On the spatial behaviour in the bending theory of porous thermoelastic plates

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Abstract

In this paper we consider the initial-boundary value problem which describes the bending of Mindlin type thermoelastic plates with voids. In terms of some appropriate time-weighted surface power functions, we study the spatial behaviour of thermodynamic processes for a large class of thermoelastic materials with voids. Bounded domains and unbounded domains are considered.

Keywords:
Plates
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1. Introduction

To take into account the microstructure of materials, special attention has been paid to include some terms in the basic formulation of thin body theories[1–5,28,29,39].

The theory of thermoelastic materials with voids is a special theory in the framework of theories of materials with microstructure[6,7]. Ieşan[22] used the representation proposed by Goodman and Cowin[20] (see also[15,30]) in order to elaborate a theory for the treatment of porous thermoelastic solids. The main idea of this theory is to suppose that there is a distribution of voids throughout the body. In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. The volume fraction field is an additional degree of kinematic freedom and represents a kinematical variable assigned to each material particle. The intended applications of this theory are to geological materials and to manufactured porous materials, as well as granular materials. A presentation of this theory can be found in[23].

In the present paper we consider the bending theory of porous thermoelastic plates introduced by Bîrsan[21]. Chiriţă and Ciarletta[9] introduced a method, called the time-weighted surface power method, which was applied in the study of spatial behaviour of elastic, viscoelastic and thermoelastic processes outside of the support of the external given data. We use this method in the framework of the bending theory of porous thermoelastic plates. To this aim, we study the spatial behaviour of dynamic processes in terms of some appropriate time-weighted surface power functions. For bounded domains we establish decay estimates of Saint-Venant type, while for unbounded domains, we establish alternatives of Phragmèn–Lindelöf type.

Moreover, for thermoelastic plates whose middle surface is like a semi-infinite strip, we can improve the above spatial decay estimate. To this aim, we follow the research line initiated by Horgan et al.[21] (for parabolic equations) and by Quintanilla[36,37] (for coupled systems of parabolic and hyperbolic equations). Using the maximum principle, we establish...
an estimate which proves that the mechanical and thermal effects are controlled by an exponential decay estimate in terms of the square of the distance from the support of the external given data.

The results are established for a large subclass of the class of strongly elliptic materials with voids. The method presented here is believed to be successfully used for the study of materials with negative Poisson’s ratio (anti-rubber, dilational materials or auxetic materials) which are most useful in biomechanics [24–26]. Necessary and sufficient conditions characterizing the strong ellipticity of materials with voids were obtained by Chirita and Ghiba [10]. These materials were then considered in several works in order to study the propagation of progressive waves [10], inhomogeneous waves [11] and seismic waves [8] within the framework of the linear theory of porous media. The problem of strong ellipticity in elastic bodies with microstructure was also considered in [32].

In the theories of porous isotropic elastic plates, the spatial behaviour of steady-state solutions under relaxed hypotheses upon the constitutive coefficients, was studied by Ghiba [18]. Similar relaxed conditions upon the constitutive coefficients were also considered in the study of the temporal behaviour in the bending theory of porous thermoelastic plates [19].

Regarding the study of spatial behaviour in plate theories, we mention that Passarella and Zampoli [33,34] established results concerning the spatial behaviour of transient and steady-state solutions in the theory of transversely isotropic elastic plates under relaxed hypothesis on the positive definiteness of the elasticity tensor, while the case of rhombic systems was considered in [31]. Ciarletta [13] studied the spatial behaviour of transient and steady-state solutions in thin plates with shear deformation. Some results concerning the state of bending for transversely isotropic plates were established in [33]. D’Apice and Chirita [17] and D’Apice [16] presented some methods for the study of the spatial behaviour of transient solutions in the bending theory of Mindlin type thermoelastic plates.

### 2. Preliminaries and auxiliary results

We consider a fixed system of rectangular Cartesian axes $Ox_i$ ($i = 1, 2, 3$). Throughout this paper, Latin subscripts take the values 1, 2, 3, Greek indices have the range 1, 2 and summation is carried out over repeated indices. Typical conventions for differential operations are implied such as a superposed dot or comma followed by a subscript in order to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

Let us consider the region $\Sigma \times [-\frac{h}{2}, \frac{h}{2}]$ of the physical space $\mathbb{R}^3$, where $\Sigma$ is a domain in $\mathbb{R}^2$ whose boundary $\partial \Sigma$ is a simple $C^2$-curve and $0 < h = \text{constant} \ll \text{diam } \Sigma$. We call this region plate with the thickness $h$.

We assume that $B$ is the interior of the right cylinder $\Sigma \times [-\frac{h}{2}, \frac{h}{2}]$ and we assume that it is filled by an isotropic and homogeneous thermoelastic material with voids.

#### 2.1. Three-dimensional approach

Let $u = (u_1, u_2, u_3)$ be the displacement field, $\varphi$ the change in volume fraction from the reference volume [20], $T_0$ the absolute temperature in the reference state and $\theta$ the temperature variation from the absolute temperature $T_0$ over $B$. In what follows, we will assume that the functions $u_i, \varphi$ and $\theta$ are so that:

(i) $u_i$ and $\varphi$ are of class $C^2$ on $B \times [0, \infty)$;
(ii) $\theta$ is of class $C^{2,1}$ on $B \times [0, \infty)$;
(iii) $u_i, \varphi$ and $\theta$ are of class $C^1$ on $\overline{B} \times [0, \infty)$.

In the context of linear theory of porous thermoelastic solid, as described by Leșan [22], the equations of motion are

$$
\begin{align*}
\dot{t}_{ij, j} + \rho f^*_{ij} &= \rho \ddot{u}_i, \\
\dot{h}_{ij} + g + \rho \ell^* &= \rho \dot{\chi} \dot{\varphi}
\end{align*}
$$

while the energy equation is

$$
\rho T_0 \dot{\eta} = q_{ii} + \rho S^*.
$$

in $B \times [0, \infty)$. Here $t_{ij}$ are the components of the stress tensor, $h_i$ are the components of the equilibrated stress vector, $g$ is the intrinsic equilibrated force, $f^*_{ij}$ are the components of the body force, $\ell^*$ is the extrinsic equilibrated body force, $\rho$ and $\chi$ are the bulk mass density and the equilibrated inertia in the reference state, $\eta$ is the specific entropy and $S^*$ is the extrinsic heat supply.

The constitutive equations of the linear theory of isotropic and homogeneous thermoelastic porous continuum [22] are

$$
\begin{align*}
\tau_{ij} &= \lambda \epsilon_{ij} \delta_{ij} + 2 \mu \epsilon_{ij} + b \varphi \delta_{ij} - \beta \theta \delta_{ij}, \\
\ell_i &= \alpha \dot{\varphi} i, \\
g &= -bu_{i, r} - \xi \varphi + m \theta, \\
\rho \eta &= \beta e_i + m \varphi + a \theta, \\
q_i &= k \theta_i,
\end{align*}
$$

where $\lambda, \mu, \alpha, b, \beta, \xi, m, a, k, \lambda, \mu$ are the material constants.
where
\[ e_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), \]  
(2.4)
the quantities \( \lambda, \mu, b, \beta, \alpha, \xi, m \) and \( k \) are constitutive constants and \( \delta_{ij} \) is the Kronecker’s delta.

The surface force, the equilibrated surface force and the heat flux at a regular point of \( \partial B \) are given by
\[ t_i = t_j n_j, \quad h = h_i n_i, \quad q = q_i n_i \]
respectively. We assume that the functions \( t_i, h \) and \( q \) are prescribed on the surfaces \( x_3 = \pm h \).

The specific internal energy \( W(0) \) for an isotropic and homogeneous elastic material with voids is defined by
\[ 2W(0) = 2\mu e_{rr}e_{ss} + \lambda e_{rr}e_{ss} + \xi \psi^2 + \alpha \varphi, \varphi, + 2b\psi e_{rr}. \]  
(2.5)

The specific internal energy \( W(0) \) is positive definite if and only if
\[ \mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad (3\lambda + 2\mu) \xi > 3b^2. \]  
(2.6)

An elastic material with void is called strongly elliptic material [10] if and only if
\[ \mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad (\lambda + 2\mu) \xi > b^2. \]  
(2.7)

We denote the class of strongly elliptic materials by \( C \) and the class of materials for which the internal energy is positive definite by \( C^+ \). From the inequalities (2.6) and (2.7) we can easily remark that \( C^+ \subseteq C \).

2.2. Reduction of the three-dimensional equations by through-the-thickness integration

In the bending theory of Mindlin type plates with voids [2], an admissible process is a state of bending on \( \overline{B} \times [0, \infty) \) provided
\[ \begin{align*}
    u_\alpha &= x_3 v_\alpha(x_1, x_2, t), \\
    u_3 &= w(x_1, x_2, t), \\
    \varphi &= x_3 \psi(x_1, x_2, t), \\
    \theta &= x_3 T(x_1, x_2, t)
\end{align*} \]  
(2.8)
on \( \overline{B} \times [0, \infty) \) In view of (2.4) we have
\[ e_{\alpha\beta} = x_3 e_{\alpha\beta}, \quad 2e_{\alpha 3} = \gamma_\alpha, \quad e_{33} = 0 \]  
(2.9)
where
\[ e_{\alpha\beta} = \frac{1}{2} (v_{\alpha,\beta} + v_{\beta,\alpha}), \quad \gamma_\alpha = v_\alpha + w_\alpha. \]  
(2.10)

The quantities \( \gamma_\alpha \) represent the angles of rotation of the cross-sections \( x_\alpha = \) constant about the middle surface (see e.g. [2,14]).

By the through-the-thickness integration [2,39], we have the following basic system of partial differential equations
\[ \begin{align*}
    N^{(1)}_{\alpha,\beta} - hN_{\alpha 3} + F_\alpha &= \rho I \ddot{v}_\alpha, \\
    N_{\beta 3,\beta} + F_3 &= \rho \ddot{v}_3, \\
    H_{\alpha,\alpha} + G - h\Gamma + L &= \rho \chi I \ddot{\psi}, \\
    Q_{\alpha,\alpha} - hR + S &= T_0 \dot{\sigma}, \quad \text{in } \Sigma \times (0, \infty),
\end{align*} \]  
(2.11)
where \( I = \frac{h^3}{12} \) and
\[ \begin{align*}
    N^{(1)}_{\alpha,\beta} &= I [\lambda e_{\alpha\beta} + 2\mu e_{\alpha\beta} + b\psi \delta_{\alpha\beta} - \beta T \delta_{\alpha\beta}], \\
    N_{\alpha 3} &= \mu \gamma_\alpha, \\
    H_\beta &= \alpha I \psi, \\
    \Gamma &= \beta \psi, \\
    G &= -I (b e_{\phi\phi} + \xi \psi - m T), \\
    \sigma &= I (\beta e_{\phi\phi} + m \psi + a T), \\
    Q_\alpha &= k T_0, \\
    R &= k T, \quad \text{in } \Sigma \times [0, \infty).
\end{align*} \]  
(2.12)
The quantities $F_i$, $L$ are related to the averaged body or extrinsic forces and moments and to the resultant average forces and moments acting on the faces $x_3 = \pm \frac{b}{2}$ [2].

In the equations of motion (2.11) and in the constitutive equations (2.12), $N_\alpha, N^{(1)}_\alpha$ ($\alpha$ not summed) and $N^{(1)}_{12} = N^{(1)}_{21}$ characterize, respectively, the average transverse shear forces and average bending and twisting moments with respect to the middle plane, acting on the face of a vertical cross-sectional element of the plane perpendicular to the $x_\alpha$-axis [14,27,38,40,41]. Similar arguments for $H_\alpha, \Gamma, Q_\alpha$ and $R$.

The Eqs. (2.11) can be expressed in terms of $v_\alpha, w, \psi$ and $T$. Thus, we obtain

$$
\begin{align*}
I[\mu \Delta v_\alpha + (\lambda + \mu) v_{\alpha,\alpha} + b \psi_{,\alpha} - \beta T_{,\alpha}] - \mu h(v_\alpha + w_{,\alpha}) &= \rho I \ddot{v}_\alpha, \\
\mu \Delta w + \mu v_{,\alpha,\alpha} &= \rho \ddot{w}, \\
I[\alpha \Delta \psi - b v_{,\alpha} + mT] - (\xi l + \alpha h) \psi &= \rho \chi \dot{\psi}, \\
I[k \Delta T - \beta T_{,\alpha} \ddot{v}_{,\alpha} - m \dot{T}_0 \dot{\psi}] - k h T &= a \dot{T}_0 \dot{T}, \quad \text{in } \Sigma \times (0, \infty),
\end{align*}
$$

(2.14) where $\Delta$ is the Laplace operator.

To the field Eqs. (2.14) we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions

$$
\begin{align*}
v_\alpha &= \ddot{v}_\alpha, & w &= \ddot{w} \\
\psi &= \ddot{\psi}, & T &= \ddot{T}
\end{align*}
$$

(2.15)
on $\partial \Sigma$.

We assume that the prescribed functions $\ddot{v}_\alpha, \ddot{w}, \ddot{\psi}$ and $\ddot{T}$ are continuous on their domain of definition.

The initial conditions consist in

$$
\begin{align*}
v_\alpha(x_\gamma, 0) &= v^0_\alpha(x_\gamma), & \dot{v}_\alpha(x_\gamma, 0) &= v^1_\alpha(x_\gamma) \\
w(x_\gamma, 0) &= w^0(x_\gamma), & \dot{w}(x_\gamma, 0) &= w^1(x_\gamma) \\
\psi(x_\gamma, 0) &= \psi^0(x_\gamma), & \dot{\psi}(x_\gamma, 0) &= \psi^1(x_\gamma) \\
T(x_\gamma, 0) &= T^0(x_\gamma), & (x_\gamma) &\in \Sigma,
\end{align*}
$$

(2.16)

where the functions $v^0_\alpha, v^1_\alpha, w^0, w^1, \psi^0, \psi^1, T^0$ are given continuous functions.

Without losing the generality, we can assume that all the quantities involved in the present paper are dimensionless. If we do not assume this, in view of the linearity of the problem, some new notations which involve the constitutive coefficients can be made in order to obtain a dimensionless form of the equations.

The internal energy density $W^{(1)}$ per unit area of the middle plane associated with the kinematic fields $v_\alpha, w$ and $\psi$ is defined by

$$
2W^{(1)} = I \left[ \lambda \varepsilon_{\alpha\beta} \varepsilon_{\beta\gamma} + 2 \mu \varepsilon_{\alpha\beta} \varepsilon_{\gamma\beta} + \xi \psi^2 + \alpha \psi,_{\beta} \psi,_{\beta} + b \psi \varepsilon_{\beta\gamma} + h(\mu \gamma_{\alpha} \gamma_{\alpha} + \alpha \psi^2) \right].
$$

(2.17)

Let us remark that the conditions

$$
\mu > 0, \quad \alpha > 0, \quad \lambda + \mu > 0, \quad \xi (\lambda + \mu) > b^2
$$

(2.18)

are enough to assure the positive definiteness of the quadratic form $W^{(1)}$.

The above inequalities are more restrictive than (2.7) and less restrictive than (2.6). We denote by $\mathcal{P}$, the initial–boundary value problem defined by the relations (2.14)–(2.16).

We associate with the solution of the problem $\mathcal{P}$, the following quantities

$$
\begin{align*}
\mathcal{K}(t) &= \frac{1}{2} \rho \left[ I \ddot{v}_\alpha(t) \ddot{v}_\alpha(t) + h \ddot{w}^2(t) + \chi \dot{\psi}^2(t) \right], \\
\mathcal{D}(t) &= \frac{k}{\dot{T}_0} \left[ I T_{,\alpha}(t) T_{,\alpha}(t) + h T^2(t) \right], \\
\delta(t) &= \frac{1}{2} a \dot{T}^2(t).
\end{align*}
$$

(2.19)

2.3. Auxiliary estimates

In this section we establish some estimates we will be used further.

For any vector $\mathbf{M}$ of the form

$$
\mathbf{M} = \left\{ \frac{1}{\sqrt{l}} \tilde{\xi}_{11}, \frac{1}{\sqrt{l}} \tilde{\xi}_{22}, \frac{1}{\sqrt{l}} \tilde{\xi}_{12}, \frac{1}{\sqrt{k}} \tilde{\xi}_{21}, \frac{1}{\sqrt{k}} \eta_1, \frac{1}{\sqrt{k}} \eta_2, \sqrt{\tilde{h}} \xi_1, \sqrt{k} \tilde{\xi}_2 \right\}
$$

(2.20)
Thus, we deduce
\[ |\mathbf{M}| = \left( \frac{1}{I} \xi_{\alpha\beta} \xi_{\alpha\beta} + h \zeta_{\alpha} \zeta_{\alpha} + \frac{1}{\chi I} \eta_{\beta} \eta_{\beta} \right)^{1/2}. \] (2.21)

**Lemma 1.** Let assume that the constitutive coefficients satisfy the inequalities
\[ \mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad (3\lambda + 2\mu) \xi > 3\lambda^2. \] (2.22)

Then, there are two positive constants \( \mu^{(0)}_M \) and \( M^{(0)} \) so that the magnitude of the vector \( \mathbf{N}^{(1)} = \left\{ \frac{1}{\sqrt{I}} H_1^{(1)}, \frac{1}{\sqrt{I}} H_2^{(1)}, \frac{1}{\sqrt{M}} N_{11}^{(1)}, \frac{1}{\sqrt{M}} N_{22}^{(1)}, \frac{1}{\sqrt{M}} N_{12}^{(1)}, \frac{1}{\sqrt{M}} N_{21}^{(1)} \right\} \) satisfies the inequality
\[ |\mathbf{N}^{(1)}|^2 \leq 2\mu^{(0)}_M (1 + \varepsilon^{(0)}) W^{(1)} + \left( 1 + \frac{1}{\varepsilon^{(0)}} \right) (M^{(0)})^2 l^2 \] (2.23)

for all \( \varepsilon^{(0)} > 0 \).

**Proof.** The inequalities (2.22) imply that the specific internal energy \( W^{(0)} \) is a positive definite quadratic form in terms of the variables \( (e_1, e_2, e_3, \varphi_1, \varphi_2, \varphi_3) \). In consequence, there is a positive constant \( \mu^{(0)}_M \) [12] so that
\[ W^{(0)} \leq \mu^{(0)}_M (e_i e_i + \varphi^2 + \chi \varphi \varphi_i). \] (2.24)

Chiriță and Scalia [12] established the following inequality
\[ t_{ij} t_{ij} \geq 2\mu^{(0)}_M (1 + \varepsilon^{(0)}) W^{(0)} + \left( 1 + \frac{1}{\varepsilon^{(0)}} \right) (M^{(0)})^2 \beta^2. \] (2.25)

where
\[ (M^{(0)})^2 = 3\beta^2 \]
and \( \varepsilon^{(0)} \) is an arbitrary positive number.

From (2.3), (2.12) and (2.9), it is easy to remark that there are the following relations between the quantities from the three-dimensional approach and from the two-dimensional approximation theory
\[ t_{\alpha\beta} = \frac{x_3}{I} N_{\alpha\beta}^{(1)}, \quad t_{\alpha 3} = N_{\alpha}, \quad t_{33} = \frac{x_3}{I} N_{33}, \]
\[ h_{\beta} = \frac{x_3}{I} H_{\beta}, \quad h_3 = \Gamma, \]
\[ g = \frac{x_3}{I} G, \]
\[ q_{\alpha} = \frac{x_3}{I} Q_{\alpha}, \quad q_3 = R, \]
\[ \rho_{\beta} \eta = \frac{x_3}{I} \sigma, \]
where
\[ N_{33} = I(\lambda \gamma_{\alpha} + b \psi - \beta T). \] (2.27)

We further substitute the relation (2.26) into (2.25) and then we integrate the result with respect to \( x_3 \) over \([-h/2, h/2]\). Thus, we deduce
\[ \frac{1}{I} h_{\alpha\beta}^{(1)} N_{\alpha\beta}^{(1)} + h N_\alpha N_\alpha + \frac{1}{I} N_{33} N_{33} + \frac{1}{\chi I} H_{\beta} H_{\beta} + h \Gamma^2 \leq 2\mu^{(0)}_M (1 + \varepsilon^{(0)}) W^{(1)} + \left( 1 + \frac{1}{\varepsilon^{(0)}} \right) (M^{(0)})^2 l^2 \] (2.28)
for all positive real numbers \( \varepsilon^{(0)} \), and the proof is complete. \( \square \)

The method used in the proof of the above lemma was previously used by Ciarletta [13] and by D’Apice and Chiriță [17]. The following lemma gives a similar estimate with (2.23) but in some milder conditions upon the constitutive coefficients. The below lemma covers a class of materials which is larger than the class \( C^+ \) and it is included in the class \( C \) of strongly elliptic materials.
Lemma 2. Let assume that the constitutive coefficients satisfy the inequalities
\[ \mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad (\lambda + \mu) \xi > b^2. \] (2.29)

Then, there are two positive constants \( \mu_M^{(1)} \) and \( M \) so that the magnitude of the vector \( N^{(1)} = \left\{ \frac{1}{\sqrt{\chi}}, \frac{1}{\sqrt{\mu}}, \frac{1}{\sqrt{\gamma}}, \frac{1}{\sqrt{\mu_1}}, \frac{1}{\sqrt{\mu_2}}, \sqrt{\gamma}N_1^{(1)}, \sqrt{\gamma}N_2^{(1)} \right\} \) satisfies the inequality
\[ |N^{(1)}|^2 \leq 2\mu_M^{(1)}(1 + \epsilon^{(1)})W^{(1)} + \left( 1 + \frac{1}{\epsilon^{(1)}} \right) M^2 T^2 \] (2.30)
for all \( \epsilon^{(1)} > 0 \).

Proof. Let us consider the following bilinear form
\[ F(\xi^{(1)}, \xi^{(2)}) = \lambda \xi_{\alpha\alpha}^{(1)} \xi_{\beta\beta}^{(2)} + 2\mu \xi_{\alpha\beta}^{(1)} \xi_{\alpha\beta}^{(2)} + \xi \phi^{(1)}(\phi^{(2)} + \alpha \phi^{(1)} \phi^{(2)}) \]
\[ + b(\phi^{(1)} \xi_{\beta\beta}^{(2)} + \phi^{(2)} \xi_{\beta\beta}^{(1)}) + \mu \theta^{(1)} \theta^{(2)} + \alpha \tau^{(1)} \tau^{(2)}, \] (2.31)
for all \( \xi^{(a)} = \left\{ \xi_{11}^{(a)}, \xi_{22}^{(a)}, \xi_{12}^{(a)}, \xi_{21}^{(a)}, \phi_1^{(a)}, \phi_2^{(a)}, \phi^{(a)}, \theta_1^{(a)}, \theta_2^{(a)}, \tau^{(a)} \right\} \in \mathbb{R}^{10} \).

We have that
\[ F(e, e) = 2W^{(1)}, \] (2.32)
for
\[ e = \left\{ \sqrt{I_{11}}, \sqrt{I_{22}}, \sqrt{I_{12}}, \sqrt{I_{21}}, \sqrt{I_X}, \sqrt{I_X}, \sqrt{Y}, \sqrt{Y_1}, \sqrt{Y_2}, \sqrt{P}, \sqrt{N}, \sqrt{H} \right\}. \] (2.33)

In view of the assumptions (2.29) we have that \( F(e, e) \) is positive definite and in consequence there is a positive constant \( \mu_M^{(1)} \) so that
\[ F(\xi, \xi) \leq \mu_M^{(1)}(\xi_{\alpha\beta} \xi_{\alpha\beta} + \phi^2 + \phi_\beta \phi_\beta + \theta_\alpha \theta_\alpha + \tau^2), \] (2.34)
for all \( \xi = \left\{ \xi_{11}, \xi_{22}, \xi_{12}, \phi_1, \phi_2, \phi, \theta_1, \theta_2, \tau \right\} \in \mathbb{R}^{10} \).

We define the following quantities
\[ T^{(1)}_{\alpha\beta} = N^{(1)}_{\alpha\beta} + I \beta T \delta_{\alpha\beta}, \]
\[ \Gamma' = \alpha \psi, \]
\[ P = G + \text{Im} T, \quad \text{in } \Sigma \times [0, T]. \]

Then, on the basis of the relation (2.12), we deduce that
\[ \frac{1}{I} T^{(1)}_{\alpha\beta} T^{(1)}_{\alpha\beta} + hN_a N_a + \frac{1}{\chi} H_\beta H_\beta + h \Gamma'^2 + \frac{1}{I} P^2 \]
\[ = T^{(1)}_{\alpha\beta} [\lambda e_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu e_{\alpha\beta} + b \psi \delta_{\alpha\beta} + h N_a \mu_\gamma + H_\beta \alpha \psi_\beta + h \Gamma' \alpha \psi + P (b e_{\gamma\gamma} + \xi \psi) \]
\[ = F(e, T), \] (2.36)
where
\[ e = \left\{ \sqrt{I_{11}}, \sqrt{I_{22}}, \sqrt{I_{12}}, \sqrt{I_{21}}, \sqrt{I_X}, \sqrt{I_X}, \sqrt{Y}, \sqrt{Y_1}, \sqrt{Y_2}, \sqrt{P}, \sqrt{N}, \sqrt{H} \right\}. \]
\[ T^{(1)} = \left\{ \frac{1}{\sqrt{I}}, \frac{1}{\sqrt{I}} T^{(1)}_{11}, \frac{1}{\sqrt{I}} T^{(1)}_{22}, \frac{1}{\sqrt{I}} T^{(1)}_{12}, \frac{1}{\sqrt{I}} H_1, \frac{1}{\sqrt{I}} H_2, \frac{1}{\sqrt{I}} P, \sqrt{N_1}, \sqrt{N_2}, \sqrt{H} \right\}. \]

By means of the Schwarz’s inequality and by using the relation (2.34), we get
\[ \frac{1}{I} T^{(1)}_{\alpha\beta} T^{(1)}_{\alpha\beta} + hN_a N_a + \frac{1}{\chi} H_\beta H_\beta + h \Gamma'^2 + \frac{1}{I} P^2 \leq \left[ F(e, e) \right]^2 \left[ F(T^{(1)}, T^{(1)}) \right]^2 \]
\[ \leq \left[ F(e, e) \right]^2 \left[ \mu_M^{(1)} \left( \frac{1}{I} T^{(1)}_{\alpha\beta} T^{(1)}_{\alpha\beta} + hN_a N_a + \frac{1}{\chi} H_\beta H_\beta + h \Gamma'^2 + \frac{1}{I} P^2 \right) \right]^2. \] (2.37)
Moreover, we remark that
\[ N_{\alpha\beta}^{(1)} N_{\alpha\beta}^{(1)} \leq (1 + \varepsilon^{(1)}) T_{\alpha\beta}^{(1)} T_{\alpha\beta}^{(1)} + \left( 1 + \frac{1}{\varepsilon^{(1)}} \right) M^2 I^2 T^2 \]
for all \( \varepsilon^{(1)} > 0 \), where
\[ M^2 = 2 \beta^2. \] (2.38)

Hence, we have
\[ \frac{1}{I} T_{\alpha\beta}^{(1)} N_{\alpha\beta}^{(1)} + h N_{\alpha} N_{\alpha} + \frac{1}{\chi I} H_{\beta} H_{\beta} \leq 2(1 + \varepsilon^{(1)}) \mu_M^{(1)} W^{(1)} + \left( 1 + \frac{1}{\varepsilon^{(1)}} \right) M^2 I^2 T^2 \] (2.39)
and the proof is complete. \( \square \)

In the following we obtain other estimates of the above type assuming some milder or complementary conditions upon the constitutive coefficients.

We write the basic equations (2.14)_1 in the following equivalent forms
\[ N_{\rho a, \beta}^{(i)} - h N_{\alpha} = \rho I \tilde{v}_{\alpha}, \quad i = 2, 3, \] (2.40)
where
\[ N_{\beta a}^{(2)} = [I [\mu v_{\alpha, \beta} + (\lambda + \mu) v_{\gamma, \gamma} \delta_{\alpha\beta} + b \psi \delta_{\alpha\beta} - \beta T \delta_{\alpha\beta}], \]
\[ N_{\beta a}^{(3)} = [I [\mu v_{\alpha, \beta} + (\lambda + \mu) v_{\beta, \alpha} + b \psi \delta_{\alpha\beta} - \beta T \delta_{\alpha\beta}]. \] (2.41) (2.42)

Let us define
\[ 2 W^{(i)}(t) = N_{\rho a, \beta}^{(i)}(t) v_{\alpha, \beta}(t) + h N_{\alpha}(t) \gamma_{\alpha}(t) + H_{\alpha}(t) \psi_{\alpha, \alpha}(t) - G(t) \psi(t) + h \Gamma(t) \psi(t)
+ I (\beta v_{\gamma, \gamma}(t) + m \psi(t)) T(t), \quad i = 2, 3. \] (2.43)

**Lemma 3.** (i) If the constitutive coefficients satisfy the inequalities
\[ \alpha > 0, \quad \mu > 0, \quad 2 \lambda + 3 \mu > 0, \quad \xi (2 \lambda + 3 \mu) > 2 b^2 \] (2.44)
then, there are two positive constants \( \mu_M^{(2)} \) and \( M \) so that the magnitude of the vector \( N^{(2)} = \left\{ \frac{1}{\sqrt{\lambda}} N_{11}^{(2)}, \frac{1}{\sqrt{\lambda}} N_{22}^{(2)}, \frac{1}{\sqrt{\lambda}} N_{12}^{(2)} \right\} \) satisfies the inequality
\[ |N^{(2)}|^2 \leq 2 \mu_M^{(2)} (1 + \varepsilon^{(2)}) W^{(2)} + \left( 1 + \frac{1}{\varepsilon^{(2)}} \right) M^2 I^2 T^2 \] (2.45)
for all \( \varepsilon^{(2)} > 0. \)

(ii) If the constitutive coefficients satisfy the inequalities
\[ \alpha > 0, \quad \mu > 0, \quad \lambda < 0, \quad \lambda + 2 \mu > 0, \quad \xi (\lambda + 2 \mu) > 2 b^2 \] (2.46)
then, there are two positive constants \( \mu_M^{(3)} \) and \( M \) so that the magnitude of the vector \( N^{(3)} = \left\{ \frac{1}{\sqrt{\lambda}} N_{11}^{(3)}, \frac{1}{\sqrt{\lambda}} N_{22}^{(3)}, \frac{1}{\sqrt{\lambda}} N_{12}^{(3)} \right\} \) satisfies the inequality
\[ |N^{(3)}|^2 \leq 2 \mu_M^{(3)} (1 + \varepsilon^{(3)}) W^{(3)} + \left( 1 + \frac{1}{\varepsilon^{(3)}} \right) M^2 I^2 T^2 \] (2.47)
for all \( \varepsilon^{(3)} > 0. \)
Proof. In order to proof the part (i) of the lemma, let us consider the bilinear forms

\[ F_1(\xi^{(1)}, \xi^{(2)}) = (\lambda + 2\mu)(\xi^{(1)}_1 \xi^{(2)}_1 + \xi^{(1)}_2 \xi^{(2)}_2) + (\lambda + \mu)(\xi^{(1)}_1 \xi^{(2)}_2 + \xi^{(1)}_2 \xi^{(2)}_1) + \xi \phi^{(1)} \phi^{(2)} + b[\tau^{(2)}(\xi^{(1)}_1 + \xi^{(2)}_2) + \tau^{(1)}(\xi^{(1)}_2 + \xi^{(2)}_1)] + \alpha \phi^{(1)} \phi^{(2)} + \mu \theta^{(1)}_\alpha \theta^{(2)}_\beta + \alpha \tau^{(1)} \tau^{(2)}, \]

\[ F_2(\xi^{(1)}, \xi^{(2)}) = \mu(\xi^{(1)}_2 \xi^{(2)}_1 + \xi^{(1)}_1 \xi^{(2)}_2), \]

in the variables \( \xi^{(a)} = \left\{ \xi^{(a)}_{11}, \xi^{(a)}_{12}, \xi^{(a)}_{21}, \xi^{(a)}_{22}, \phi^{(a)}, \phi^{(a)}_\alpha, \phi^{(a)}_\beta, \theta^{(a)}_\alpha, \theta^{(a)}_\beta, \tau^{(a)} \right\} \in \mathbb{R}^{10}. \)

In view of the assumptions \((2.44)\), we can say that the quadratic forms \( F_1(\xi, \xi) \) and \( F_2(\xi, \xi) \) are positive definite, where \( \xi = \{\xi^{(1)}, \xi^{(2)}, \phi^{(1)}, \phi^{(2)}, \theta^{(1)}, \theta^{(2)} \} \in \mathbb{R}^{10}. \)

Clearly, we can find a positive constant \( \mu^{(2)} \) such that

\[ F_1(\xi, \xi) + F_2(\xi, \xi) \leq \mu^{(2)}(\xi \phi^{(1)} \phi^{(2)} + \xi \phi^{(1)} + \theta^{(1)} \theta^{(2)}). \]

Moreover, we have

\[ 2W^{(2)} = F_1(\mathcal{V}, \mathcal{V}) + F_2(\mathcal{V}, \mathcal{V}), \]

where

\[ \mathcal{V} = \left\{ \sqrt{I} v_{11}, \sqrt{I} v_{12}, \sqrt{I} v_{21}, \sqrt{I} v_{22}, \sqrt{I} \psi_{11}, \sqrt{I} \psi_{12}, \sqrt{I} \psi_{21}, \sqrt{I} \psi_{22}, \sqrt{H} y_{11}, \sqrt{H} y_{12}, \sqrt{H} y_{21}, \sqrt{H} y_{22} \right\}. \]

We define the following quantities

\[ T^{(2)}_{\alpha \beta} = N^{(2)}_{\alpha \beta} + \beta T \delta_{\alpha \beta}, \quad \text{in } \Sigma \times [0, T]. \]

Then, on the basis of the relation \((2.35)_{2,3}, (2.52)\), by means of the Schwarz's inequality and by using the relations \((2.12)\) and \((2.41)\), we get

\[ \frac{1}{I} T^{(2)}_{\alpha \beta} T^{(2)}_{\alpha \beta} + h N_{\alpha \beta} N_{\alpha \beta} + \frac{1}{\chi L} H_{\beta} H_{\beta} + h l^2 + \frac{1}{I} P^2 \]

\[ \leq [F_1(\mathcal{V}, \mathcal{V}) + F_2(\mathcal{V}, \mathcal{V})]^2 \left[ F_1(T^{(2)}, T^{(2)}) + F_2(T^{(2)}, T^{(2)}) \right]^2 \]

\[ \leq [F_1(\mathcal{V}, \mathcal{V}) + F_2(\mathcal{V}, \mathcal{V})]^2 \left[ \mu^{(2)} \left( \frac{1}{I} T^{(2)}_{\alpha \beta} T^{(2)}_{\alpha \beta} + h N_{\alpha \beta} N_{\alpha \beta} + \frac{1}{I} H_{\alpha} H_{\alpha} + h l^2 + \frac{1}{I} P^2 \right) \right]^2, \]

where

\[ T^{(2)} = \left\{ \frac{1}{\sqrt{I}} T^{(2)}_{11}, \frac{1}{\sqrt{I}} T^{(2)}_{12}, \frac{1}{\sqrt{I}} T^{(2)}_{21}, \frac{1}{\sqrt{I}} T^{(2)}_{22}, \frac{1}{\sqrt{I}} H_{11}, \frac{1}{\sqrt{I}} H_{12}, \frac{1}{\sqrt{I}} H_{21}, \frac{1}{\sqrt{I}} H_{22}, \frac{1}{\sqrt{I}} P, \sqrt{H} N_{11}, \sqrt{H} N_{12}, \sqrt{H} N_{21}, \sqrt{H} N_{22} \right\}. \]

So, in view of \((2.50)\), we deduce

\[ \frac{1}{I} T^{(2)}_{\alpha \beta} T^{(2)}_{\alpha \beta} + h N_{\alpha \beta} N_{\alpha \beta} + \frac{1}{I} H_{\alpha} H_{\alpha} + h l^2 + \frac{1}{I} P^2 \leq 2 \mu^{(2)} W^{(2)}. \]

We also have

\[ N^{(2)}_{\alpha \beta} N^{(2)}_{\alpha \beta} \leq (1 + \epsilon^{(2)}) T^{(2)}_{\alpha \beta} T^{(2)}_{\alpha \beta} + \left( 1 + \frac{1}{\epsilon^{(2)}} \right) M^2 l^2 T^2. \]

Hence, we have

\[ \frac{1}{I} N^{(2)}_{\alpha \beta} N^{(2)}_{\alpha \beta} + h N_{\alpha \beta} N_{\alpha \beta} + \frac{1}{I} H_{\alpha} H_{\alpha} \leq 2(1 + \epsilon^{(2)}) M^{(2)} W^{(2)} + \left( 1 + \frac{1}{\epsilon^{(2)}} \right) M^2 l^2 T^2 \]

and the proof of the first part of the lemma is complete.

The proof of the second part of the lemma is quite similar with the proof of the first part, and uses some estimates in terms of the following bilinear forms

\[ Q_1(\xi^{(1)}, \xi^{(2)}) = (\lambda + 2\mu)(\xi^{(1)}_1 \xi^{(2)}_1 + \xi^{(1)}_2 \xi^{(2)}_2) + \xi \phi^{(1)} \phi^{(2)} + b(\phi^{(1)} \xi^{(2)}_\beta + \phi^{(2)} \xi^{(1)}_\beta) + \alpha \phi^{(1)} \phi^{(2)} + \mu \theta^{(1)}_\alpha \theta^{(2)}_\beta + \alpha \tau^{(1)} \tau^{(2)}, \]

\[ Q_2(\xi^{(1)}, \xi^{(2)}) = \mu(\xi^{(1)}_2 \xi^{(2)}_1 + \xi^{(1)}_1 \xi^{(2)}_2) + (\lambda + \mu)(\xi^{(1)}_1 \xi^{(2)}_2 + \xi^{(1)}_2 \xi^{(2)}_1), \]

for all \( \xi^{(a)} = \left\{ \xi^{(a)}_{11}, \xi^{(a)}_{12}, \xi^{(a)}_{21}, \xi^{(a)}_{22}, \phi^{(a)}, \phi^{(a)}_\alpha, \phi^{(a)}_\beta, \theta^{(a)}_\alpha, \theta^{(a)}_\beta, \tau^{(a)} \right\} \in \mathbb{R}^{10}. \)

This is why we skip the proof of the point (ii). □
2.4. Formulation of the problem

In order to study the spatial behaviour of the solutions, we first have to introduce the support of the given data. For fixed \( t^* > 0 \), we consider the support \( D_{t^*} \) of the initial and boundary data and of the body supplies on the time interval \([0, t^*]\) and further, we assume that it is a regular bounded set.

Following the method developed by Chiriță and Ciarletta [9], \( D_{t^*} \) is the set of all \( x \in \Sigma \) such that

(i) if \( x \in \Sigma \), then
\[
\begin{align*}
& v_0^0(x) \neq 0 \quad \text{or} \quad v_1^0(x) \neq 0 \quad \text{or} \quad w_0^0(x) \neq 0 \quad \text{or} \quad w_1^0(x) \neq 0 \quad \text{or} \quad \psi_0^0(x) \neq 0 \quad \text{or} \quad \psi_1^0(x) \neq 0 \\
\quad \text{or} \quad F_i(x, t) \neq 0 \quad \text{or} \quad \lambda_i(x, t) \neq 0 \quad \text{or} \quad S(x, t) \neq 0 \quad \text{for some} \ t \in [0, t^*];
\end{align*}
\]

(ii) if \( x \in \partial \Sigma \), we have
\[
\begin{align*}
\tilde{v}_\alpha(x, t) \neq 0 \quad \text{or} \quad \tilde{w}(x, t) \neq 0 \quad \text{or} \quad \tilde{\psi}(x, t) \neq 0 \quad \text{or} \quad \tilde{T}(x, t) \neq 0 \quad \text{for some} \ t \in [0, t^*].
\end{align*}
\]

We introduce the set \( D_r, r \geq 0 \), by
\[
D_r = \{ x \in \Sigma, D_{t^*} \cap \bar{S}(x, r) \neq 0 \},
\]
where \( S(x, r) \) is the open ball with radius \( r \) and centre \( x \).

We also introduce the sets:
\[
\Sigma_r = \Sigma \setminus D_r, \quad \Sigma(r_1, r_2) = \Sigma_{r_2} \setminus \Sigma_{r_1}, \quad r_1 \geq r_2
\]
and \( S_r \) denotes the subcurve \( \partial \Sigma_r \) contained inside \( \Sigma \) and whose outward unit normal vector is directed to the exterior of \( D_r \).

Our first aim is to study the behaviour of the solution of the problem \( \mathcal{P} \) with respect to the distance at the support \( D_{t^*} \) of the initial and boundary data and of the body supplies. The second aim is to obtain information about the spatial behaviour for a large class of materials. In fact, our results cover the class \( \mathcal{C}_{\text{max}} \) of materials defined by

\[
\mathcal{C}_{\text{max}}: \alpha > 0, \mu > 0, \xi > 0, \xi(\lambda + \mu) > b^2.
\]

This is the reason why we have considered the following classes of materials with voids:

\[
\begin{align*}
\mathcal{C}_1: & \quad \alpha > 0, \mu > 0, \xi > 0, \xi(\lambda + \mu) > b^2, \\
\mathcal{C}_2: & \quad \alpha > 0, \mu > 0, \xi > 0, \xi(2\lambda + 3\mu) > 2b^2, \\
\mathcal{C}_3: & \quad \alpha > 0, \mu > 0, \lambda < 0, \xi > 0, \xi(\lambda + 2\mu) > 2b^2.
\end{align*}
\]

**Remark 1.** The above classes of materials have the following properties:

(i) \( \mathcal{C}_{\text{max}} = \bigcup_{k=2}^{3} \mathcal{C}_k \);

(ii) \( \mathcal{C}^+ \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_{\text{max}} \subset \mathcal{C} \).

**Proof.** It is easy to see that
\[
\mathcal{C}_{\text{max}} = \mathcal{C}_3 \cup \mathcal{H}_1
\]
where \( \mathcal{H}_1 \) contains those materials from \( \mathcal{C}_2 \) for which \( \lambda \geq 0 \).

Hence, we have
\[
\mathcal{C}_{\text{max}} \subseteq \bigcup_{k=2}^{3} \mathcal{C}_k.
\]

On the other hand, it is clear that if a material belongs in the class \( \mathcal{C}_2 \) or in the class \( \mathcal{C}_3 \), then it belongs to the class \( \mathcal{C}_{\text{max}} \).

The proof of the second part of the remark is obvious. \( \square \)
3. Spatial behaviour for bounded plates

In the case of a bounded plates, \( r \) ranges on \([0, \ell]\), \( \ell < \infty \), where

\[
\ell = \max\{\min\{(x_a - y_a)(x_a - y_a)\}^{1/2} : y \in D_r \cup x \in \Sigma\}. \tag{3.1}
\]

Corresponding to the solution of the initial–boundary value problem \( \mathcal{P} \), for one class \( \mathcal{C}_\kappa \) \( (\kappa = 1, 2, 3) \), we introduce the following time-weighted surface power function

\[
J^{(\kappa)}(r, t) = -\int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \frac{N^{(\kappa)}_{\ell \alpha}}{\rho} \dot{y}_a + hN_\beta \dot{w} + H_\beta \dot{\psi} + \frac{1}{T_0} Q_\beta T \right] n_\beta d\sigma d\tau, \quad r \geq 0, \; t \in [0, t^*], \; \kappa = 1, 2, 3 \tag{3.2}
\]

where \( \gamma \) is a positive parameter at our disposal. Further, we introduce the quantities

\[
\tilde{J}^{(\kappa)}(r, t) = \int_0^t J^{(\kappa)}(r, \tau) d\tau, \quad \kappa = 1, 2, 3. \tag{3.3}
\]

We obtain the following result.

**Theorem 1.** Suppose that \( \Sigma \) is a bounded regular region. Then, for the class \( \mathcal{C}_\kappa \), \( \kappa = 1, 2, 3 \) the corresponding quantities \( J^{(\kappa)}(r, t) \) and \( \tilde{J}^{(\kappa)}(r, t) \) are two acceptable measures and for each fixed \( t \in [0, t^*] \) we have the following estimates

\[
J^{(\kappa)}(r, t) \leq J^{(\kappa)}(0, t) e^{-\frac{r}{c^{(\kappa)}(t)}}, \quad 0 \leq r \leq \ell \tag{3.4}
\]

where the constant \( c^{(\kappa)} \) depends on the constitutive coefficients, and

\[
\tilde{J}^{(\kappa)}(r, t) \leq \tilde{J}^{(\kappa)}(0, t) e^{-\frac{1}{c^{(\kappa)(t)}}}, \quad 0 \leq r \leq \ell \tag{3.5}
\]

where \( \varepsilon^{(\kappa)}(t) \) depends on the constitutive coefficients and on time \( t \).

**Proof.** By taking into account the definition of \( S_r, \Sigma_r \) and \( \Sigma(t_1, t_2) \), the divergence theorem and relations (2.12), (2.12) and (2.40)–(2.42) we have

\[
J^{(\kappa)}(r_1, t) - J^{(\kappa)}(r_2, t) = -\int_0^t \int_{\Sigma(t_1, t_2)} e^{-\gamma \tau} \left[ \frac{N^{(\kappa)}_{\ell \alpha}}{\rho} \dot{y}_a + hN_\beta \dot{w} + H_\beta \dot{\psi} + \frac{1}{T_0} Q_\beta T \right] n_\beta d\sigma d\tau
\]

\[
= -\int_{\Sigma(t_1, t_2)} e^{-\gamma \tau} \left[ \mathcal{K}(t) + \delta(t) + \mathcal{W}^{(\kappa)}(t) \right] d\sigma
\]

\[
- \int_0^t \int_{\Sigma(t_1, t_2)} e^{-\gamma \tau} \left[ \gamma (\mathcal{K}(\tau) + \delta(\tau) + \mathcal{W}^{(\kappa)}(\tau)) + \mathcal{D}(\tau) \right] d\sigma d\tau, \quad r_1 \geq r_2 \geq 0, \; t \in [0, t^*]. \tag{3.6}
\]

In consequence

\[
\frac{d}{dt} J^{(\kappa)}(r, t) = -\int_{S_r} e^{-\gamma \tau} \left[ \mathcal{K}(t) + \delta(t) + \mathcal{W}^{(\kappa)}(t) \right] d\sigma
\]

\[
- \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \gamma (\mathcal{K}(\tau) + \delta(\tau) + \mathcal{W}^{(\kappa)}(\tau)) + \mathcal{D}(\tau) \right] d\sigma d\tau, \quad r \geq 0, \; t \in [0, t^*]. \tag{3.7}
\]

In view of the Schwarz’s inequality and of the arithmetic–geometric mean inequality, we deduce

\[
|J^{(\kappa)}(r, t)| \leq \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \frac{1}{\varepsilon^{(\kappa)}_1} \mathcal{K}(t) + \frac{\varepsilon^{(\kappa)}_2}{2\rho} |\mathcal{N}^{(\kappa)}|^2 + \frac{1}{\varepsilon^{(\kappa)}_2 T_0} \delta(t) + \frac{\varepsilon^{(\kappa)}_2}{2\rho T_0} Q_\beta T_\beta \right] d\sigma d\tau. \tag{3.8}
\]

For each classes \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \mathcal{C}_3 \) we can use the Lemmas 2 and 3, respectively. Thus, using (2.12) we will have

\[
|J^{(\kappa)}(r, t)| \leq \int_0^t \int_{S_r} e^{-\gamma \tau} \left[ \frac{1}{\varepsilon^{(\kappa)}_1} \mathcal{K}(t) + \frac{\varepsilon^{(\kappa)}_2}{2\rho T_0} \delta(t) + \frac{\varepsilon^{(\kappa)}_2}{2\rho T_0} k^2 T_\beta T_\beta \right.
\]

\[
+ \frac{\varepsilon^{(\kappa)}_1}{2\rho} \left( (1 + \varepsilon^{(\kappa)}) 2\mu^{(\kappa)}_M \mathcal{W}^{(\kappa)} + \left( 1 + \frac{1}{\varepsilon^{(\kappa)}} \right) M^2 T^2 \right) \left] d\sigma d\tau. \tag{3.9}
\]
As in [12], we choose $\varepsilon^{(k)}_\alpha$, $\alpha = 1, 2$ to be

$$\varepsilon^{(k)}_1 = \frac{1}{c^{(k)}}, \quad \varepsilon^{(k)}_2 = \frac{2ac^{(k)}}{\gamma k}, \quad \varepsilon^{(k)} = \varepsilon^{(k)}_0$$

where

$$c^{(k)} = \sqrt{\frac{(1 + \varepsilon^{(k)}_0)\mu^{(k)}_M}{\rho}}$$

and $\varepsilon^{(k)}_0$ is the positive root of the algebraic equation

$$(\varepsilon^{(k)}_0)^2 + \varepsilon^{(k)}_0 \left(1 - \frac{M^2}{a\mu^{(k)}_M} - \frac{\gamma \rho k}{2a\mu^{(k)}_M} \right) - \frac{M^2}{a\mu^{(k)}_M} = 0.$$  

(3.12)

Thus, we obtain that $f^{(k)}(r, t)$ satisfies the following first-order differential inequality

$$\frac{\gamma}{c^{(k)}}|r^{(k)}(r, t)| + \frac{\partial f^{(k)}}{\partial r}(r, t) \leq 0, \quad r \geq 0, \quad t \in [0, t^*].$$

(3.13)

Because the body is bounded, $r \in [0, \ell]$, $\ell < \infty$ and in view of the definition of $D_{t^*}$ we have

$$f^{(k)}(\ell, t) = 0, \quad t \in [0, t^*].$$

(3.14)

But the function $f^{(k)}(r, t)$ is a decreasing function with respect to $r$ and thus, by a direct integration of relation (3.13), we obtain the estimate (3.4).

In order to establish the estimate (3.5), let us take into account that for all $f \in C([0, t^*])$ the following inequality holds

$$\int_0^t \int_0^s f^2 \, d\tau \, ds \leq t \int_0^t f^2 \, d\tau, \quad t \in [0, t^*].$$

(3.15)

Hence, we can write

$$\int_0^t \int_0^s |\tilde{r}^{(k)}(r, t)| \leq \sqrt{t} \int_0^t \int_{S_\ell} e^{-\gamma \tau} \left[\frac{1}{\varepsilon^{(k)}_1} \sqrt{I} K(\tau) + \sqrt{\frac{1}{\varepsilon^{(k)}_2}} |N^{(k)}|^2 + \frac{1}{\varepsilon^{(k)}_2} \frac{1}{I_0} \frac{1}{\mu^{(k)}_M} Q_0 \right] d\sigma d\tau.$$  

(3.16)

Moreover, using the Lemmas 2 and 3, respectively, we have

$$\int_0^t \int_0^s |\tilde{r}^{(k)}(r, t)| \leq \sqrt{t} \int_0^t \int_{S_\ell} e^{-\gamma \tau} \left[\frac{1}{\varepsilon^{(k)}_1} \sqrt{I} K(\tau) + \frac{1}{\varepsilon^{(k)}_2} \frac{1}{I_0} \frac{1}{\mu^{(k)}_M} Q_0 \right] d\sigma d\tau + \frac{1}{\varepsilon^{(k)}_2} \frac{1}{I_0} \frac{1}{\mu^{(k)}_M} Q_0 \int_0^t \int_{S_\ell} e^{-\gamma \tau} \left[1 + \varepsilon^{(k)}_2 \right] d\sigma d\tau.$$  

(3.17)

Let choose $\delta^{(k)}(t)$ to be the positive root of the algebraic equation

$$(\delta^{(k)}(t))^2 + \left(1 - \frac{k \rho}{2a I_0 \mu^{(k)}_M} - \frac{M^2}{\mu^{(k)}_M a} \right) \delta^{(k)} - \frac{M^2}{\mu^{(k)}_M a} = 0.$$  

(3.18)

and

$$\varepsilon^{(k)}_1 = \frac{1}{\varepsilon^{(k)}_1(T_0)}, \quad \varepsilon^{(k)}_2 = \frac{2a}{k} \varepsilon^{(k)}_1(T_0) \sqrt{t}, \quad \varepsilon^{(k)} = \varepsilon^{(k)}(t)$$

(3.19)

where

$$\varepsilon^{(k)}(t) = \sqrt{\frac{\mu^{(k)}_M (1 + \delta^{(k)}(t))}{\rho}}.$$  

(3.20)

In view of (3.7) and (3.17) we deduce the first-order differential inequality

$$t \varepsilon^{(k)}(t) \frac{\partial}{\partial \tau} \tilde{r}^{(k)}(r, t) + \tilde{r}^{(k)}(r, t) \leq 0, \quad r \geq 0, \quad t \in [0, t^*].$$

(3.21)

The integration of the differential inequality (3.21) leads to the estimate (3.5) and the proof is complete. \( \square \)
4. Spatial behaviour for unbounded plates

Let us consider an unbounded body, that is we assume that \( \Sigma \) is an unbounded regular region. We derive a result of Phragmén–Lindelöf type. Then, for the plate whose middle surface has a specific shape, we prove that some measures associated with the thermodynamic process decay faster than an exponential of a polynomial of second degree.

4.1. Phragmén–Lindelöf alternatives

**Theorem 2.** Let be a fixed time \( t \in [0, t^*] \) and let suppose that \( \Sigma \) is an unbounded regular region. Then, for the class \( E_\kappa \), \( \kappa = 1, 2, 3 \) the corresponding quantities \( f^{(\kappa)}(r, t) \) satisfy the following alternative

(i) If \( f^{(\kappa)}(r, t) \geq 0 \) for all \( r \geq 0 \), then \( f^{(\kappa)}(r, t) \) is an acceptable measures and we have the estimate

\[
J^{(\kappa)}(r, t) \leq J^{(\kappa)}(0, t)e^{-\frac{r}{c^{(\kappa)}}}, \quad r \geq 0; \tag{4.1}
\]

(ii) If there is a value \( r_t \geq 0 \) so that \( J^{(\kappa)}(r_t, t) < 0 \), then we obtain

\[
-J^{(\kappa)}(r, t) \geq -J^{(\kappa)}(r_t, t)e^{-\frac{r}{c^{(\kappa)}}(t-t)}, \quad r \geq r_t. \tag{4.2}
\]

**Proof.** For the proof of this theorem we follow the method presented in [9] and the method used in the proof of the previous theorem.

In the case of unbounded bodies, we cannot conclude that the functions \( f^{(\kappa)}(r, t) \) \( (\kappa = 1, 2, 3) \) are positive functions. However, from (3.7) we have only the following two possibilities:

(i) \( f^{(\kappa)}(r, t) \geq 0 \) for all \( r \in [0, \infty) \), or

(ii) for the fixed time \( t \) there exists \( r_t \in [0, \infty) \) such that \( f^{(\kappa)}(r_t, t) < 0 \).

In the first case, because the inequality (3.13) holds true for the unbounded bodies, we obtain easily the estimate (4.1). In the second case, we have

\[
J^{(\kappa)}(r, t) \leq J^{(\kappa)}(r_t, t) < 0, \quad \text{for all } r \geq r_t \tag{4.3}
\]

so that the inequality (3.13) implies that

\[
-\frac{\gamma}{c^{(\kappa)}} J^{(\kappa)}(r, t) + \frac{\partial J^{(\kappa)}}{\partial r}(r, t) \leq 0, \quad r \geq r_t \tag{4.4}
\]

and by integration, we obtain the estimate (4.2). \( \square \)

4.2. Proof of a faster decay

In this section we consider an unbounded plate and we present an estimate in the sense described by Horgan, Payne and Wheeler [21] and by Quintanilla [36,37].

To this end, we assume that the support \( D^t \) of the initial and boundary data, defined in the previous section, is enclosed in the half-space \( x_2 < 0 \). We introduce the notation \( S_z \) for the open cross-section of \( \Sigma \) for which \( x_2 = z \), \( z \geq 0 \) and whose unit normal vector is \((0, 1, 0)\). We assume that the unbounded set \( \Sigma \) is so that \( S_z \) is bounded for all finite \( z \in [0, \infty) \). We denote by \( \Sigma_z \) that portion of \( \Sigma \) for which \( x_2 > z \).

Inspired by the works [21,36,37], we introduce the following functions, for the classes \( E^{(\kappa)} \), \( \kappa = 1, 2, 3 \), respectively,

\[
H^{(\kappa)}(z, t) = -\int_0^t \int_{\Sigma_z} \left[ N^{(\kappa)}(\tau) \hat{v}_u(\tau) + hN_2(\tau) \hat{w}(\tau) + H_2(\tau) \hat{\psi}(\tau) + \frac{1}{T_0} Q_2(\tau) T(\tau) \right] d\sigma d\tau. \tag{4.5}
\]

We assume further the following asymptotic behaviour (see [36,37])

\[
H^{(\kappa)}(z, t) \rightarrow 0 \tag{4.6}
\]

uniformly in \( t \), as \( z \rightarrow \infty \).

Using Eqs. (2.11), (2.12), (2.40), (2.41), (2.43) the divergence theorem and having in mind the definition of \( S_z \) and \( \Sigma_z \), we deduce that the function \( H^{(\kappa)}(z, t) \) can be written in the form

\[
H^{(\kappa)}(z, t) = \int_{\Sigma_z} \left[ J(t) + \delta(t) + W^{(\kappa)}(t) \right] d\sigma + \int_0^t \int_{\Sigma_z} D(\tau) d\sigma d\tau. \tag{4.7}
\]

Further, we introduce the functions

\[
E^{(\kappa)}(z, t) = \int_{\Sigma_z} H^{(\kappa)}(p, t) dp. \tag{4.8}
\]
**Theorem 3.** For the class \( C_a \), the corresponding function \( E^{(\kappa)} \) decays exponentially in terms of the square of the distance \( z \) from the support of the external given data when \( z > t \sqrt{\frac{\mu_{\kappa} a + M^2}{\rho a}} \).

**Proof.** It is easy to see that
\[
\frac{\partial E^{(\kappa)}}{\partial z} = -H^{(\kappa)}(z, t) \tag{4.9}
\]
and
\[
\frac{\partial^2 E^{(\kappa)}}{\partial z^2} = \int_{S_z} \left[ K(t) + \delta(t) + W^{(\kappa)}(t) \right] d\sigma + \int_{0}^{t} \int_{S_z} D(\tau) d\sigma d\tau. \tag{4.10}
\]

On the other hand, from the relations (2.12), (2.41), (2.42) and the relations (4.5) and (4.8), we have
\[
\frac{\partial E^{(\kappa)}}{\partial t} = -\int_{S_z} \left[ N^{(\kappa)}(t) \dot{v}_w(t) + hN_2(t) \dot{w}(t) + H_2(t) \dot{\psi}(t) + \frac{1}{T_0} Q_2(t) \bar{T}(t) \right] da. \tag{4.11}
\]

The next step is to estimate the time derivative of \( E^{(\kappa)} \) in terms of the two first spatial derivatives of \( E^{(\kappa)} \). We remark that
\[
\int_{S_z} Q_2(t) \bar{T}(t) da = -\int_{\mathbb{R}^2} \int_{S_z} kI_{T_2}(0) T(t) d\sigma dz = \frac{1}{2} \int_{S_z} kI T^2(t) d\sigma. \tag{4.12}
\]
Moreover, using the above inequality and the Lemmas 2 and 3, we have
\[
\int_{S_z} \left[ N^{(\kappa)}(t) \dot{v}_w(t) + hN_2(t) \dot{w}(t) + H_2(t) \dot{\psi}(t) + \frac{1}{T_0} Q_2(t) \bar{T}(t) \right] da \\
\leq A_1^{(\kappa)} \int_{S_z} \left[ K(t) + \delta(t) + W^{(\kappa)}(t) \right] da + A_2 \int_{S_z} \delta(t) d\sigma \tag{4.13}
\]
where
\[
A_1^{(\kappa)} = \sqrt{\frac{\mu_{\kappa} a + M^2}{\rho a}}, \quad A_2 = \frac{k}{a T_0}. \tag{4.14}
\]

In view of the relations (4.9)-(4.12) we can conclude that
\[
\frac{\partial E^{(\kappa)}}{\partial t} \leq -A_1^{(\kappa)} \frac{\partial E^{(\kappa)}}{\partial z} + A_2 \frac{\partial^2 E^{(\kappa)}}{\partial z^2}. \tag{4.15}
\]

Following [21], such inequality can be further treated by the Comparison Principle [42]. On this basis we can conclude that
\[
E^{(\kappa)}(z, t) \leq \left( \max_{\tau \in [0, t]} E^{(\kappa)}(0, \tau) \right) \exp \left( \frac{A_1^{(\kappa)} z}{2A_2} \right) G^{(\kappa)}(z, t) \tag{4.16}
\]
where
\[
G^{(\kappa)}(z, t) = \frac{1}{2 \sqrt{A_2^{(\kappa)} \pi}} \int_{0}^{t} \frac{1}{\tau^{3/2}} \exp \left( -\frac{z^2}{4A_2 \tau} - \frac{A_1^{(\kappa)} \tau}{4A_2} \right) d\tau. \tag{4.17}
\]

Using the estimate discussed by Pompei and Scalia [35], we have
\[
G^{(\kappa)}(z, t) \leq \frac{2z (A_2 t / \pi)^{1/2}}{z^2 - (A_1^{(\kappa)} t)^2} \exp \left( -\frac{z^2}{4A_2 t} \right) \tag{4.18}
\]
for \( z > A_1^{(\kappa)} t \) and \( t > 0 \).
Thus, we find the estimate

$$E^{(z)}(t, z) \leq \left( \max_{\tau \in [0,t]} E^{(z)}(0, \tau) \right) \frac{1}{z^2 - (A_1^{(z)})^2 t^2} 2z \left( \frac{A_2 t}{\pi} \right)^2 \exp \left( -\frac{(A_1^{(z)})^2}{4A_2 t} \right) \exp \left( \frac{A_1^{(z)} z}{2A_2} - \frac{z^2}{4A_2 t} \right) \tag{4.19}$$

for $z > A_1^{(z)} t$ and $t > 0$. □

**Remark 2.** The estimate (4.19) proves that, for a fixed time, at large distance to the support $D_\tau$ of the given data the dominant term is $\exp \left( -\frac{z^2}{4A_2 t} \right)$. Moreover, $A_2$ depends only on the thermal coefficients. We can conclude that at large distance from the support of the external given data, the spatial decay of processes is influenced only by the thermal effect.

**References**


