabla u), \text{Curl } \text{sym}(\nabla u)) \) and the following new form of the virtual work principle

\[
\mathcal{P}^{\text{int}} = \mathcal{P}^{\text{ext}}, \quad (3.60)
\]

where

\[
\mathcal{P}^{\text{int}} = \frac{d}{dt} \left. \int_{\Omega} W(\text{sym}(\nabla u + t\nabla \delta u), \text{Curl } \text{sym}(\nabla u + t\nabla \delta u)) \right|_{t=0},
\]

\[
\mathcal{P}^{\text{ext}} = \int_{\Omega} \langle \text{body forces } f, u \rangle dv + \int_{\Omega} \langle \text{body couple } \mathcal{M}, \text{sym } \nabla u \rangle dv
\]

\[
+ \int_{\partial \Omega} \langle \text{surface traction } t, u \rangle da + \int_{\partial \Omega} \langle \text{surface double tractions } g, (\mathbb{I} - n \otimes n) \nabla u.n \rangle da
\]

\[
+ \oint_{\partial \Gamma} \langle \text{edge line force } \pi, \nu \rangle ds,
\]

with \( f : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), \( t : \partial \Omega \setminus \Gamma \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), \( \mathcal{M} : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \), \( g : \partial \Omega \setminus \Gamma \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) and \( \pi : \partial \Gamma \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \).

From this virtual work principle, we obtain the equilibrium equations

\[
\text{Div } (\sigma + \mathcal{T} + \mathcal{M}) + \text{Div } \mathcal{M} + f \overset{\text{the total body force}}{=} 0, \quad (3.62)
\]
where

\[ \sigma = D_{\text{sym}} \nabla u W(\text{sym} \nabla u, \text{Curl} (\text{sym} \nabla u)), \]
\[ \tilde{\tau} = \text{sym}[\text{Curl} \tilde{m}] \in \text{Sym}(3), \]
\[ \tilde{m} = D_{\text{Curl}(\text{sym} \nabla u)} W(\text{sym} \nabla u, \text{Curl} (\text{sym} \nabla u)) \]

and the following traction boundary conditions

\[ [(\sigma + \tilde{\tau}).n - \nabla[(\text{sym} M)(\mathbb{I} - n \otimes n)] : (\mathbb{I} - n \otimes n)] = t - M.n, \quad \text{the total traction condition} \]
\[ [(\mathbb{I} - n \otimes n)(\text{sym} \tilde{M}).n] = [(\mathbb{I} - n \otimes n) g], \]
\[ [(\text{sym} \tilde{M}).\nu]_{\partial \Gamma} = \pi, \]

where \( \Gamma \) is an arbitrary open subset of \( \partial \Omega \).

In Subsection 3.3 and Appendix A.1 we show that the traction boundary conditions in the \( \text{Curl} (\text{sym} (\nabla u)) \)-formulation do not coincide pointwise with those arising from \( \nabla[\text{axl}(\text{skew} \nabla u)] \)-formulation, even if the boundary virtual power works are identical.

## 4 Relation to the Cosserat-micropolar and micromorphic model

We have seen that it is irrelevant whether we take \( \nabla[\text{axl}(\text{skew} \nabla u)] \) or \( \text{Curl} (\text{sym} \nabla u) \) as basic curvature measures for the indeterminate couple stress model as long as consistent requirements on \( \Gamma = \partial \Omega \) are considered and the following Dirichlet conditions are used both together

\[ u \big|_{\Gamma} = u_{\text{ext}}, \quad (\mathbb{I} - n \otimes n) \nabla u.n \big|_{\Gamma} = (\mathbb{I} - n \otimes n).a_{\text{ext}} \]
\[ \Leftrightarrow u \big|_{\Gamma} = u_{\text{ext}}, \quad (\mathbb{I} - n \otimes n).\text{curl} u \big|_{\Gamma} = (\mathbb{I} - n \otimes n).b_{\text{ext}}. \]

The difference of the formulation appears only when considering mixed Dirichlet-Neumann boundary conditions. However, when we want to switch from a 4th-order (gradient elastic) problem to a 2nd-order micromorphic model or Cosserat model [15, 81, 28], we need to introduce new independent variables and decide about the useful coupling conditions in terms of adding a penalty term. It is also clear that when adding more variables it depends on the number of the added fields whether the new formulation is weaker softer in the language of a finite element context. In general, more degrees of freedom mean softer response, at the price of needing to specify more boundary conditions.

We discuss the following cases:

i) [Cosserat] \( \text{skew} \nabla u \mapsto \bar{A} \in \mathfrak{so}(3) \). In the case \( \nabla[\text{axl}(\text{skew} \nabla u)] \) we are led to introduce a skew-symmetric variable \( \bar{A} \in \mathfrak{so}(3) \) instead of \( \text{skew} \nabla u \), thus using the curvature tensor \( \nabla \text{axl}(\bar{A}) \) together with the coupling

\[ \mu_c \| \text{skew}(\nabla u) - \bar{A} \|^2 = \frac{\mu_c}{2} \| \text{curl} u - 2 \text{axl}(\bar{A}) \|^2, \]
Figure 4: Different possibilities of lifting the variants of the 4th.-order indeterminate couple stress model to a 2nd.-order micromorphic or Cosserat-type formulation formulation. In the penalty case, all the considered alternatives lead to the same limit model provided only geometric boundary conditions are imposed. It is not surprising that the limit model has some peculiarities since the limit procedure is itself singular. Note that different micromorphic or Cosserat type formulations generate different sets of boundary conditions. Here $\tau_\alpha$, $\alpha = 1, 2$ denote two independent tangential vectors on the boundary.
\[
\n\nabla [\text{axl} (\text{skew} \nabla u)] = [\text{Curl} (\text{sym} \nabla u)]^T \quad \text{but} \quad \nabla \text{axl} (\text{skew} p) \neq [\text{Curl} \text{ sym} p]^T \quad \text{for} \quad p \neq \nabla u
\]

Figure 5: Integrability conditions.

leading to the classical Cosserat model, with a new penalty parameter \( \mu_c > 0 \) known as the Cosserat couple modulus. To be more precise, the corresponding minimization problem becomes

\[
I(u, A) = \int_{\Omega} \left[ \mu \|\text{sym} \nabla u\|^2 + \frac{\lambda}{2} [\text{tr} (\text{sym} \nabla u)]^2 + \mu_c \|\text{skew} (\nabla u) - A\|^2 + \mu L_c^2 (\alpha_1 \|\text{dev} \text{ sym} \nabla \text{axl}(A)\|_2^2 + \alpha_2 \|\text{skew} \text{ axl} (A)\|_2^2) \right] dV
\]

min. w.r.t \( u \in H^1_0 (\Omega), \; A \in H^1_0 (\Omega) \). In this case, the force-stress tensor is clearly non-symmetric

\[
\sigma = 2 \mu \text{ sym} \nabla u + 2 \mu_c (\text{skew} \nabla u - A) + \lambda \text{ tr} (\nabla u) \; 1 \notin \text{Sym}(3), \quad (4.2)
\]

and the couple stress tensor (hyperstress tensor) is given by

\[
\tilde{m} = 2 \mu L_c^2 (\alpha_1 \|\text{dev} \text{ sym} \nabla \text{axl}(A)\|_2 + \alpha_2 \|\text{skew} \nabla \text{axl}(A)\|_2), \quad (4.3)
\]

which is also in general non-symmetric. Note that \( \tilde{m} \) has now 3 independent length scale parameters.

ii) [microstrain] \( \text{sym} \nabla u \rightarrow \hat{\varepsilon} \in \text{Sym}(3) \). In the case of starting with the representation \( \text{Curl} (\text{sym} \nabla u) \) we are led to introduce a symmetric tensor variable \( \hat{\varepsilon} \in \text{Sym}(3) \) instead of \( \text{sym} \nabla u \), thus using the curvature measure \( \text{Curl} \hat{\varepsilon} \) together with the coupling

\[
\kappa^+ \|\text{sym} \nabla u - \hat{\varepsilon}\|^2,
\]

leading to a “microstrain” theory [33, 79], the minimization problem is now

\[
I(u, \hat{\varepsilon}) = \int_{\Omega} \left[ \mu \|\text{sym} \nabla u\|^2 + \frac{\lambda}{2} [\text{tr} (\text{sym} \nabla u)]^2 + \kappa^+ \|\text{sym} \nabla u - \hat{\varepsilon}\|^2 + \mu L_c^2 (\beta_1 \|\text{dev} \text{ Curl} \hat{\varepsilon}\|_2^2 + \beta_2 \|\text{skew} \text{ Curl} \hat{\varepsilon}\|_2^2) \right] dV \rightarrow \text{min.}
\]

w.r.t \( u \in H^1_0 (\Omega), \; \hat{\varepsilon} \in H^1_0 (\text{Curl}; \Omega) \), and, in this case, the force-stress tensor is symmetric

\[
\sigma = 2 \mu \text{ sym} \nabla u + 2 \kappa^+ (\text{sym} \nabla u - \hat{\varepsilon}) + \lambda \text{ tr} (\nabla u) \; 1 \in \text{Sym}(3), \quad (4.4)
\]

and the hyperstress-tensor is given by

\[
\tilde{m} = 2 \mu L_c^2 (\beta_1 \text{ dev} \text{ Curl} \hat{\varepsilon} + \beta_2 \text{ skew} \text{ Curl} \hat{\varepsilon}), \quad (4.5)
\]

which is non-symmetric in general, depending on the material parameters. Note again that \( \text{tr} (\text{Curl} \hat{\varepsilon}) = 0 \), thus the spherical part of the hyperstress tensor vanishes and \( \tilde{m} \) features only 2 independent length scale parameters.
iv) the relaxed micromorphic model
\[ \sigma = 2 \mu \text{sym} \nabla \umu + 2 \kappa^+ \text{sym}(\nabla \umu - p) \in \text{Sym}(3) \]
\[ m = 2 \mu L_2^g \text{ Curl } p \in \mathfrak{gl}(3) \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ Curl } p \|^2 \]
well-posed:
\[ u \mapsto u + \bar{W} x + \bar{b}, \bar{b} \in \mathbb{R}^3 \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ Curl } p \|^2 \]
well-posed:
\[ \Gamma \in H^1(\Omega), p \in H(\text{Curl}; \Omega) \]
\[ 3 + 3 \cdot 2 = 9 \text{ geometric bc: } u_{\Gamma}, p, \tau_{\alpha}|_\Gamma \]
\[ 3 + 3 \cdot 2 = 9 \text{ traction bc: } \sigma.n|_{\partial \Omega}, m, \tau_{\alpha}|_{\partial \Omega} \]

v) the further relaxed micromorphic model
\[ \sigma = 2 \mu \text{sym} \nabla \umu + 2 \kappa^+ \text{sym}(\nabla \umu - p) \in \text{Sym}(3) \]
\[ m = 2 \mu L_2^g \text{ dev Curl } p \in \mathfrak{gl}(3) \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ dev Curl } p \|^2 \]
well-posed:
\[ u \mapsto u + \bar{W} x + \bar{b}, \bar{b} \in \mathbb{R}^3 \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ dev Curl } p \|^2 \]
well-posed:
\[ \Gamma \in H^1(\Omega), p \in H(\text{Curl}; \Omega) \]
\[ 3 + 3 \cdot 2 = 9 \text{ geometric bc: } u_{\Gamma}, p, \tau_{\alpha}|_\Gamma \]
\[ 3 + 3 \cdot 2 = 9 \text{ traction bc: } \sigma.n|_{\partial \Omega}, m, \tau_{\alpha}|_{\partial \Omega} \]

vi) another relaxed micromorphic model
\[ \sigma = 2 \mu \text{sym} \nabla \umu + 2 \kappa^+ \text{sym}(\nabla \umu - p) \in \text{Sym}(3) \]
\[ m = 2 \mu L_2^g \text{ sym Curl } p \in \mathfrak{sym}(3), \text{ symmetric} \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ sym Curl } p \|^2 \]
well-posed not clear:
\[ u \in H^1(\Omega), p \in H(\text{Curl}; \Omega) \]
\[ 3 + 3 \cdot 2 = 9 \text{ traction bc: } \sigma.n|_{\partial \Omega}, m, \tau_{\alpha}|_{\partial \Omega} \]

vii) relaxed micromorphic model with integral boundary coupling
\[ \sigma = 2 \mu \text{sym} \nabla \umu + 2 \kappa^+ \text{sym}(\nabla \umu - p) \in \text{Sym}(3) \]
\[ m = 2 \mu L_2^g \text{ Curl } p \in \mathfrak{gl}(3) \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ Curl } p \|^2 \]
\[ + \int_{\Gamma} \| \nabla \umu - p \| n \|^2 \text{da} \]
well-posed:
\[ u \mapsto u + \bar{W} x + \bar{b}, \bar{b} \in \mathbb{R}^3 \]
\[ \mathcal{E} \sim \mu (\text{sym} \nabla \umu)^2 + \kappa^+ \| \text{sym}(\nabla \umu - p) \|^2 + \mu L_2^g \| \text{ Curl } p \|^2 \]
well-posed:
\[ \Gamma \in H^1(\Omega), p \in H(\text{Curl}; \Omega) \]
\[ 3 + 3 \cdot 2 = 9 \text{ traction bc: } \sigma.n|_{\partial \Omega}, m, \tau_{\alpha}|_{\partial \Omega} \]

Figure 6: For comparison, the same situation as in Fig. 4 for the relaxed micromorphic model and the further relaxed micromorphic model [81]. The first two micromorphic formulations are well-posed. In our view these models are advantageous as compared to the models in Fig. 4. Note also that the boundary conditions for the new microdistortion field \( p \) has 6 degrees of freedom, the only physical and mathematical possible choice is \( p \times n \mid _{\Gamma} = 0 \) on \( \Gamma \), where Dirichlet-boundary conditions \( u_{\Gamma} = 0 \) are prescribed. In the case of non-homogeneous (non-zero) boundary prescription \( u \mid _{\Gamma} \), we would need to modify the total energy by adding as a boundary term \( \int_{\Gamma} \| \nabla \umu - p \| n \|^2 \text{ds} \). This will introduce a certain coupling at the boundary. The model in this format is still well-posed and awaits to be further investigated. A nonlinear modification of this model is investigated in [55].
Figure 7: Some models involving size effects and their interrelation. The given energy expressions are only meant to represent the main features of the models: $\mu \| \text{sym} \nabla u \|^2$ represents the linear elastic energy which may be anisotropic $\langle C, \text{sym} \nabla u, \text{sym} \nabla u \rangle$, while $\mu L_2^2 \| \text{dev} \nabla \text{axl} \{ \text{skew} \nabla u \} \|^2$ represents the curvature energy, which could also be anisotropic.
\[ \hat{\tau} = \mu L_c^2 \text{sym} \text{Curl} \{2 \alpha_1 \text{dev sym} \text{Curl} (\text{sym} \nabla u) + 2 \alpha_2 \text{skew} \text{Curl} (\text{sym} \nabla u)\} = \text{sym} \text{Curl} (\hat{m}) \in \text{Sym}(3) \]

\[ \hat{\tau} = \mu L_c^2 \text{anti Div} \{\alpha_1 \text{dev sym} \nabla \text{axl} (\text{skew} \nabla u) + \alpha_2 \text{skew} \nabla [\text{axl} (\text{skew} \nabla u)]\} = \text{anti Div}(\hat{m}) \in \mathfrak{so}(3) \]

\[ \text{Div}(\hat{\tau} - \hat{\tau}) = 0 \]

Figure 8: The two possibilities of defining a nonlocal force stress tensor: either \(\hat{\tau}\) is symmetric in the Curl (sym \nabla u) formulation or \(\hat{\tau}\) is antisymmetric in the \(\nabla [\text{axl} (\text{skew} \nabla u)]\) formulation. The difference between both stresses is a divergence-free stress field (a self-equilibrated force field).

iii) [micromorphic] \(\nabla u \mapsto p\). In this case we may introduce a tensor \(p \in \mathbb{R}^{3 \times 3}\) instead of \(\nabla u\), and use the coupling

\[ \kappa^+ \|\nabla u - p\|^2 \]

leading to a micromorphic theory [28, 81], the minimization problem being

\[ I(u, p) = \int_\Omega \left[ \mu \|\text{sym} \nabla u\|^2 + \frac{\lambda}{2} [\text{tr} (\text{sym} \nabla u) ]^2 + \kappa^+ \|\nabla u - p\|^2 \right. \]

\[ + \mu L_c^2 (\gamma_1 \|\text{dev sym} \text{Curl} (\text{sym} p)\|^2 + \gamma_3 \|\text{skew} \text{Curl} (\text{sym} p)\|^2) \left.] dV \rightarrow \min. \]

w.r.t \(u \in H_0^1(\Omega)\), \(p \in H_0^1 (\text{Curl} ; \Omega)\). We also point out that the force-stress tensor in this formulation will be non-symmetric

\[ \sigma = 2 \mu \text{sym} \nabla u + 2 \kappa^+ (\nabla u - P) + \lambda \text{tr} (\nabla u) \mathbb{I} \]

\[ = 2 \mu \text{sym} \nabla u + 2 \kappa^+ \text{skew} (\nabla u - P) + 2 \kappa^+ \text{sym} (\nabla u - P) + \lambda \text{tr} (\nabla u) \mathbb{I} \notin \text{Sym}(3) \]

and the hyperstress tensor (non-symmetric) is given by

\[ \hat{m} = 2 \mu L_c^2 (\gamma_1 \text{dev sym} \text{Curl} (\text{sym} p) + \gamma_3 \text{skew} \text{Curl} (\text{sym} p)). \]

(4.7)

In this formulation \(\text{tr} (\text{Curl} \text{sym} p)\) does not appear since \(\text{tr} (\text{Curl} \text{sym} p) = 0, \forall p \in \mathbb{R}^{3 \times 3}\). Thus the spherical part of the hyperstress tensor vanishes, and \(\hat{m}\) features only 2 independent length scale parameters.

iv) [relaxed micromorphic] for comparison with other extended continuum models we present the relaxed micromorphic model [81, 35, 64, 63, 80]. In the relaxed micromorphic model, the minimization problem is of the type

\[ I(u, p) = \int_\Omega \left[ \mu \|\text{sym} \nabla u\|^2 + \frac{\lambda}{2} [\text{tr} (\text{sym} \nabla u) ]^2 + \kappa^+ \|\text{sym} (\nabla u - p)\|^2 \right. \]

\[ + L_c^2 (\alpha_1 \|\text{dev sym} \text{Curl} p\|^2 + \alpha_2 \|\text{skew} \text{Curl} p\|^2 + \alpha_3 [\text{tr} (\text{Curl} p)]^2) \left.] dV \]

43
w.r.t $u \in H^1_0(\Omega)$, $p \in H^1_0(\text{Curl}; \Omega)$, and the corresponding force-stress tensor is symmetric
\[ \sigma = 2\mu \text{sym} \nabla u + 2 \sigma^+ \text{sym}(\nabla u - p) + \lambda \text{tr}(\nabla u) \mathbb{1} \in \text{Sym}(3) \quad (4.8) \]
and the hyperstress tensor is given by
\[ \hat{m} = 2\mu L^2_c (\alpha_1 \text{dev sym Curl} p + \alpha_2 \text{skew Curl} p + \alpha_3 \text{tr}[\text{Curl} p] \mathbb{1}), \quad (4.9) \]
with a non-vanishing spherical part of the hyperstress tensor. Note that $\hat{m}$ has 3 independent material parameters.

v) we have also proposed a further relaxed micromorphic model [81, 35, 64, 63, 80], in which case the minimization problem is of the type
\[ I(u, p) = \int_{\Omega} \left[ \mu \|\text{sym} \nabla u\|^2 + \lambda \left( \text{tr}(\text{sym} \nabla u) \right)^2 + \sigma^+ \|\text{sym}(\nabla u - p)\|^2 + \mu L^2_c (\alpha_1 \|\text{dev sym Curl} p\|^2 + \alpha_2 \|\text{skew Curl} p\|^2) \right] dV \]
w.r.t $u \in H^1_0(\Omega)$, $p \in H^1_0(\text{Curl}; \Omega)$, the corresponding force-stress tensor is symmetric
\[ \sigma = 2\mu \text{sym} \nabla u + 2 \sigma^+ \text{sym}(\nabla u - p) + \lambda \text{tr}(\nabla u) \mathbb{1} \in \text{Sym}(3), \quad (4.10) \]
and the hyperstress $\hat{m}$ is trace free
\[ \hat{m} = 2\mu L^2_c (\alpha_1 \text{dev Curl} p + \alpha_2 \text{skew Curl} p). \quad (4.11) \]
The further relaxed micromorphic model remains well-posed [35]. A still weaker variant is v) with $\alpha_2 = 0$. Whether this choice is mathematically well-posed is yet unclear.

5 Conclusion

Our new symmetric-conformal $\text{Curl}(\text{sym} \nabla u)$-reformulation has the following crucial properties setting it apart from existing formulations of couple-stress models:

- the local and the nonlocal force stress tensors ($\sigma$, $\tilde{\sigma}$) are both symmetric, while the couple stress tensor $\hat{m}$ is symmetric in the conformally-invariant model.
- the curvature energy is conformally invariant and the couple stress tensor $\hat{m}$ vanishes for conformal displacement.
- the model has only one additional length scale parameter, similar to the modified couple stress model.
- the model is derived with consistent boundary conditions: either 5 geometrical conditions or 5 mechanical (traction) conditions. The mechanical conditions are separated into force stress tractions and couple stress tractions and correspond to completely orthogonal boundary conditions in the sense of Definition 1.1.
- for mixed Dirichlet-Neumann boundary conditions the model does not reduce to the modified indeterminate couple stress model.
Standard boundary conditions in the indeterminate couple stress model [73]

Geometric (essential) boundary conditions (3+2) [independent, not orthogonal]
\[ u|_{\Gamma} = u^{ext} \in \mathbb{R}^3, \quad (\mathbf{1} - n \otimes n) \nabla u.n|_{\Gamma} = (\mathbf{1} - n \otimes n) \tilde{a}^{ext} \in \mathbb{R}^3, \quad \text{or} \quad (\mathbf{1} - n \otimes n) . \text{curl} u|_{\Gamma} = (\mathbf{1} - n \otimes n) \tilde{f}^{ext} \]

Mechanical (traction) boundary conditions (3+2)
\[ ((\sigma - \tilde{\tau}).n - \frac{1}{2} n \times \nabla[(n, (\text{sym} \tilde{m}).n)])|_{\partial \Omega \setminus \Gamma} = \tilde{f}^{ext}, \quad \tilde{\tau} = \text{Div} \tilde{m} = \frac{1}{2} \text{anti}\{\text{Div} \tilde{m}\} \in \mathfrak{so}(3) \quad \text{3 bc} \]

Boundary virtual work
\[ - \int_{\partial \Omega} (\langle \sigma - \tilde{\tau} \rangle \cdot n, \delta u) \, da - \int_{\partial \Omega} (\tilde{m} \cdot n, \text{axl}(\text{skew} \nabla \delta u)) \, da = 0 \quad \Leftrightarrow \]
\[ - \int_{\partial \Omega} \left\{ \langle \sigma - \tilde{\tau} \rangle \cdot n - \frac{1}{2} n \times \nabla[(n, (\text{sym} \tilde{m}).n)] \right\}, \delta u \, da - \int_{\partial \Omega} (\tilde{m} \cdot n, \left\{ (\mathbf{1} - n \otimes n) \text{axl}(\text{skew} \nabla \delta u) \right\} ) \, da = 0 \]

Standard boundary conditions in the indeterminate couple stress model, index-format

Geometric (essential) boundary conditions (3+2) [independent, not orthogonal]
\[ u|_{\Gamma} = u^{ext} \in \mathbb{R}^3, \quad (\epsilon_{ijkl} u^i_k - \epsilon_{ijkl} u^j_k n_j n_i)|_{\Gamma} = \epsilon_{ijkl} u^{ext}_{i,k} - \epsilon_{ijkl} u^{ext}_{j,k} n_j n_i \]

or \[ (u_{i,k} n_k - u_{j,k} n_j n_j)|_{\Gamma} = u^{ext}_{i,k} n_k - u^{ext}_{j,k} n_j n_j \]

Mechanical (traction) boundary conditions (3+2)
\[ ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j - \frac{1}{2} \epsilon_{ijkl} (\tilde{m}_{ij} n_j n_i))|_{\partial \Omega \setminus \Gamma} = \epsilon^{ext}_{ij}, \quad \tilde{\tau}_{ij} = \frac{1}{2} \epsilon_{ijkl} \tilde{m}_{kl,j} \in \mathfrak{so}(3) \quad \text{3 bc} \]

Boundary virtual work
\[ - \int_{\partial \Omega} ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j) \delta u_i \, da - \int_{\partial \Omega} (\tilde{m} \cdot n, \text{axl}(\text{skew} \nabla \delta u)) \, da = 0 \quad \Leftrightarrow \]
\[ - \int_{\partial \Omega} ((\sigma_{ij} - \tilde{\tau}_{ij}) n_j - \frac{1}{2} \epsilon_{ijkl} (\tilde{m}_{ij} n_j n_i)) \delta u_i \, da - \frac{1}{2} \int_{\partial \Omega} (\tilde{m}_{ik} n_k - \tilde{m}_{jk} n_k n_j) \left( \epsilon_{ijkl} \delta u_{i,k} - \epsilon_{ijkl} \delta u_{j,k} n_j n_i \right) \, da = 0 \]

Figure 9: The standard boundary conditions in the indeterminate couple stress model which have been employed hitherto by all authors to our knowledge. The virtual displacement is denoted by \( \delta u \in C^\infty(\Omega) \). The number of traction boundary conditions is correct, but the split into independent variations at the boundary is not orthogonal.

We also conclude that:

- All the differences between the various alternative couple stress models can be traced back to the appearance of null-Lagrangians, either on the level of the total stresses or on the level of the moment stresses.
- Null-Lagrangians leave the Euler-Lagrange equations invariant but alter the boundary conditions. For each choice of Null-Lagrangian different “material parameters” intervene.
- The question arises: how can we identify boundary value problem-independent parameters in the indeterminate couple stress model when the only effect different representations have is
### Boundary conditions in the indeterminate couple stress model in terms of gradient elasticity and third order moment tensors

**Geometric (essential) boundary conditions** \((3+2)[\text{independent} + \text{orthogonal}]\)

\[
\mathbf{u}\big|_{\Gamma} = u_{\text{ext}} \in \mathbb{R}^3, \quad (\mathbf{I} - n \otimes n)\nabla \mathbf{u} \cdot n\big|_{\Gamma} = (\mathbf{I} - n \otimes n)\tilde{\mathbf{u}}_{\text{ext}} \in \mathbb{R}^3, \quad \text{or} \quad (\mathbf{I} - n \otimes n) \mathbf{\nabla} \mathbf{u}\big|_{\Gamma} = (\mathbf{I} - n \otimes n)\tilde{\mathbf{u}}_{\text{ext}}
\]

**Mechanical (traction) boundary conditions** \((3+2)\)

\[
((\sigma - \text{Div} \mathbf{m}) . n - \nabla ((\tilde{\mathbf{m}} . n) (\mathbf{I} - n \otimes n)) \big|_{\partial\Omega_{\text{Gamma}}} = t_{\text{ext}}, \quad \mathbf{m} = D \mathbf{\nabla} \mathbf{u} [\nabla [\text{axl}(\text{skew} \nabla \mathbf{u})]] \quad (3 \text{bc})
\]

\[
(\mathbf{I} - n \otimes n) ([\tilde{\mathbf{m}} . n]) . n\big|_{\partial\Omega_{\text{Gamma}}} = (\mathbf{I} - n \otimes n) \mathbf{m}_{\text{ext}} \quad (2 \text{bc})
\]

\[
[[\tilde{\mathbf{m}} . n], \nu]\big|_{\partial\Omega} = \pi_{\text{ext}} \quad \text{“edge line force” on } \partial\Gamma \quad (3 \text{bc})
\]

**Boundary virtual work**

\[
- \int_{\partial\Omega} (((\sigma - \text{Div} \mathbf{m}) . n, \delta u) \, da - \int_{\partial\Omega} (\tilde{\mathbf{m}} . n, \nabla \delta u) \, da = 0 \quad \Leftrightarrow
\]

\[
- \int_{\partial\Omega} (((\sigma - \text{Div} \mathbf{m}) . n - \nabla ((\tilde{\mathbf{m}} . n) (\mathbf{I} - n \otimes n)), \delta u) \, da - \int_{\partial\Omega} (\tilde{\mathbf{m}} . n, \nabla \delta u) \, da = 0
\]

\[
- \int_{\partial\Omega} (\tilde{\mathbf{m}} . n, \delta u) \, ds = 0
\]

\[\updownarrow \text{ equivalent}\]

### Boundary conditions \((3+2)\) in the indeterminate couple stress model in terms of gradient elasticity, third order moment tensors, and written in indices

**Geometric (essential) boundary conditions** \((3+2)[\text{independent} + \text{orthogonal}]\)

\[
u_i = u^\text{ext}, \quad (u_{i,k} n_k - u_{j,k} n_j n_k n_i)\big|_{\Gamma} = u^\text{ext}_{i,k} n_k - u^\text{ext}_{j,k} n_j n_k, \quad \text{or} \quad (\epsilon_{i,k,l} u_{i,k} - \epsilon_{j,k,l} u_{j,k} n_j n_i)\big|_{\Gamma} = (\epsilon_{i,k,l} u^\text{ext}_{i,k} - \epsilon_{j,k,l} u^\text{ext}_{j,k} n_j n_i)
\]

**Mechanical (traction) boundary conditions** \((3+2)\)

\[
[[\sigma_{ij} - \tilde{m}_{ijk,k}] n_j - \tilde{m}_{ijp} n_k - \tilde{m}_{ij,k} n_j n_i, p\big|_{\partial\Omega_{\text{Gamma}}} = \rho_{\text{ext}}, \quad \tilde{m}_{ijk} = D_{n,jkh} \mathbf{\nabla} \mathbf{u} \cdot [\nabla [\text{axl}(\text{skew} \mathbf{\nabla} \mathbf{u})]] \quad (3 \text{bc})
\]

\[
(\tilde{m}_{ijp} n_j - \tilde{m}_{jp,k} n_k n_j n_i) \big|_{\partial\Omega_{\text{Gamma}}} = m^\text{ext}_{i} - m^\text{ext}_{j} n_p n_i \quad (2 \text{bc})
\]

\[
[[\tilde{m}_{ijp} n_k, \nu_j]\big|_{\partial\Omega} = \pi^\text{ext} \quad \text{“edge line force” on } \partial\Gamma \quad (3 \text{bc})
\]

---

**Figure 10:** The orthogonal-standard boundary conditions in the indeterminate couple stress model showing that the way \(\tilde{\mathbf{m}}\) is chosen to \(\tilde{\mathbf{m}}\) in the classical linear isotropic elasticity theory this sort of freedom to play with Null-Lagrangians does not exist and the two Lamé-parameters are really material parameters, independent of the specific boundary value problem [54].

**on the boundary conditions? We believe that this is one of the fundamental issues in linear, isotropic gradient elasticity.**

- Every real advance in the subject will be connected to choosing the “right” Null-Lagrangian.
- It is well known that in the classical linear isotropic elasticity theory this sort of freedom to play with Null-Lagrangians does not exist and the two Lamé-parameters are really material parameters, independent of the specific boundary value problem [54].
- The new variant model in the indeterminate couple stress theory shows that the way chosen in order to obtain the equilibrium equations is a constitutive choice, a modeling choice.
- There are some possible boundary conditions which arise naturally in the indeterminate couple stress theory from the full gradient model and which are not the same as in the direct approach of Mindlin and Tiersten.
• However, the a priori information on the loads applied on the boundary seem to be the same in these two alternative form of the boundary conditions.

• We are expecting that some physical or experimental reason are in favor of our orthogonal boundary conditions.

• There is no need to take the Hadjesfandiari and Dargush’s model as a final answer to everything. Their claim is just a constitutive choice which may be applied in some specific problems, their corresponding boundary value problem is still mathematically well posed but the raison d’être of this model is not justified.

• The format of boundary conditions may be different, even for the same energy and the same balance equations: the resulting models are physically different.

• Our new model yields a symmetric total force stress tensor and a symmetric trace free moment stress tensor with a least number of material parameters. It is conformally invariant and well-posed.

• Our development shows clearly that one may always safely fall back to symmetric total force stress tensor and symmetric moment stress tensors at the expense of dealing with modified boundary conditions. What, then, is the meaning of the symmetry or non-symmetry of stress tensors in these higher gradient linear elasticity models?

• We surmise that the symmetry of the total stress tensor should be used to fix the traction boundary conditions. In that sense, the symmetry requirement acts like a gauge condition fixing one preferred possibility.

• Moreover, in the absence of electromagnetic interaction and if there is no real necessity for non-symmetric stress tensors, we should use Occham’s razor and discard these possibilities in favour of the simplest approach:

  **the Cauchy–Boltzmann-axiom of symmetric total force stress tensors**

The energies in both possible formulations (in terms of $\nabla[axl(\text{skew } \nabla u)]$ or $\text{Curl (sym } \nabla u)$) are the same, differences appear only once traction boundary conditions are specified. The need for prescribing this or that boundary conditions determines which model should be used.

In a polar gradient elasticity model we could influence directly continuum rotations without prescribing $u_{\Gamma} = 0$. But this should only be possible in a theory which extends beyond mechanics: for example to magnetic or electric effects, i.e. needed for particular loading and boundary conditions which excite particular micro-rotations (“polarization”). In contrast, in a non-polar elasticity model it is not possible to influence directly continuum rotations but a non-polar model is applicable and much more appropriate in a purely mechanical context (see Figure 7). The case iv) in Fig. 6 needs mathematical discussion. The extension of the well-posedness to the finite strain case in which the corresponding Lagrangian may be written as $W = W(U) + W_{\text{curv}}(U, \text{Curl } U)$, where $F = RU$ is the polar decomposition is yet missing. Some steps in this direction are presented in [55].
6 Epilogue: Much ado about nothing

We have seen how much effort it took us to derive the consistent orthogonal boundary conditions in the indeterminate couple stress model. The conceptual advantage of not having to discuss the physical meaning of independent degrees of freedom is, now, more than outweighed by the burdensome interpretation of traction boundary conditions. Nevertheless, all presented formulations are shown to be mathematically well-posed. In the last part of the paper we have had a look at 2nd-order (micromorphic) approximations of the given gradient elastic models. In these micromorphic models, the boundary conditions are completely transparent and orthogonal. However, it seems that in this larger class of models there is yet another variant (the relaxed micromorphic model with integral boundary coupling) which combines conceptual simplicity, symmetry of force stress tensor and symmetry of moment stress tensor, simplicity of traction boundary conditions and well-posedness to make it superior to all other presented formulations. With hindsight, we understand why the indeterminate couple stress model had been abandoned in the late ’60ies. For us it is a mystery how it was possible at all to identify material parameters in a theory in which boundary conditions had not been conclusively settled?

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References


In this section we prove the claim from Subsection (3.3), i.e. we show that the possible traction boundary conditions are different.

On one hand, we have

\[
\begin{align*}
\text{sym } \vec{M} \cdot e_1 &= \text{sym } \left( \begin{array}{c}
\hat{m}_{11} e_1 + \hat{m}_{12} e_2 + \hat{m}_{13} e_3 \\
\hat{m}_{21} e_1 + \hat{m}_{22} e_2 + \hat{m}_{23} e_3 \\
\hat{m}_{31} e_1 + \hat{m}_{32} e_2 + \hat{m}_{33} e_3
\end{array} \right) \cdot e_1 \\
&= \text{sym } \left( \begin{array}{c}
\hat{m}_{13} \\
\hat{m}_{23} \\
\hat{m}_{33}
\end{array} \right) \cdot e_1 = \frac{1}{2} \left( \begin{array}{c}
\hat{m}_{13} \\
\hat{m}_{23} \\
\hat{m}_{33}
\end{array} \right) \cdot e_1 = \frac{1}{2} \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right) \\
&= \frac{1}{2} \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right) \\
&= \frac{1}{2} \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right) \\
&= \frac{1}{2} \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right)
\end{align*}
\]

and therefore

\[
\begin{align*}
\text{sym } \vec{M} \cdot e_1 &= \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 \\
\hat{m}_{13} \\
-\hat{m}_{12}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 \\
\hat{m}_{13} \\
-\hat{m}_{12}
\end{pmatrix}.
\end{align*}
\]

On the other hand, we obtain

\[
\text{anti}(\vec{M}) \cdot e_1 = \text{anti}(\vec{M}_{11}, \vec{M}_{21}, \vec{M}_{31}) \cdot e_1 = \begin{pmatrix}
0 & \hat{m}_{31} & -\hat{m}_{21} \\
-\hat{m}_{31} & 0 & \hat{m}_{11} \\
\hat{m}_{21} & -\hat{m}_{11} & 0
\end{pmatrix} \cdot e_1 = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Hence, we deduce

\[
\begin{align*}
\text{sym } \vec{M} \cdot e_1 &= \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 \\
\hat{m}_{31} \\
\hat{m}_{21}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\end{align*}
\]
We may conclude that (2) = (2') implies

\[ \hat{m}_{13} = -\hat{m}_{31}, \quad \hat{m}_{23} = -\hat{m}_{32}. \] (A.6)

Let us now point out that \( \hat{m} \) and \( \hat{m} \) are not independent, see Figure 7. Considering the case \( \alpha_2 = 0 \) we have \( \hat{m} = \hat{m} \in \text{Sym}(3) \), while considering \( \alpha_1 = 0 \) we have \( \hat{m} = -\hat{m} \in \phi(3) \). Therefore, in the conformal invariant case \( \alpha_2 = 0 \) and also in the case \( \alpha_1 = 0 \), since the condition (A.6) does not hold true, it follows that (2) ≠ (2').

If the boundary conditions imply the continuity of \( \hat{m} \) and \( \hat{m} \), then (3) = (3') = 0. However, if \( \hat{m} \) and \( \hat{m} \) are not continuous across the curve \( \Gamma \), then the system of coordinates is initially chosen such that the normal vector on the boundary at this point is \( e_1 := (1,0,0) \) and \( \nu = e_3 := (0,0,1) \), we prove that

\[ (\text{sym } \hat{m}).e_3 \neq \text{anti}(\hat{m}.e_1).e_3. \] (A.7)

Doing similar calculations as above, we deduce

\[ (\text{sym } \hat{m}).e_3 = \frac{1}{2} \begin{pmatrix} 0 & \tilde{m}_{13} & -\tilde{m}_{12} \\ \tilde{m}_{13} & \tilde{m}_{23} & -\tilde{m}_{22} + \tilde{m}_{33} \\ -\tilde{m}_{12} & -\tilde{m}_{22} + \tilde{m}_{33} & -\tilde{m}_{32} \end{pmatrix} \cdot \hat{m}_{31} \quad \hat{m}_{11} \quad \hat{m}_{31} \quad \hat{m}_{11} \quad \hat{m}_{31} \quad \hat{m}_{11} \end{pmatrix} \] (A.8)

and

\[ \text{anti}(\hat{m}.e_1).e_3 = \begin{pmatrix} 0 & -\tilde{m}_{21} \\ -\tilde{m}_{21} & \tilde{m}_{21} \end{pmatrix} \cdot \hat{m}_{21} \] (A.9)

We remark that in both particular cases, the conformal invariance model \( \alpha_2 = 0 \) and the case \( \alpha_1 = 0 \), we deduce that \( (\text{sym } \hat{m}).e_3 \neq \text{anti}(\hat{m}.e_1).e_3 \), in general. However, even if \( (\text{sym } \hat{m}).e_3 \neq \text{anti}(\hat{m}.e_1).e_3 \), the jump (c) may coincide with the jump (c').

Let us remark that in order to compare (1) and (1') we may not proceed as above. However, we will prove that (1) ≠ (1') in a specific situation. We assume that there is an open subset \( \omega \subset \partial \Omega \) such that on \( \omega \) the normal vector \( n \) is constant. Let us consider a point \( P \in \omega \). We may assume for simplicity that \( n = e_1 \) at all points \( P \in \omega \). Upon this assumption on the domain \( \Omega \), at the point \( P \in \omega \) we obtain

\[ (\nabla [(\text{sym } \hat{m})T]) = \nabla \frac{1}{2} \begin{pmatrix} 0 & \tilde{m}_{13} & -\tilde{m}_{12} \\ \tilde{m}_{13} & \tilde{m}_{23} & -\tilde{m}_{22} + \tilde{m}_{33} \\ -\tilde{m}_{12} & -\tilde{m}_{22} + \tilde{m}_{33} & -\tilde{m}_{32} \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (A.10)

and

\[ \text{sym Curl } (\hat{m}).n = \text{sym} \begin{pmatrix} \tilde{m}_{13} - \tilde{m}_{31} & \tilde{m}_{11} - \tilde{m}_{31} & \tilde{m}_{12} - \tilde{m}_{31} \\ \tilde{m}_{23} - \tilde{m}_{32} & \tilde{m}_{21} - \tilde{m}_{32} & \tilde{m}_{22} - \tilde{m}_{32} \\ \tilde{m}_{33} - \tilde{m}_{31} & \tilde{m}_{31} - \tilde{m}_{31} & \tilde{m}_{32} - \tilde{m}_{31} \end{pmatrix} \cdot e_1 \] (A.11)

\[ = \frac{1}{2} \begin{pmatrix} \tilde{m}_{13} - \tilde{m}_{31} & \tilde{m}_{11} - \tilde{m}_{31} & \tilde{m}_{12} - \tilde{m}_{31} \\ \tilde{m}_{23} - \tilde{m}_{32} & \tilde{m}_{21} - \tilde{m}_{32} & \tilde{m}_{22} - \tilde{m}_{32} \\ \tilde{m}_{33} - \tilde{m}_{31} & \tilde{m}_{31} - \tilde{m}_{31} & \tilde{m}_{32} - \tilde{m}_{31} \end{pmatrix} \cdot e_1 \]
Therefore, we deduce

\[
\text{sym Curl} \{\tilde{m}\}, n - (\nabla [(\text{sym} \tilde{M})] T) : T = \frac{1}{2} \begin{pmatrix}
\tilde{m}_{13,2} - \tilde{m}_{12,3} \\
\tilde{m}_{11,3} - \tilde{m}_{13,1} + \tilde{m}_{23,2} - \tilde{m}_{22,3} \\
\tilde{m}_{12,1} - \tilde{m}_{11,2} + \tilde{m}_{33,2} - \tilde{m}_{32,3}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\tilde{m}_{13,2} - \tilde{m}_{12,3} \\
\tilde{m}_{23,2} - \tilde{m}_{22,3} + \tilde{m}_{33,3} \\
-\tilde{m}_{22,2} + \tilde{m}_{33,2} - \tilde{m}_{32,3}
\end{pmatrix}
\]

\[
= \frac{1}{2} \begin{pmatrix}
\tilde{m}_{13,2} - \tilde{m}_{12,3} - \tilde{m}_{11,3} + \tilde{m}_{13,1} + \tilde{m}_{23,2} - \tilde{m}_{22,3} - \tilde{m}_{23,2} - \tilde{m}_{22,3} + \tilde{m}_{33,3} - \tilde{m}_{33,2} + \tilde{m}_{32,3}
\end{pmatrix} \]  \quad (A.12)

Moreover, we obtain

\[
\frac{1}{2} \nabla [(\text{anti}((\tilde{m} \cdot n)) T) : T = \frac{1}{2} \nabla [(\text{anti}(\tilde{m}_{11}, \tilde{m}_{21}, \tilde{m}_{31})) T] : T = \frac{1}{2} \nabla [(\text{anti}(\tilde{m}_{11}, \tilde{m}_{21}, \tilde{m}_{31})) T] : T \quad (A.13)
\]

\[
= \frac{1}{2} \nabla \begin{pmatrix}
0 & \tilde{m}_{31} & -\tilde{m}_{21} \\
-\tilde{m}_{31} & 0 & \tilde{m}_{11} \\
\tilde{m}_{21} & -\tilde{m}_{11} & 0
\end{pmatrix} \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \frac{1}{2} \begin{pmatrix}
\tilde{m}_{31,2} - \tilde{m}_{21,3} \\
\tilde{m}_{11,3} \\
-\tilde{m}_{11,2}
\end{pmatrix}
\]

and

\[
\frac{1}{2} \text{anti Div}[\tilde{m}], n = \frac{1}{2} \text{anti}([\tilde{m}_{1,j}, \tilde{m}_{2,j}, \tilde{m}_{3,j}] \cdot e_1) = \frac{1}{2} \begin{pmatrix}
0 & \tilde{m}_{3,j} & -\tilde{m}_{2,j} \\
-\tilde{m}_{3,j} & \tilde{m}_{1,j} & 0 \\
\tilde{m}_{2,j} & -\tilde{m}_{1,j} & 0
\end{pmatrix} \cdot e_1 \quad (A.14)
\]

Hence, it follows that

\[
-\frac{1}{2} \text{anti Div}[\tilde{m}], n - \frac{1}{2} \nabla [(\text{anti}((\tilde{m} \cdot n)) T) : T = -\frac{1}{2} \begin{pmatrix}
0 & -\tilde{m}_{3,j} \\
-\tilde{m}_{3,j} & \tilde{m}_{1,j} \\
\tilde{m}_{2,j} & -\tilde{m}_{1,j}
\end{pmatrix} - \frac{1}{2} \begin{pmatrix}
\tilde{m}_{31,2} - \tilde{m}_{21,3} \\
\tilde{m}_{11,3} \\
-\tilde{m}_{11,2}
\end{pmatrix} \]  \quad (A.15)

Concluding, (1) = (1') if and only if

\[
\begin{pmatrix}
\tilde{m}_{13,2} - \tilde{m}_{12,3} \\
\tilde{m}_{11,3} - \tilde{m}_{13,1} + \tilde{m}_{23,2} - \tilde{m}_{22,3} \\
\tilde{m}_{12,1} - \tilde{m}_{11,2} + \tilde{m}_{33,2} - \tilde{m}_{32,3}
\end{pmatrix} = \begin{pmatrix}
\tilde{m}_{31,2} + \tilde{m}_{21,3} \\
\tilde{m}_{3,j} - \tilde{m}_{1,3} \\
-\tilde{m}_{2,j} + \tilde{m}_{11,2}
\end{pmatrix} \quad (A.16)
\]

which holds true, in general. In the conformal case, the above condition reads

\[
\tilde{m}_{13,2} = \tilde{m}_{12,3}, \quad 2 \tilde{m}_{11,3} - 2 \tilde{m}_{13,1} = \tilde{m}_{23,2} + \tilde{m}_{33,3} \quad 2 \tilde{m}_{11,2} - 2 \tilde{m}_{12,1} = \tilde{m}_{33,2} + \tilde{m}_{22,2} \quad (A.17)
\]

while in the case \( \alpha_1 = 0 \) it becomes

\[
\tilde{m}_{23,2} = 0, \quad \tilde{m}_{32,3} = 0 \quad \Rightarrow \quad \tilde{m}_{23} = \tilde{m}_{23}(e_1),
\]

which is clearly not satisfied, in general.
A.2 From second order couple stress tensors to third order moment stress tensors and back

Let us consider the general anisotropic case and

\[ W_{\text{curv}}(D^2u) = \langle \zeta, D^2u, D^2u \rangle_{\mathbb{R}^{3\times3 \times3}}, \quad \hat{W}_{\text{curv}}(\text{Curl (sym \nabla u)}) = \langle \zeta, \text{Curl (sym \nabla u)}, \text{Curl (sym \nabla u)} \rangle_{\mathbb{R}^{3\times3}} \]

where

\[ C = (C_{ijklmn}) : \mathbb{R}^{3\times3 \times3} \to \mathbb{R}^{3\times3 \times3} \quad \text{and} \quad \zeta = (\zeta_{ijkl}) : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3} \]

Let us also consider the tensors

\[ m := D_{ijkl}W_{\text{curv}}(D^2u) \quad \text{and} \quad \hat{m} := D_{\text{Curl (sym \nabla u)}}W_{\text{curv}}(\text{Curl (sym \nabla u)}) \]

which for our anisotropic case are

\[ m := C.D^2u \quad \text{and} \quad \hat{m} := L \text{Curl (sym \nabla u)}. \]

Since

\[ D^2u = \text{Lin}(\nabla (\text{sym \nabla u})) = A. (\nabla (\text{sym \nabla u})), \quad u_{k,ij} = \varepsilon_{ik,j} - \varepsilon_{jk,i} \]

where \( \varepsilon = \text{sym \nabla u} \), we obtain

\[ m := Ca. (\nabla (\text{sym \nabla u})) = B. (\nabla (\text{sym \nabla u})). \]

The next problem is to find particular form of the tensor \( C \), for which we have

\[ \langle m, D^2u \rangle_{\mathbb{R}^{3\times3 \times3}} = \langle \hat{m}, \text{Curl (sym \nabla u)} \rangle_{\mathbb{R}^{3\times3}} \]

or equivalently

\[ \langle Ca. (\nabla (\text{sym \nabla u})), A. (\nabla (\text{sym \nabla u})) \rangle_{\mathbb{R}^{3\times3 \times3}} = \langle L \text{Curl (sym \nabla u)}, \text{Curl (sym \nabla u)} \rangle_{\mathbb{R}^{3\times3}}. \]

This is equivalent to

\[ \langle A^T Ca. (\nabla (\text{sym \nabla u})), (\nabla (\text{sym \nabla u})) \rangle_{\mathbb{R}^{3\times3 \times3}} = \langle L \text{Curl (sym \nabla u)}, \text{Curl (sym \nabla u)} \rangle_{\mathbb{R}^{3\times3}}. \]

Let us denote in the following

\[ B := A^T Ca. \]

We consider a specific form of the tensor \( B \) in terms of another tensor \( L : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3} \) such that

\[ \langle B. (\nabla (\text{sym \nabla u})), (\nabla (\text{sym \nabla u})) \rangle_{\mathbb{R}^{3\times3 \times3}} = \langle L \text{Curl (sym \nabla u)}, \text{Curl (sym \nabla u)} \rangle_{\mathbb{R}^{3\times3}}. \]

Let us show how to obtain the tensor \( B \) if \( L \) is given, such that the last identity holds true. We first remark that a tensor \( B : \mathbb{R}^{3\times3 \times3} \to \mathbb{R}^{3\times3 \times3} \) is uniquely defined by the fourth order tensors

\[ B_{im} = (B_{ijklmn})_{im}, \quad B_{im} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}, \quad (A.1) \]

and

\[ \langle B. \nabla (\text{sym \nabla u}), \nabla (\text{sym \nabla u}) \rangle_{\mathbb{R}^{3\times3 \times3}} := \langle B_{im}, \nabla (\text{sym \nabla u})_m, \nabla (\text{sym \nabla u})_i \rangle_{\mathbb{R}^{3\times3}}, \]

where Einstein’s summation rule is used. Let \( L : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3} \) be a given fourth order tensor. We may write this tensor in the form

\[ L = (\hat{L}^1, \hat{L}^2, \hat{L}^3), \quad \hat{L}^i : \mathbb{R}^{3} \to \mathbb{R}^{3}. \quad (A.2) \]

Let us define the tensor \( B \) by (A.1) where

\[ B_{im} = \begin{cases} 2 \text{skew anti } \hat{L}^i \text{ and skew} & \text{for } i = m \\ 0 & \text{for } i \neq m. \end{cases} \quad (A.3) \]

56
For the particular form (A.3) of $B_{im}$, using the formula 2 $axl\text{skew}\nabla (sym \nabla u)_m$, we obtain

$$
\langle B_{im}, \nabla (sym \nabla u)_m, \nabla (sym \nabla u)_i \rangle_{R^3x3} = 2 \langle \text{skew anti } \hat{L}^i, axl \text{skew} \nabla (sym \nabla u)_i, \nabla (sym \nabla u)_i \rangle_{R^3x3}
$$

$$
= 2 \langle \text{anti } \hat{L}^i, axl \text{skew} \nabla (sym \nabla u)_i, \text{skew} \nabla (sym \nabla u)_i \rangle_{R^3x3}
$$

$$
= 4 \langle \hat{L}^i, axl \text{skew} \nabla (sym \nabla u)_i, axl \text{skew} \nabla (sym \nabla u)_i \rangle_{R^3}
$$

$$
= \langle \hat{L}^i, \text{curl} (sym \nabla u)_i, \text{curl} (sym \nabla u)_i \rangle_{R^3}
$$

Now, we conclude that for a given fourth order tensor $L : R^{3x3} \rightarrow R^{3x3}$ the tensor $\hat{L}$ defined by (A.1) and (A.3) is such that

$$
\langle B, \nabla (sym \nabla u), \nabla (sym \nabla u) \rangle_{R^{3x3x3}} = \langle B_{im}, \nabla (sym \nabla u)_m, \nabla (sym \nabla u)_i \rangle_{R^{3x3x3}}
$$

$$
= \langle \hat{L}^i, \text{curl} (sym \nabla u)_i, \text{curl} (sym \nabla u)_i \rangle_{R^{3x3}} = \langle \hat{L}, \text{Curl} (sym \nabla u), \text{Curl} (sym \nabla u) \rangle_{R^{3x3}}.
$$

In conclusion, we have found a tensor $C$ given by

$$
C := ABA^T,
$$

where $B$ is given by (A.1) and (A.3), such that

$$
\langle C, D^2u, D^2u \rangle_{R^{3x3x3}} = \langle \hat{L}, \text{Curl} (sym \nabla u), \text{Curl} (sym \nabla u) \rangle_{R^{3x3}},
$$

or equivalently

$$
\langle m, D^2u \rangle_{R^{3x3x3}} = \langle \hat{m}, \text{Curl} (sym \nabla u) \rangle_{R^{3x3}},
$$

or equivalently

$$
W_{\text{curv}}(D^2u) = \tilde{W}_{\text{curv}}(\text{Curl} (sym \nabla u)).
$$

### A.3 The name of the indeterminate couple stress model

Regarding the name of the indeterminate couple stress model, Paria [91, p. 1] writes: “...it has led to the difficulties that the anti-symmetric part of the stress dyadic as well as the isotropic part of the couple-stress dyadic remain indeterminate. These indeterminacies are perhaps due to the fact that the rotation vector, defined above, is not independent but depends on the displacement vector”. The theory has a variety of names, such as “Cosserat theory with constrained rotations” (Toupin, 1964), “Couple stress theory” (Koiter, 1964), “Indeterminate couple stress theory” (Eringen, 1968), “Cosserat pseudo-continuum” (Nowacki, 1968). Eringen writes [28]: “At this time [in the 1960, our addition] also popular was a theory of indeterminate couple stress which is mostly abandoned now [1998]. In this theory, the axi-symmetric [skew-symmetric] part of the stress tensor is redundant and it remains indeterminate”. Schäfer [98] called “indeterminate couple stress model” as pseudo-Cosserat-continuum of the trièdre caches (see also [89]).

If the microrotations $\hat{A} \in so(3)$ are constrained to be equal to the macrorotations $\text{skew} \nabla u$, the Cosserat model reduces to the couple stress theory. This corresponds to the case $\mu_c \rightarrow \infty$, for which the antisymmetric part of the strain tensor $\text{skew} (\nabla u - \hat{A})$ and the spherical part of the curvature tensor $\text{tr} (\nabla \text{skew} (\nabla \hat{A}))$ tend to zero. Consequently, by energetic duality the antisymmetric part of the Cauchy stress tensor $\text{skew} (\sigma)$ in the Cosserat model, and the first invariant of the couple stress, namely $tr(\hat{m})$ do not appear in the formulation of the virtual work principle as well as in the constitutive equations. The first invariant of the couple stress remains “indeterminate” and it is taken to be equal to zero [51]. Now, the skew-symmetric part of the total force stress tensor is not constitutively determined, but can be obtained from balance of momentum.
Orthogonal boundary conditions in the indeterminate couple stress model

Geometric (essential) boundary conditions (3+2)

\[ u|_\Gamma = u^{\text{ext}} \in \mathbb{R}^3, \quad (\mathbf{I} - n \otimes n) \nabla u.n|_\Gamma = (\mathbf{I} - n \otimes n) . a^{\text{ext}} \in \mathbb{R}^3, \text{ or } (\mathbf{I} - n \otimes n) . \text{curl } u|_\Gamma = (\mathbf{I} - n \otimes n) . f^{\text{ext}} \]

Mechanical (traction) boundary conditions (3+2)

\[ \left( (\sigma - \tau) . n - \frac{1}{2} \nabla [(\text{anti}((\mathbf{I} - n \otimes n) \tilde{m}) . n)] \right)|_{\partial \Omega \setminus \Gamma} = t^{\text{ext}}, \]

\[ (\mathbf{I} - n \otimes n) \text{anti}[(\mathbf{I} - n \otimes n) \tilde{m}] . n|_{\partial \Omega \setminus \Gamma} = (\mathbf{I} - n \otimes n) m^{\text{ext}} \]

Boundary virtual work

\[ - \int_{\partial \Omega \setminus \Gamma} \left( (\sigma - \tau) . n - \frac{1}{2} \nabla [(\text{anti}((\mathbf{I} - n \otimes n) \tilde{m}) . n)] \right) . \delta u \, ds + \int_{\partial \Omega \setminus \Gamma} (\mathbf{I} - n \otimes n) \text{anti}[(\mathbf{I} - n \otimes n) \tilde{m}] . n \nabla \delta u . n \, ds = 0 \]

\[ \text{equivariant} \]

Alternative equivalent orthogonal boundary conditions in the indeterminate couple stress model

Geometric (essential) boundary conditions (3+2)

\[ u|_\Gamma = u^{\text{ext}} \in \mathbb{R}^3, \quad (\mathbf{I} - n \otimes n) \nabla u.n|_\Gamma = (\mathbf{I} - n \otimes n) . a^{\text{ext}} \in \mathbb{R}^3, \text{ or } (\mathbf{I} - n \otimes n) . \text{curl } u|_\Gamma = (\mathbf{I} - n \otimes n) . f^{\text{ext}} \]

Mechanical (traction) boundary conditions (3+2)

\[ \left( (\sigma - \tau) . n - \frac{1}{2} \nabla [(\text{anti}((\mathbf{I} - n \otimes n) \tilde{m}) . n)] \right)|_{\partial \Omega \setminus \Gamma} = t^{\text{ext}}, \]

\[ (\mathbf{I} - n \otimes n) \text{anti}[(\mathbf{I} - n \otimes n) \tilde{m}] . n|_{\partial \Omega \setminus \Gamma} = (\mathbf{I} - n \otimes n) m^{\text{ext}} \]

Boundary virtual work

\[ - \int_{\partial \Omega \setminus \Gamma} \left( (\sigma - \tau) . n - \frac{1}{2} \nabla [(\text{anti}((\mathbf{I} - n \otimes n) \tilde{m}) . n)] \right) . \delta u \, ds + \int_{\partial \Omega \setminus \Gamma} (\mathbf{I} - n \otimes n) \text{anti}[(\mathbf{I} - n \otimes n) \tilde{m}] . n \nabla \delta u . n \, ds = 0 \]

\[ \text{equivariant} \]

Alternative equivalent correct boundary conditions, index-format

Geometric (essential) boundary conditions (3+2)

\[ u_i|_\Gamma = u_i^{\text{ext}} \in \mathbb{R}^3, \quad \epsilon_{ijk} u_{l,k} - \epsilon_{jkl} u_{i,k} n_j n_i \big|_\Gamma = a_i^{\text{ext}} \]

or \[ (u_{i,k} n_k - u_{j,k} n_j n_i) \big|_\Gamma = b_i^{\text{ext}} \]

Mechanical (traction) boundary conditions (3+2)

\[ \left( (\sigma_{ij} - \tau_{ij}) n_j + \frac{1}{2} (\epsilon_{ipk} \tilde{m}_{k} n_s - \epsilon_{ipk} \tilde{m}_{k} n_s n_i) p (\delta_{kp} - n_k n_p) \right)|_{\partial \Omega \setminus \Gamma} = t_i^{\text{ext}}, \]

\[ (\epsilon_{ipk} \tilde{m}_{k} n_s - \epsilon_{ipk} \tilde{m}_{k} n_s n_i) n_p|_{\partial \Omega \setminus \Gamma} = m_{i,p} - m_{p}^{\text{ext}} n_n n_i, \]

\[ \epsilon_{ipk} \tilde{m}_{k} n_s|_{\partial \Omega \setminus \Gamma} = \pi_{i,p}^{\text{ext}} \text{ "edge line force" on } \partial \Gamma \]

\[ \text{equivariant} \]

Figure 11: The possible boundary conditions in the indeterminate couple stress model. The equivalence of the geometric boundary condition is clear. The virtual displacement is denoted by \( \delta u \in C^\infty(\Omega) \).
Orthogonal boundary conditions in the Curl (sym $\nabla u$)–formulation

Geometric (essential) boundary conditions (3+2) [independent + orthogonal]

$u|_\Gamma = \hat{u}^{\text{ext}} \in \mathbb{R}^3$, \hspace{2mm} $(\mathbf{I} - n \otimes n) \nabla u|_\Gamma = (\mathbf{I} - n \otimes n) \hat{\delta}^{\text{ext}} \in \mathbb{R}^3$, \hspace{2mm} or \hspace{2mm} $(\mathbf{I} - n \otimes n).\text{curl}\; u|_\Gamma = (\mathbf{I} - n \otimes n) \hat{\delta}^{\text{ext}} \in \mathbb{R}^3$

Mechanical (traction) boundary conditions (3+2)

$(\sigma + \hat{\tau}).n - \nabla [(\text{sym } \hat{M})(\mathbf{I} - n \otimes n)] : (\mathbf{I} - n \otimes n)|_{\partial\Omega \cap \Gamma} = \hat{\tau}^{\text{ext}}, \hspace{2mm} \hat{M} = \begin{pmatrix} \hat{m}_1 \times n \\ \hat{m}_2 \times n \\ \hat{m}_3 \times n \end{pmatrix}, \hspace{2mm} \hat{\delta} = \begin{pmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \\ \hat{\delta}_3 \end{pmatrix}$

Boundary virtual work

$- \int_{\partial\Omega} ((\sigma + \hat{\tau}).n, \text{d}u)\text{d}a - \int_{\partial\Omega} \sum_{i=1}^{3} (\hat{m}_i \times n_i, (\text{sym } \nabla \delta u)_i)\text{d}a = 0 \hspace{2mm} \Leftrightarrow$

$- \int_{\partial\Omega} ((\sigma + \hat{\tau}).n - \nabla [(\text{sym } \hat{M})(\mathbf{I} - n \otimes n)] : (\mathbf{I} - n \otimes n), \delta u)\text{d}a - \int_{\partial\Omega} ((\mathbf{I} - n \otimes n)(\text{sym } \hat{M}).n, (\nabla \delta u).n)\text{d}a$

$\Leftrightarrow \int_{\partial\Omega} [(\text{sym } \hat{M}).\nu, \delta u] \text{d}a = 0$

$\Leftrightarrow$ equivalent

Orthogonal boundary conditions in the Curl (sym $\nabla u$)–formulation, written in indices

Geometric (essential) boundary conditions (3+2) [independent + orthogonal]

$u|_\Gamma = \hat{u}^{\text{ext}} \in \mathbb{R}^3$, \hspace{2mm} $(\epsilon_{ijkl}u_{l,k} - \epsilon_{ijkl}u_{l,k})|_\Gamma = \epsilon_{ijkl}\hat{u}^{0}_{l,k} - \epsilon_{ijkl}u^{0}_{l,k}n_jn_i$

Mechanical (traction) boundary conditions (3+2)

$(\delta_{ij} + \hat{\delta}_{ij}).n_j - \frac{1}{2}(\epsilon_{ijkl}\hat{m}_{ik}n_i + \epsilon_{ijkl}\tilde{m}_{ik}n_i - \epsilon_{ijkl}\hat{m}_{ik}n_i n_i n_i - \epsilon_{ijkl}\tilde{m}_{ik}n_i n_i n_i)\nu_p |_{\partial\Omega \cap \Gamma} = \hat{\delta}_i$, \hspace{2mm} 3 bc

$\frac{1}{2}(\epsilon_{ijkl}\hat{m}_{ik}n_i + \epsilon_{ijkl}\tilde{m}_{ik}n_i - \epsilon_{ijkl}\hat{m}_{ik}n_i n_i n_i - \epsilon_{ijkl}\tilde{m}_{ik}n_i n_i n_i)\nu_p |_{\partial\Omega \cap \Gamma} = \hat{\delta}_i$, \hspace{2mm} 2 bc

Figure 12: The possible boundary conditions in the $\nabla [\text{axl}(\text{skew } \nabla u)]$ and Curl (sym $\nabla u$)–formulation. The equivalence of the geometric boundary condition is clear. The virtual displacement is denoted by $\delta u \in C^\infty(\Omega, \mathbb{R}^3)$. 59