On uniqueness and continuous dependence of solutions in viscoelastic mixtures

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Abstract This note deals with the isothermal linear theory of porous viscoelastic mixtures. Questions of uniqueness and continuous dependence for solutions of various classes of initial boundary value problems in mixtures consisting of two constituents: a porous elastic solid and a porous Kelvin–Voigt material are studied. The Lagrange identity and Logarithmic convexity methods are used to establish uniqueness and continuous dependence results, with no definiteness assumptions upon the internal energy.

Keywords Lagrange identity · Logarithmic convexity · Uniqueness · Continuous dependence · Viscoelastic mixtures

1 Introduction

Various theories of mixtures developed in Eulerian or Lagrangian description have been proposed in literature for describing the thermomechanical and chemical behavior of interacting continua (see for example the works cited in [4–6, 9, 14, 15, 19]). In recent years a great attention has been given to the theory of viscoelastic mixtures.

One of the mathematical models which includes viscoelastic effects has been proposed by Ieșan [9]. In [9], the basic equations of a theory of binary mixture where the individual components are: a porous elastic solid and a porous Kelvin–Voigt material have been established. In this theory the dissipation effects are determined by the viscosity of rate type of a constituent and the relative velocity. The theory takes into account the effects of porosity [7, 13], by considering the volume fraction of each constituent as an independent kinematic variable.

For the linear theory of viscoelastic mixtures proposed in [9], various qualitative properties of solutions have been investigated. Thus, a uniqueness result and a stability result have been derived in [9], the spatial behavior problem has been presented in [4], the existence of solutions and the temporal behavior in terms of the Cesáro means of various parts of the total energy have been studied in [5]. All these results were derived under positive definiteness assumptions upon the internal energy.

In the Sects. 2–4 of the present work we treat the uniqueness and the continuous data dependence problems without positive definiteness assumptions on the internal energy. Thus, the results hold for the entire class of porous viscoelastic mixtures. The definiteness of internal energy is a strong condition. In several situations (see [10–12, 16, 17]) this assumption
is not needed to study the uniqueness and the continuous data dependence of solutions. When the internal energy is not definite, the problem becomes ill posed [1, 10]. We recall that these situations is natural in the study of pre-stressed solids [8]. The methods adopted here are based on Lagrange identity and logarithmic convexity arguments. We prove that the approach of the linear theory of viscoelastic mixture depends continuously against errors in the external given data.

2 Basic equations

We consider a body that in the reference configuration taken at time \( t = 0 \), occupies the bounded regular region \( \Omega \) of Euclidian three-dimensional space and assume that its boundary \( \partial \Omega \) is a piecewise smooth surface. A chemically inert mixture consisting of two constituents: a Kelvin–Voigt porous elastic material \( s_1 \) and a porous elastic solid \( s_2 \), fills \( \Omega \).

We refer the motion of the body to a fixed system of rectangular Cartesian axes. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood (unless otherwise specified) to range over the integers \( (1, 2, 3) \), summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a supersposed dot denotes time differentiation. Greek indices understood to range over the integers \( (1, 2) \) and summation convention are not used for these indices.

The present study is devoted to the study of some mathematical problems of the theory of viscoelastic mixtures developed by Ieşan [9] using the Lagrangian description. The theory includes also effects of porosity [7, 13]. According to the isothermal linear theory, the behavior of the mixture is characterized by the displacement vectors \( u \) and \( w \) associated with the constituents \( s_1 \) and \( s_2 \) respectively, and the change in volume fractions from reference configuration for each constituent \( \phi \) and \( \psi \) respectively.

As discussed by Ieşan [9], the fundamental system of field equations that governs the motion of an anisotropic and centrosymmetric mixture consists of:

- the equations of motion

\[
\begin{align*}
t_{ij,j} - p_i + \rho_1^0 F_i^{(1)} &= \rho_1^0 \dot{u}_i, \\
s_{ij,j} + p_1 + \rho_2^0 F_i^{(2)} &= \rho_2^0 \dot{w}_i, \\
h_{ij}^{(1)} + g^{(1)} + \rho_1^0 L^{(1)} &= \rho_1^0 \varsigma_{ij} \dot{\phi}, \\
h_{ij}^{(2)} + g^{(2)} + \rho_2^0 L^{(2)} &= \rho_2^0 \varsigma_{ij} \dot{\psi}
\end{align*}
\]

in \( \Omega \times (0, \infty) \), where \( \rho_1^0 \) and \( \rho_2^0 \) are the densities at time \( t = 0 \) of the two constituents, \( t_{ij} \) and \( s_{ij} \) are the partial stress tensors, \( F_i^{(\alpha)} \), \( \alpha = 1, 2 \) are the body forces, \( p_i \) is the internal body force characterizing the mechanical interaction between constituents, \( h_{ij}^{(\alpha)} \) is the partial equilibrated stress vector associated with \( s_{\alpha} \), \( g^{(\alpha)} \) and \( L^{(\alpha)} \) are the intrinsic and extrinsic equilibrated body forces for the constituent \( s_{\alpha} \), and \( \kappa_{ij} \) is the equilibrated inertia associated with the constituent \( s_{\alpha} \);

- the constitutive equations

\[
\begin{align*}
t_{ij} &= (A_{jirs} + B_{jirs})e_{rs} + (B_{jirs} + C_{jirs})g_{rs} \\
&+ (D_{ji} + M_{ji})\phi + (E_{ji} + N_{ji})\psi + A_{jirs}^* \dot{e}_{rs}, \\
s_{ij} &= B_{rjsij}e_{rs} + C_{rjsirs}g_{rs} + M_{ij} \phi + N_{ij} \psi, \\
p_i &= a_{ij}d_j + b_{ij}\phi,j + c_{ij}\psi,j + a_{ij}^* \dot{d}_j, \\
h_{ij}^{(1)} &= a_{ij}\phi,j + \eta_{ij}\psi,j + b_{ij}d_j, \\
h_{ij}^{(2)} &= \eta_{ij}\phi,j + \gamma_{ij}\psi,j + c_{ij}d_j, \\
g^{(1)} &= -D_{ij} e_{ij} - M_{ij} g_{ij} - \zeta^{(1)} \phi - \zeta^{(3)} \psi, \\
g^{(2)} &= -E_{ij} e_{ij} - N_{ij} g_{ij} - \zeta^{(3)} \phi - \zeta^{(2)} \psi
\end{align*}
\]

in \( \Omega \times (0, \infty) \), where \( e_{ij}, g_{ij} \) and \( d_i \) are defined by

- the geometric equations

\[
\begin{align*}
e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}), \\
g_{ij} &= w_{i,j} + u_{j,i}, \\
d_i &= u_i - w_i
\end{align*}
\]

and \( A_{ijrs}, B_{ijrs}, C_{ijrs}, D_{ij}, M_{ij}, E_{ij}, N_{ij}, A_{ijrs}^*, a_{ij}, b_{ij}, c_{ij}, a_{ij}^*, \alpha_{ij}, \eta_{ij}, \gamma_{ij} \) and \( \varsigma^{(i)} \) are the constitutive coefficients.

The above coefficients have the following symmetries

\[
\begin{align*}
A_{ijrs} &= A_{jirs} = A_{rsij}, & B_{ijrs} &= B_{jirs}, \\
A_{ijrs}^* &= A_{jirs}^* = A_{rsij}^*, & C_{ijrs} &= C_{rsij}, \\
D_{ij} &= D_{ji}, & E_{ij} &= E_{ji}, & \alpha_{ij} &= \alpha_{ji}, \\
\gamma_{ij} &= \gamma_{ji}, & a_{ij} &= a_{ji}
\end{align*}
\]
and moreover, the dissipation inequality implies
\[ \Phi \equiv A_{ijrs}e_{ij}\dot{e}_{rs} + a_{ij}^*d_i\dot{d}_j \geq 0. \] (2.5)

Let us consider the subsets \( S_p \) (\( p = 1, 2, 3, 4 \)) of \( \partial \Omega \) so that \( \overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \partial \Omega, S_1 \cap S_2 = S_3 \cap S_4 = \emptyset \).

To above equations we have to adjoin boundary conditions and initial conditions. We consider the following boundary conditions:
\[ u_i = \bar{u}_i, \quad w_i = \bar{w}_i \quad \text{on} \quad \overline{S}_1 \times I, \]
\[ (t_{ji} + s_{ji})n_j = \bar{t}_i, \quad d_i = \bar{d}_i \quad \text{on} \quad S_2 \times I, \]
\[ \varphi = \bar{\varphi}, \quad \psi = \bar{\psi} \quad \text{on} \quad \overline{S}_3 \times I, \]
\[ (h_i^{(1)} + h_i^{(2)})n_i = \bar{h}, \quad \varphi - \psi = \bar{\delta} \quad \text{on} \quad S_4 \times I, \] (2.6)

where \( \bar{u}_i, \bar{w}_i, \bar{t}_i, \bar{d}_i, \bar{\varphi}, \bar{\psi}, \bar{\delta} \) are given, \( I = [0, \infty) \) and \( n_i \) are the components of the outward unit normal vector. We denote by \( (P) \) the initial boundary value problem defined by the basic equations (2.1), the constitutive equations (2.2), the geometrical equations (2.3), the boundary conditions (2.6) and the initial conditions
\[ u_i(x, 0) = \bar{u}_i(x), \quad w_i(x, 0) = \bar{w}_i(x), \]
\[ \varphi(x, 0) = \bar{\varphi}(x), \quad \psi(x, 0) = \bar{\psi}(x), \]
\[ \dot{u}_i(x, 0) = \bar{\dot{u}}_i(x), \quad \dot{w}_i(x, 0) = \bar{\dot{w}}_i(x), \]
\[ \dot{\varphi}(x, 0) = \bar{\dot{\varphi}}(x), \quad \dot{\psi}(x, 0) = \bar{\dot{\psi}}(x), \quad x \in \overline{\Omega}, \] (2.7)

where \( \bar{u}_i, \bar{w}_i, \bar{t}_i, \bar{d}_i, \bar{\varphi}, \bar{\psi}, \bar{\delta} \) and \( \bar{\chi} \) are prescribed functions.

3 Lagrange–Brun integral identities

Let us denote by \( W \) the internal energy density, that is
\[ W = \frac{1}{2}A_{ijrs}e_{ij}e_{rs} + B_{ijrs}e_{ij}g_{rs} + C_{ijrs}g_{ij}g_{rs} + D_{ij}e_{ij}\varphi + E_{ij}e_{ij}\psi + M_{ij}g_{ij}\varphi + N_{ij}g_{ij}\psi + \frac{1}{2}\zeta^{(1)}\varphi^2 + \frac{1}{2}\zeta^{(2)}\psi^2 + 2\zeta^{(3)}\varphi\psi + \frac{1}{2}\alpha_{ij}\varphi_i\psi_j + \frac{1}{2}\gamma_{ij}\varphi_i\psi_j + \frac{1}{2}a_{ij}d_id_j + b_{ij}d_i\varphi_j + c_{ij}d_i\psi_j \] (3.1)

and let us introduce the following energies:
– the kinetic energy
\[ \mathcal{K}(t) = \frac{1}{2}\int_\Omega \left( \rho_1^0\dot{u}_i(t)\dot{u}_i(t) + \rho_2^0\dot{w}_i(t)\dot{w}_i(t) + \rho_1^0\kappa_1\varphi^2(t) + \rho_2^0\kappa_2\psi^2(t) \right)dv, \] (3.2)
– the internal energy
\[ \mathcal{W}(t) = \int_\Omega W(t)dv, \] (3.3)
– the dissipated energy
\[ \Delta(t) = \int_0^t \int_\Omega \Phi(\tau)dvd\tau. \] (3.4)
– the total energy
\[ \mathcal{E}(t) = \mathcal{K}(t) + \mathcal{W}(t) + \Delta(t). \] (3.5)

and the functions
\[ I(t) = \frac{1}{2}\int_\Omega \left[ \rho_1^0u_i(t)u_i(t) + \rho_2^0w_i(t)w_i(t) + \rho_1^0\kappa_1\varphi^2(t) + \rho_2^0\kappa_2\psi^2(t) \right]dv \]
\[ + \int_0^t \int_\Omega \left[ A_{ijrs}e_{ij}(\tau)e_{rs}(\tau) + a_{ij}d_i(\tau)d_j(\tau) \right]dvdt, \] (3.6)
\[ P(t, \tau) = \int_\Omega \left[ \rho_1^0F_i^{(1)}(t)u_i(\tau) + \rho_2^0F_i^{(2)}(t)w_i(\tau) \right. \]
\[ \left. + \rho_1^0L^{(1)}(t)\dot{\varphi}(\tau) + \rho_2^0L^{(2)}(t)\dot{\psi}(\tau) \right]dv \]
\[ + \int_{\partial\Omega} \left[ t_{ji}(t)\dot{u}_i(\tau) + s_{ji}(t)\dot{w}_i(\tau) \right. \]
\[ \left. + h_j^{(1)}(t)\dot{\varphi}(\tau) + h_j^{(2)}(t)\dot{\psi}(\tau) \right]n_jda, \] (3.7)
\[ Q(t, \tau) = \int_\Omega \left[ \rho_1^0F_i^{(1)}(t)u_i(\tau) + \rho_2^0F_i^{(2)}(t)w_i(\tau) \right. \]
\[ \left. + \rho_1^0L^{(1)}(t)\varphi(\tau) + \rho_2^0L^{(2)}(t)\psi(\tau) \right]dv \]
\[ + \int_{\partial\Omega} \left[ t_{ji}(t)u_i(\tau) + s_{ji}(t)w_i(\tau) \right. \]
\[ \left. + h_j^{(1)}(t)\varphi(\tau) + h_j^{(2)}(t)\psi(\tau) \right]n_jda. \] (3.8)

The following first three lemmas were proved in [5] for homogeneous boundary conditions and in the absence of the body forces. Following the same strategy
we can extend them to include the prescribed boundary data and body loads. Thus, we have

**Lemma 3.1** (Conservation law of total energy) If \( \mathcal{U} = \{u, w, \varphi, \psi\} \) is solution of the problem \((\mathcal{P})\), then the following conservation energy law holds:

\[
\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t P(\tau, t) d\tau. \tag{3.9}
\]

**Lemma 3.2** Let \( \mathcal{U} = \{u, w, \varphi, \psi\} \) be solution of the problem \((\mathcal{P})\). Then for every \( t \in [0, \infty) \), the following identity holds

\[
\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + 2 \int_0^t [\mathcal{K}(\tau) - \mathcal{W}(\tau)] d\tau + \int_0^t Q(\tau, \tau) d\tau. \tag{3.10}
\]

**Lemma 3.3** Suppose that \( \{u, w, \varphi, \psi\} \) is solution of the problem \((\mathcal{P})\). Then for every \( t \in [0, \infty) \), we have

\[
\frac{dI}{dt}(t) = \mathcal{L}(t) + \mathcal{A}(t) + \frac{1}{2} \int_0^t [Q(t, t, \tau + \tau) - Q(t, \tau, \tau + \tau)] d\tau, \tag{3.11}
\]

where

\[
\mathcal{L}(t) = \frac{1}{2} \int_\Omega \left[ \rho_0^0 \left( \dot{u}_i(t)u_i(2t) + \dot{w}_i(t)w_i(0) \right) + \rho_0^1 \left( \dot{w}_i(t)w_i(2t) + \dot{w}_i(t)w_i(0) \right) + \rho_0^1 \kappa_1(\dot{\varphi}(0)\dot{\varphi}(2t) + \dot{\varphi}(2t)\varphi(0)) + \rho_0^2 \kappa_2(\dot{\psi}(0)\dot{\psi}(2t) + \dot{\psi}(2t)\psi(0)) \right] dv, \tag{3.12}
\]

\[
\mathcal{A}(t) = \frac{1}{2} \int_\Omega \left[ A_{ijrs} \dot{e}_{ij}(0)e_{rs}(2t) + A_{ijrs} \dot{d}_{ij}(0)d_{rs}(2t) \right] dv. \tag{3.13}
\]

Moreover, we have the following result:

**Lemma 3.4** Assume that \( \mathcal{U} = \{u, w, \varphi, \psi\} \) is solution of the problem \((\mathcal{P})\). Then for every \( t \in [0, \infty) \), we have

\[
\mathcal{W}(t) - \mathcal{K}(t) = \frac{1}{2} \int_0^t \left[ P(t + \tau, t) - P(t - \tau, t) \right] d\tau + \mathcal{G}(t) - H(t), \tag{3.14}
\]

where

\[
H(t) = \frac{1}{2} \int \rho_0^0 \left( \dot{u}_i(0)\dot{u}_i(2t) + \dot{u}_i(2t)\dot{u}_i(0) \right) + \rho_0^1 \left( \dot{\psi}(0)\dot{\psi}(2t) + \dot{\psi}(2t)\psi(0) \right) + \rho_0^2 \kappa_2(\dot{\psi}(0)\dot{\psi}(2t) + \dot{\psi}(2t)\psi(0)) + \mathcal{G}(t). \tag{3.15}
\]

\[
\mathcal{G}(t) = \frac{1}{2} \int_\Omega \left[ B_{ijrs} \delta_{ij} e_{ij}(0)e_{rs}(2t) + C_{ijrs} \delta_{ij} g_{ij}(0)g_{rs}(2t) + D_{ij} \delta_{ij} e_{ij}(0)e_{ij}(2t) + E_{ij} \delta_{ij} \varphi(2t) + F_{ij} \delta_{ij} \psi(2t) + G_{ij} \delta_{ij} \varphi(0) \right] dv. \tag{3.16}
\]

**Proof** Let us introduce the notation:

\[
\mathcal{R}(t, \tau) = t_{ji}(t)\dot{u}_i,j(\tau) + s_{ji}(t)\dot{w}_i,j(\tau) + p_i(t)d_i(\tau) + h^{(1)}(t)\dot{\psi}(\tau) + h^{(2)}(t)\dot{\psi}(\tau) + g^{(1)}(t)\dot{\varphi}(\tau). \tag{3.17}
\]

Moreover, for the sake of simplicity, let us use the following convention: if \( f \) is a function of time, then we use the notations \( f^- \) and \( f^+ \) for the values of
the function \( f \) in the points \( t - \tau \) and \( t + \tau \), namely
\[ f^- = f(t - \tau) \quad \text{and} \quad f^+ = f(t + \tau). \]

Then, by means of the constitutive equations (2.2), the geometric equations (2.3) and the symmetry relations (2.4) we deduce
\[
\mathcal{R}(t - \tau, t + \tau) - \mathcal{R}(t + \tau, t - \tau)
= \frac{\partial}{\partial \tau} \left[ A_{ijsr} e_{ij}^{+} e_{rs}^{+} + B_{ijsr} (e_{ij}^{+} g_{rs}^{+} + e_{ij}^{+} g_{rs}^{-}) + C_{ijsr} g_{ij}^{+} g_{rs}^{+} + D_{ijsr} (e_{ij}^{+} g_{rs}^{+} + e_{ij}^{+} g_{rs}^{-}) + E_{ij} (e_{ij}^{+} \psi^+ + e_{ij}^{+} \psi^-) + M_{ij} (g_{ij}^{+} \psi^+ + g_{ij}^{+} \psi^-) + N_{ij} (g_{ij}^{+} \psi^+ + g_{ij}^{+} \psi^-) + \xi(\psi^- \psi^+ - \psi^+ \psi^-) \right]
= \frac{\partial}{\partial \tau} \left[ \mathcal{R}(t - \tau, t + \tau) - \mathcal{R}(t + \tau, t - \tau) \right] d\nu d\tau
= -2K(t) + 2H(t)
+ \int_0^t [P(t - \tau, t + \tau) - P(t + \tau, t - \tau)] d\tau.
\]

From (3.19) and (3.21) we deduce the identity (3.14).

Various Lagrange identities may be obtained by combining the identities (3.9), (3.10), (3.11) and (3.14). In the next sections we use the following identities which follow from (3.9) and (3.10) (the first remark), and from (3.9) and (3.14) (the second remark).

**Remark 1** Let \( \mathcal{U} = \{u, w, \varphi, \psi\} \) be solution of the problem \((\mathcal{P})\). Then for every \( t \in [0, \infty) \), the following identity holds
\[
\frac{dI}{dt}(t) = \frac{dI}{dt}(0) + \int_0^t \left[ 4K(t) + 2\Delta(t) \right] d\tau - 2rE(0) - 2 \int_0^t P(s, s) ds d\tau + \int_0^t Q(\tau, \tau) d\tau.
\]

**Remark 2** Assume that \( \mathcal{U} = \{u, w, \varphi, \psi\} \) is solution of the problem \((\mathcal{P})\). Then for every \( t \in [0, \infty) \), we have
\[
2\mathcal{W}(t)
= \mathcal{W}(0) + 2{\mathcal{K}}(0)
+ \frac{1}{2} \int_0^t [P(t + \tau, t - \tau) - P(t - \tau, t + \tau)] d\tau
+ \int_0^t P(\tau, \tau) d\tau - \Delta(t) + \Gamma(t) - H(t)
\]
and
\[
2{\mathcal{K}}(t)
= {\mathcal{K}}(0) + \mathcal{W}(0)
- \frac{1}{2} \int_0^t [P(t + \tau, t - \tau) - P(t - \tau, t + \tau)] d\tau
+ \int_0^t P(\tau, \tau) d\tau - \Delta(t) - \Gamma(t) + H(t).
\]

The Remark 2 leads to a separation of energies in the sense described by Brun [2].
In the hypothesis that the internal energy density $W$ is a positive definite quadratic form in terms of $e_{ij}$, $g_{ij}$, $\varphi$, $\psi$, $\varphi_i$, $\psi_i$, and $d_{ij}$, Lemma 3.1 leads to a counterpart of the Neumann uniqueness theorem of linear elastodynamics and to a Liapounov stability result. Using these arguments Ieşan [9] investigated the uniqueness and continuous dependence problems in the thermo-viscoelastic case.

Using the identities from Remark 2 we deduce the following:

**Theorem 3.1** Suppose that $\rho_1^0$, $\rho_2^0$, $\kappa_1$ and $\kappa_2$ are strictly positive. Then, the initial-boundary value problem $(P)$ has at most one solution.

The above uniqueness result is established without any restriction conditions on the elastic coefficients and without any a priori assumptions on the solutions.

### 4 Continuous dependence

In this section we obtain estimates describing continuous dependence for a class of solutions with respect to body forces, initial and boundary data under mild assumptions on the characteristic coefficients. In this aim we use the identities (3.11) and (3.22). The approach based upon the identity (3.22) and convexity arguments yields H"older continuous dependence valid on all compact intervals. This method leads to results of the kind given in [3, 11] for linear elasticity and viscoelasticity. The other method leads to estimates similar to those derived in [18] for materials with memory.

In the following we shall use the notation:

- $(P_f)$ is the boundary value problem corresponding to homogeneous boundary data and homogeneous initial data.
- $(P_b)$ is the boundary value problem with homogeneous initial data and vanishing body loads.
- $(P_i)$ is the boundary value problem corresponding to homogeneous boundary data and vanishing body loads.

In the following, we discuss the continuous dependence for the solutions included in some constraint set.

**Theorem 4.1** (Continuous dependence upon body forces) Suppose that $\rho_1^0$, $\rho_2^0$, $\kappa_1$ and $\kappa_2$ are strictly positive. Further, we assume that the solution $\mathcal{U} = \{u, w, \varphi, \psi\}$ of the problem $(P_f)$ satisfies on the interval $[0, T]$, the conditions

$$
\sup_{\Omega \times [0, T]} \{u_i u_i + w_i w_i + \kappa_1 \varphi_i \varphi_i + \kappa_2 \psi_i \psi_i\} < M_1,
$$

$$
\sup_{\Omega \times [0, T]} \{u_{i,j} u_{i,j} + w_{i,j} w_{i,j}\} < M_2,
$$

where $M_i, i = 1, 2$ are positive constants. Then, for each fixed $t \in [0, T]$ we have

$$
G(t) \leq e^{\delta(1-\delta)} M_3^2 |G(0)|^{1-\delta}.
$$

where $\delta = \frac{t}{T}$, $M_3$ is a positive constant and

$$
G(t) = \int_0^t I(\tau)d\tau + \frac{1}{2} T^4 \int_0^T \|\mathbf{f}(\tau)\|_2^2 d\tau,
$$

$$
\|\mathbf{f}(t)\|^2 = \int_{\Omega^2} \left[ \rho_1^0 F_i^{(1)}(t) F_i^{(1)}(t) + \rho_2^0 F_i^{(2)}(t) F_i^{(2)}(t) + \frac{\rho_1^0}{\kappa_1} L^{(1)}(t) L^{(1)}(t) + \frac{\rho_2^0}{\kappa_2} L^{(2)}(t) L^{(2)}(t) \right] dv.
$$

**Proof** In view of the hypothesis of the theorem and the relations (3.5) and (3.6), we get

$$
I(0) = 0, \quad \dot{I}(0) = 0, \quad \varphi(0) = 0.
$$

Then, from (3.6), (4.3) and (4.5) we deduce

$$
\ddot{G}(t) = I(t)
$$

$$
= \int_0^t I(\tau)d\tau
$$

$$
= \int_0^t \left[ \rho_1^0 u_i u_i + \rho_2^0 w_i w_i + \rho_1^0 \kappa_1 \dot{\varphi} \dot{\varphi} + \rho_2^0 \kappa_2 \dot{\psi} \dot{\psi} \right] dv d\tau
$$

$$
+ \int_0^t \int_0^\tau \int_{\Omega} [A_{i j r s} e_{i j} \dot{e}_{r s} + a_{i j} d_{i j}] d v d s d \tau.
$$

Moreover, the identity (3.22) leads to

$$
\dddot{G}(t) = \dddot{I}(t)
$$

$$
= \int_0^t \left[ 4K(\tau) + 2\Delta(\tau) \right] d\tau
$$

$$
- \int_0^t \left[ 2(t - \tau) P(\tau, \tau) - Q(\tau, \tau) \right] d\tau.
$$

\[\square\]
Now, substituting $K$ and $\Delta$ from $(3.2)$ and $(3.4)$ in $(4.7)$ and making use of the Schwarz’s inequality, we deduce

$$G\ddot{G} - \dot{G}^2 \geq T^4 \left[ \int_0^T \left\| \Phi(t) \right\|^2 d\tau \right] \left[ 2 \int_0^T K(\tau) d\tau \right] - G \int_0^t \left[ 2(t - \tau) P(\tau, \tau) - Q(\tau, \tau) \right] d\tau. \quad (4.8)$$

Applying the Schwarz’s inequality and the arithmetic–geometric mean inequality, we deduce

$$-G \int_0^t Q(\tau, \tau) d\tau \leq \frac{G}{T^2} \left[ \int_0^T \left( \rho_0^0 u_i u_i + \rho_0^2 w_i w_i + \rho_1^0 \kappa_1 \phi^2 + \rho_2^0 \kappa_2 \psi^2 \right) dv d\tau \right]^{1/2} \times \left[ T^4 \int_0^T \left\| \Phi(t) \right\|^2 d\tau \right]^{1/2} \leq T^{-2} G^2 \quad (4.9)$$

and

$$2G \int_0^t (t - \tau) P(\tau, \tau) d\tau \leq 2tG \int_0^t |P(\tau, \tau)| d\tau \leq \frac{2}{T} G \left[ T^4 \int_0^T \left\| \Phi(t) \right\|^2 d\tau \right]^{1/2} \left[ 2 \int_0^T K(\tau) d\tau \right]^{1/2} \leq T^{-2} G^2 + T^4 \left[ \int_0^T \left\| \Phi(t) \right\|^2 d\tau \right] \left[ 2 \int_0^t K(\tau) d\tau \right]. \quad (4.10)$$

Relations $(4.8)$, $(4.9)$ and $(4.10)$ imply

$$G\ddot{G} - \dot{G}^2 \geq -2T^{-2} G^2. \quad (4.11)$$

Since $G(t)$ is strictly positive unless $F_i^{(a)} = 0$, $L^{(a)} = 0$, $a = 1, 2$ (except possibly on a set of measure zero), we may divide $(4.11)$ by $G^2$ to obtain

$$\frac{d^2}{dt^2} \left[ \ln G(t) \right] \geq -2T^{-2}. \quad (4.12)$$

Since $\ln[G(t)e^{t^2/T^2}]$ is a convex function, by Jensen’s inequality we obtain the inequality $(4.2)$.

Theorem 4.2 (Continuous dependence upon initial data) Assume that $\rho_0^0, \rho_0^2, \kappa_1$ and $\kappa_2$ are strictly positive and the solution $U = \{u, w, \varphi, \psi\}$ of the problem (P1) satisfies $(4.1)$. Then for initial data with initial energy $\mathcal{E}(0) \leq 0$, we have

$$G^*(t) \leq M_4^G |G^*(0)|^{-\delta} \quad (4.13)$$

and for initial data with $\mathcal{E}(0) > 0$, we have

$$G^*(t) + T^2 \mathcal{E}(0) \leq e^{\delta(1-\delta)} [G^*(T) + T^2 \mathcal{E}(0)]^{\delta} \times [G^*(0) + T^2 \mathcal{E}(0)]^{1-\delta}, \quad (4.14)$$

where $M_4$ is a positive constant, $\delta = \frac{1}{2}, t \in [0, T]$ and

$$G^*(t) = I(t) + (T - t) \Lambda(0). \quad (4.15)$$

Proof From $(3.6)$, $(3.13)$ and $(4.15)$ we get

$$G^*(t) \geq I(t) \quad (4.16)$$

and

$$\dot{G}^*(t) = \int_\Omega \left[ \rho_0^0 u_i \dot{u}_i + \rho_2^0 w_i \dot{w}_i + \rho_1^0 \kappa_1 \phi \dot{\varphi} + \rho_2^0 \kappa_2 \psi \dot{\psi} \right] dv + \int_\Omega \int_\Omega A_{ijrs} e_{ij} \dot{e}_{rs} + a_{ij} d_i d_j] dv d\tau. \quad (4.17)$$

The identity $(3.22)$ leads to

$$\ddot{G}^*(t) = (I(t) - 2 \Delta(t) - 2 \mathcal{E}(0)). \quad (4.18)$$

By Schwarz’s inequality, from $(4.16)$–$(4.18)$ we deduce

$$G^*(t) \dot{G}^*(t) - [\dot{G}^*(t)]^2 \geq -2\mathcal{E}(0)G^*(t). \quad (4.19)$$

Now, if $\mathcal{E}(0) \leq 0$ then by using the fact that $G^*(t) > 0$, $t \in (0, T)$, we may rewrite $(4.19)$ in the form

$$\frac{d^2}{dt^2} \left[ \ln G^*(t) \right] \geq 0 \quad (4.20)$$

and thus by convexity arguments we deduce the estimate $(4.13)$.

If $\mathcal{E}(0) > 0$ then we use a similar argument applied to the function $G^*(t) + T^2 \mathcal{E}(0)$. Using the fact that

$$-2\mathcal{E}(0)[G^*(t) + T^2 \mathcal{E}(0)] \geq -2T^{-2}[G^*(t) + T^2 \mathcal{E}(0)]^2$$
we obtain for $G^*(t) + T^2E(0)$ the inequality (4.12), and thus by Jensen’s inequality we deduce the estimate (4.14).

An adequate measure and similar arguments may be used to obtain H"older continuous dependence of solutions upon boundary data.

Now we discuss continuous dependence based on the identity (3.11).

**Theorem 4.3** (Continuous dependence upon body forces) Suppose that $\rho^0_1, \rho^0_2, \kappa_1$ and $\kappa_2$ are strictly positive and the solution $U = \{u, w, \varphi, \psi\}$ of the problem $(P_f)$ satisfies (4.1) on the interval $[0, T]$. Then, for each $t \in [0, T/2]$, we have the following estimate

$$I(t) \leq \frac{1}{2} M_5 \left[ \int_0^T \|f(\tau)\|^2 d\tau \right]^{1/2},$$

where $M_5$ is a positive constant.

**Proof** In view of the hypothesis, the identity (3.11) implies

$$\frac{dI}{dt}(t) = \frac{1}{2} \int_0^T [Q(t - \tau, t + \tau) - Q(t + \tau, t - \tau)]d\tau.$$  (4.22)

Using the same notations as those utilized in the proof of the Lemma 3.4, from the Schwarz’s inequality we deduce that there is a positive constant $M_6$ such that

$$\int_0^T \left[ \rho^0_1 F^1_i u^+_i + \rho^0_2 F^2_i w^+_i + \rho^0_1 \kappa_1 L^1 \varphi^+ + \rho^0_2 \kappa_2 L^2 \psi^+ \right]d\tau \leq M_6 \left[ \int_0^T \|f(\tau)\|^2 d\tau \right]^{1/2}$$  (4.23)

and

$$\int_0^T \left[ \rho^0_1 F^1_i u^-_i + \rho^0_2 F^2_i w^-_i + \rho^0_1 \kappa_1 L^1 \varphi^- + \rho^0_2 \kappa_2 L^2 \psi^- \right]d\tau \leq M_6 \left[ \int_0^T \|f(\tau)\|^2 d\tau \right]^{1/2}.$$  (4.24)

Substituting the estimates (4.23) and (4.24) into (4.22) and integrating the result between 0 and $t, t \in [0, T/2]$ we obtain (4.21) and the proof is complete. \hfill \square

**Theorem 4.4** (Continuous dependence upon boundary data) Assume that $\rho^0_1, \rho^0_2, \kappa_1$ and $\kappa_2$ are strictly positive. Let $U = \{u, w, \varphi, \psi\}$ be solution of the problem $(P_b)$ in the constraint set for which there exists the positive constants $N_i, i = 1, 2, 3$ defined by

$$N_1 = \sup_{\partial \Omega \times [0,T]} [u_i u_i + w_i w_i],$$

$$N_2 = \sup_{\partial \Omega \times [0,T]} [u_i u_i, j + w_i w_i, j + \varphi \varphi + \psi \psi],$$

$$N_3 = \sup_{\partial \Omega \times [0,T]} [\varphi, i \varphi, i + \psi, i \psi, i].$$

Then, for each $t \in [0, T/2]$, we have the following estimate

$$I(t) \leq N_4 \left[ \int_0^T \left( \tilde{u}_i \tilde{u}_i + \tilde{w}_i \tilde{w}_i \right)adt \right]^{1/2}$$

$$+ N_5 \left[ \int_0^T \left( \tilde{u}_i \tilde{u}_i + \tilde{w}_i \tilde{w}_i \right) d\tau \right]^{1/2}$$

$$+ N_6 \left[ \int_0^T \left( \tilde{\varphi}_i \tilde{\varphi}_i + \tilde{\psi}_i \tilde{\psi}_i \right) d\tau \right]^{1/2}$$

$$+ N_7 \left[ \int_0^T \left( \tilde{\rho}_i \tilde{\rho}_i + \tilde{\rho}_i \tilde{\rho}_i \right) d\tau \right]^{1/2}.$$  (4.26)

where $N_1, i = 4, \ldots, 7$ are positive constants.

**Proof** In view of the hypothesis, the identity (3.11) reduces to

$$\frac{dI}{dt}(t) = \frac{1}{2} \int_0^T [Q(t - \tau, t + \tau) - Q(t + \tau, t - \tau)]d\tau,$$  (4.27)

where

$$Q(t, \tau) = \int_{\partial \Omega} \left[ t_{ji}(t) u_i(\tau) + s_{ji}(t) w_i(\tau) + h_j^{(1)}(t) \psi(\tau) + h_j^{(2)}(t) \psi(\tau) \right]n_2 d\tau.$$  (4.28)
Now, we substitute $Q$ in (4.27) and write the surface integral in the form:

$$
\int_{\partial \Omega} (f + g) da = \int_{S_1} f da + \int_{S_2} f da + \int_{S_3} g da + \int_{S_4} g da,
$$

(4.29)

where $f = (t_{ji} u_i + s_{ji} w_i) n_j$ and $g = (h^{(1)}_j \varphi + \psi n_j)$. By using the Schwarz’s inequality for each of the four integrals, the relations (4.25) and the equalities

$$
t_{ji} u_i + s_{ji} w_i = (t_{ji} + s_{ji}) u_i - s_{ji} d_i \quad (4.30)
$$

and

$$
h^{(1)}_j \varphi + h^{(2)}_j \psi = (h^{(1)}_j + h^{(2)}_j) \varphi - h^{(2)}_j (\varphi - \psi) \quad (4.31)
$$

for the integral on $S_3$ and $S_4$, respectively, we deduce after integration the estimate (4.26).

In the same manner we may prove the continuous dependence upon the initial data.

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References