The relaxed linear micromorphic continuum: well-posedness of the static problem and relations to the gauge theory of dislocations

Patrizio Neff∗ and Ionel-Dumitrel Ghiba† and Markus Lazar ‡ and Angela Madeo§

November 4, 2014

Abstract

In this paper we consider the equilibrium problem in the relaxed linear model of micromorphic elastic materials. The basic kinematical fields of this extended continuum model are the displacement $u \in \mathbb{R}^3$ and the non-symmetric micro-distortion density tensor $P \in \mathbb{R}^{3 \times 3}$. In this relaxed theory a symmetric force-stress tensor arises despite the presence of microstructure and the curvature contribution depends solely on the micro-dislocation tensor $\text{Curl} \, P$. However, the relaxed model is able to fully describe rotations of the microstructure and to predict non-polar size-effects. In contrast to classical linear micromorphic models, we allow the usual elasticity tensors to become positive-semidefinite. We prove that, nevertheless, the equilibrium problem has a unique weak solution in a suitable Hilbert space. The mathematical framework also settles the question of which boundary conditions to take for the micro-distortion. Similarities and differences between linear micromorphic elasticity and dislocation gauge theory are discussed and pointed out.

Key words: micromorphic elasticity, symmetric Cauchy stresses, static problem, dislocations, gradient plasticity, dislocation energy, generalized continua, microstructure, micro-elasticity, non-smooth solutions, well-posedness, Cosserat couple modulus, gauge theory of dislocations.

∗Patrizio Neff, Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: patrizio.neff@uni-due.de
†Ionel-Dumitrel Ghiba, Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany; Alexandru Ioan Cuza University of Iaşi, Department of Mathematics, Blvd. Carol I, no. 11, 700506 Iaşi, Romania; Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi Branch, 700505 Iaşi; and Institute of Solid Mechanics, Romanian Academy, 010141 Bucharest, Romania, email: dumitrel.ghiba@uni-due.de, dumitrel.ghiba@uaic.ro
‡Markus Lazar, Heisenberg Research Group, Department of Physics, Darmstadt University of Technology, Hochschulstr. 6, D-64289 Darmstadt, Germany, email: lazar@fp.tu-darmstadt.de
§Angela Madeo: Laboratoire de Génie Civil et Ingénierie Environnementale, Université de Lyon-INSIA, Bâtiment Coulomb, 69621 Villeurbanne Cedex, France; International Center M&MOCS “Mathematics and Mechanics of Complex Systems”, Palazzo Caetani, Cisterna di Latina, Italy, email: angela.madeo@insa-lyon.fr
1 Introduction

In this paper we consider the static variant of the relaxed micromorphic model introduced in [1]. This new model reconciles Kröner’s rejection [2] of antisymmetric force stresses in dislocation theory with the asymmetric dislocation model of Eringen and Claus [3] and shows that the concept of asymmetric force stress is not strictly needed in order to describe rotations of the microstructure in non-polar materials. In fact, a non symmetric local force stress tensor $\sigma$ deviates considerably from classical linear elasticity theory and indeed it does not necessarily appear in gradient elasticity [1]. After more than half a century of intensive research there is no conclusive experimental evidence for the necessity of non-symmetric force stresses, at least for what concerns a huge variety of natural and engineering micro-structured materials. However, some very special engineering meta-materials like phononic crystals and lattice structures may need the introduction of asymmetric force stress to fully describe their mechanical behavior. This fact was observed in [4], in which the presence of the Cosserat couple modulus $\mu_c > 0$ has been proved to be necessary for the physically correct description of the dynamical behavior of high-tech micro-structured materials which are known to show frequency band-gaps in the dynamic regime. The relaxed micromorphic model with positive Cosserat modulus proposed in [4] is the only generalized continuum model which is able to predict frequency band-gaps contrarily to what is possible in the Mindlin-Eringen model (see [5, 6, 7, 8]) or in so-called second-gradient models (see e.g. [9, 10, 11]). The described new meta-materials are able to stop wave propagation when excited with signals at frequencies falling in a precise range of values. Materials of this type are expected to have important technological applications for what concerns control of vibrations and would provide a valid alternative to currently used piezo-electric materials which are vastly studied in the literature (see e.g. [12, 13, 14, 15]). Therefore, it gets clear that the asymmetry of the force stress tensor in a continuum theory is not an immediate consequence of the presence of microstructure in the body, it is rather a constitutive assumption [16]. Thus, in the relaxed model we dispose of this assumption. Despite the simplification of giving rise to symmetric stress, the relaxed model preserves full kinematical freedom ($12$ degree of freedom). Moreover, the proposed relaxed model is still able to fully describe rotations of the microstructure and to fit a huge class of mechanical behaviours of materials with microstructure. Another strong point of the relaxed theory is that for the isotropic relaxed micromorphic model only $3$ curvature parameters remain to be determined, which may eventually be reduced to $2$ parameters, which are needed for fitting bending and torsion experiments (see e.g. [17, 18]). This could be a decisive step in the main problematic of determination of constitutive parameters in the micromorphic theory of elastic materials (and other more
general extended continuum models). The relaxed formulation of micromorphic elasticity has some similarities to recently studied models of gradient plasticity [19, 20, 21, 22, 23].

The mathematical analysis of general micromorphic solids is well-established for infinitesimal, linear elastic models, see, for example [24, 25, 26, 27]. The only known existence results for the static geometrically nonlinear formulation are due to Neff [28] and to Mariano and Modica [29]. Compared with [28], Mariano and Modica [29] assume much more stringent coercivity assumptions which restrict the material response. As for the numerical implementation, see e.g. [30] and the development in [31]. In [31] the original problem is decoupled into two separate problems and the corresponding domain-decomposition techniques for the subproblem related to balance of forces are investigated. On the other hand, in the classical linear theory of Mindlin-Eringen micromorphic elasticity, existence and uniqueness results were already established by Sós [24], by Hlaváček [25], by Iesan and Nappa [26] and by Iesan [27] assuming that the free energy is a pointwise positive definite quadratic form in terms of the usual set of independent constitutive variables [6, 7, 8]. Iesan [27] also gave a uniqueness result for the dynamic problem without assuming that the free energy is a positive definite quadratic form. Moreover, in order to study the existence of solution of the resulting system, Hlaváček [25], Iesan and Nappa [26] and Iesan [27] considered the strong anchoring boundary condition: the micro-distortion \( P \) is completely described at a part of the boundary. In contrast with the models considered until now, our free energy of the relaxed model is not uniformly pointwise positive definite in the control of the classical constitutive variables. We mention that the well-posedness of the dynamic problem of the relaxed model is established in [1], while a well-posed problem class of autonomous evolutionary equations from elasticity theory modeling solids with micro-structure are studied in [32].

As far as the mechanical behavior of micro-structured materials in the static regime is considered, no clear experimental evidence about the real need of introducing a non-symmetric force stress has been provided up to now. Nevertheless, some rare experiments exist on special micro-structured materials which are subjected to particular loading and boundary conditions which would show the need of Cosserat theory for the complete description of their mechanical behavior also in the static case. This is what is described e.g. in [33] in which experimental evidence of the need of Cosserat elasticity seems to be provided for what concerns small samples of compact bone under torsion. In [33] it is shown that as soon as the size of the specimen becomes so small that the influence of deformation of bone-microstructure (osteons) on macroscopic overall deformation cannot be neglected, then the use of Cosserat theory becomes necessary for the correct description of deformation of such materials. This need is related to the fact that the micro-deformation associated to relative rotations of osteons inside the considered specimens becomes comparable to the overall macroscopic deformation of the sample. It is clear that, in order to activate the deformation modes which need a Cosserat-type theory for their correct description, multiple conditions must be simultaneously satisfied (specimens of small sizes, applied external loads and boundary conditions which excite particular micro-rotations, etc.). In all other cases a symmetric force stress tensor would be sufficient to correctly describe the mechanical behavior of the same material. We can hence conclude that, except for what happens in some particular cases in which the relative micro/macro-rotations cannot be neglected, the use of a relaxed model is fully sufficient for the description of the mechanical behavior of materials with microstructure, both in the static and the dynamic case. We explicitly remark that the proposed relaxed model is thought for applications involving isotropic materials undergoing small deformations.

The main point of the present work is to prove that the static problem of the new micromorphic relaxed model [1] is well-posed, i.e. we study the existence and uniqueness of the solution in absence of inertial effects, and moreover we point out the relation with the dislocation gauge theory. All the results are obtained for a standard set of tangential boundary conditions for the micro-distortion \( P \), i.e. \( P \tau = 0 \) (\( P_i \times n = 0, i = 1, 2, 3 \)) on \( \partial \Omega \) and not the usual strong anchoring condition \( P = 0 \) on \( \partial \Omega \). The solution space for the elastic distortion and micro-distortion is only \( H_0(\text{Curl}; \Omega) \supseteq H^1_0(\Omega) \) (see [34]) and for the macroscopic displacement \( u \in H^1_0(\Omega) \). As mentioned, compared to other existence results established in the static case of the theory of micromorphic elastic materials, we allow the usual elasticity tensors to become positive-semidefinite. The main point in establishing the desired estimates is represented by the new coercive inequalities recently proved by Neff, Pauly and Witsch [35, 36, 37] and by Bauer, Neff, Pauly and Starke [38, 39].

The relaxed formulation of micromorphic elasticity has some similarities to the gauge theory of dislocations given by Lazar [40], Lazar and Anastassiadis [41, 42] and Agiasofitou and Lazar [43]. In fact, in [42, 44] a simplified static version of the isotropic Eringen-Claus model for dislocation dynamics [45] has been investigated with \( \mu = 0 \) and \( \mu_c \geq 0 \), with a focus on the gauge theory of dislocations. However, the dynamical theory of Lazar
cannot be derived from Mindlin’s dynamic theory, since in [41] there appears a (dynamical) gauge potential which has no counterpart in Mindlin’s model. In the dislocation gauge theoretical formulation dislocations arise naturally as a consequence of broken translational symmetry and therefore their existence is not required to be postulated a priori. In the last part of this paper we explain the similarities and the differences between the relaxed micromorphic elastic theory and the gauge theory of dislocations.

2 Formulation of the problem

2.1 Notation and main inequalities

For $a, b \in \mathbb{R}^3$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. We denote by $\mathbb{R}^{3 \times 3}$ the set of real $3 \times 3$ second order tensors which are denoted in the sequel by capital letters. The standard Euclidean scalar product on $\mathbb{R}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(XY^T)$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3 \times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{R}^{3 \times 3}$. The identity tensor on $\mathbb{R}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$. By $T(3)$ we denote the three-dimensional translation group. We let $\text{Sym}(3)$ denote the set of symmetric tensors. We adopt the usual abbreviations of Lie-algebra theory, i.e., $\mathfrak{so}(3) := \{ X \in \mathbb{R}^{3 \times 3} \mid X^T = -X \}$ is the Lie-algebra of skew symmetric tensors and $\mathfrak{sl}(3) := \{ X \in \mathbb{R}^{3 \times 3} \mid \text{tr}(X) = 0 \}$ is the Lie-algebra of traceless tensors. For all $X \in \mathbb{R}^{3 \times 3}$ we set $\text{sym} X = \frac{1}{2}(X + X^T) \in \text{Sym}(3)$, $\text{skew} X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3)$ and the deviatoric part $\text{dev} X = X - \frac{1}{3} \text{tr}(X) \mathbb{1} \in \mathfrak{sl}(3)$ and we have the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{g}(3)$

$$\mathfrak{g}(3) = \{ \mathfrak{sl}(3) \cap \text{Sym}(3) \} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \cdot \mathbb{1},$$

$$X = \text{dev} \text{sym} X + \text{skew} X + \frac{1}{3} \text{tr}(X) \mathbb{1}. \quad (2.1)$$

We consider a micromorphic continuum which occupies a bounded domain $\Omega$ and is bounded by the piecewise smooth surface $\partial \Omega$. The equilibrium of the body is referred to a fixed system of rectangular Cartesian axes $Ox_i, \ (i = 1, 2, 3)$. Throughout this paper (if we do not specify otherwise) Latin subscripts take the values 1, 2, 3. The micro-distortion (plastic distortion) $P = (P_{ij}) : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ is intended to describe the substructure of the material which can rotate, stretch, shear and shrink, while $u = (u_i) : \Omega \rightarrow \mathbb{R}^3$ is the displacement of the macroscopic material points. Typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate.

The quantities involved in our new relaxed micromorphic continuum model have the following physical meaning:

- $(u, P)$ are the kinematical variables,
- $\sigma$ is the force-stress tensor (the Cauchy stress tensor, second order, symmetric),
- $s$ is the microstress tensor (second order, symmetric),
- $m$ is the moment stress tensor (micro-hyperstress tensor, third order),
- $u$ is the displacement vector (translational degrees of freedom),
- $P$ is the micro-distortion tensor (“plastic distortion”, second order, non-symmetric),
- $f$ is the external body force,
- $M$ is the external body moment tensor (second order),
- $e := \nabla u - P$ is the elastic distortion (relative distortion or gauge potential, second order, non-symmetric),
- $\varepsilon_e := \text{sym} e = \text{sym}(\nabla u - P)$ is the elastic strain tensor (second order, symmetric),
- $\varepsilon_p := \text{sym} P$ is the micro-strain tensor (“plastic strain”, second order, symmetric),
- $\alpha := \text{Curl} e = -\text{Curl} P$ is the micro-dislocation density tensor (translational field strength, second order).
By \( C^\infty_0(\Omega) \) we denote the set of smooth functions with compact support in \( \Omega \). The usual Lebesgue spaces of square integrable functions, vector or tensor fields on \( \Omega \) with values in \( \mathbb{R} \), \( \mathbb{R}^3 \) or \( \mathbb{R}^{3 \times 3} \), respectively will be denoted by \( L^2(\Omega) \). Moreover, we introduce the standard Sobolev spaces \cite{46,47,34} of functions \( u \) or vector fields \( v \), respectively. Furthermore, we introduce their closed subspaces \( H^1_0(\Omega) \), and \( H_0(\text{curl}; \Omega) \) as the closure with respect to the associated graph norms of \( C^\infty_0(\Omega) \). Roughly speaking, \( H^1_0(\Omega) \) is the subspace of functions \( u \in H^1(\Omega) \) which are zero on \( \partial \Omega \), while \( H_0(\text{curl}; \Omega) \) is the subspace of vectors \( v \in \text{H(curl}; \Omega) \) which are normal at \( \partial \Omega \) (see \cite{35,36,37}). For vector fields \( v \) with components in \( H^1(\Omega) \) and tensor fields \( P \) with rows in \( \text{H(curl}; \Omega) \), i.e.,

\[
v = (v_1, v_2, v_3)^T, \quad v_i \in H^1(\Omega), \quad P = (P_1^T, P_2^T, P_3^T)^T \quad P_i \in \text{H(curl}; \Omega) \quad (2.2)
\]

we define

\[
\nabla v := ((\text{grad } v_1)^T, (\text{grad } v_2)^T, (\text{grad } v_3)^T)^T, \quad \text{Curl } P := ((\text{curl } P_1)^T, (\text{curl } P_2)^T, (\text{curl } P_3)^T)^T. \quad (2.3)
\]

We note that \( v \) is a vector field, whereas \( P, \text{Curl } P \) and \( \nabla v \) are second order tensor fields. The corresponding Sobolev spaces will be denoted by \( H^1(\Omega) \) and \( \text{H(curl}; \Omega) \). We recall that for a fourth order tensor \( C \) and \( X \in \mathbb{R}^{3 \times 3} \), we have \( C X \in \mathbb{R}^{3 \times 3} \), \( C^* X \in \mathbb{R}^{3 \times 3} \) with the components

\[
(C X)_{ij} = C_{ijkl} X_{kl}, \quad (C^* X)_{kl} = C_{ijkl} X_{ij}, \quad (2.4)
\]

while if \( L \) a sixth order tensor, then

\[
L Z \in \mathbb{R}^{3 \times 3 \times 3} \quad \text{for all } Z \in \mathbb{R}^{3 \times 3 \times 3}, \quad (L Z)_{ijk} = L_{ijklmnp} Z_{mnp}, \quad (2.5)
\]

where Einstein’s summation rule is used.

In \cite{35,36,37}, for tensor fields \( P \in \text{H(curl}; \Omega) \) the following seminorm \( \| \| \cdot \| \) is defined

\[
|||P|||^2 := \| \text{sym } P \|_{L^2(\Omega)}^2 + \| \text{Curl } P \|_{L^2(\Omega)}^2. \quad (2.6)
\]

From \cite{35,36,37} we have the following result:

**Theorem 2.1** There exists a constant \( \hat{c} \) such that

\[
\| P \|_{L^2(\Omega)} \leq \hat{c} \| | | P | |, \quad (2.7)
\]

for all \( P \in \text{H(curl}; \Omega) \) with vanishing restricted tangential trace on \( \partial \Omega \), i.e. \( P \tau = 0 \) on \( \partial \Omega \).

Moreover, we have

**Theorem 2.2** On \( H_0(\text{curl}; \Omega) \) the norms \( \| \cdot \|_{\text{H(curl}; \Omega)} \) and \( \| \| \cdot \| \) are equivalent. In particular, \( \| \| \cdot \| \) is a norm on \( H_0(\text{curl}; \Omega) \) and there exists a positive constant \( c \) such that

\[
c \| P \|_{\text{H(curl}; \Omega)} \leq \| | | P | |, \quad (2.8)
\]

for all \( P \in H_0(\text{curl}; \Omega) \).

Moreover, in a forthcoming paper \cite{38} the following results are proved:

**Theorem 2.3** There exists a positive constant \( C \), only depending on \( \Omega \), such that for all \( P \in H_0(\text{curl}; \Omega) \) the following estimate holds:

\[
\| P \|_{L^2(\Omega)} + \| \text{Curl } P \|_{L^2(\Omega)} \leq C (\| \text{sym } P \|_{L^2(\Omega)}^2 + \| \text{Curl } P \|_{L^2(\Omega)}^2).
\]

**Theorem 2.4** There exists a positive constant \( C \), only depending on \( \Omega \), such that for all \( u \in H_0^1(\Omega) \) the following estimate holds:

\[
\| \nabla u \|_{L^2(\Omega)} \leq C \| \text{sym } \nabla u \|_{L^2(\Omega)}.
\]

The estimates given by the above theorems will be essential in the study of our relaxed linear micromorphic elasticity model.
2.2 Formulation of the static problem

We consider a relaxed version of the classical micromorphic model with symmetric force stress $\sigma$. Let us first consider the general form of the elastic free energy for a micromorphic elastic material:

$$
2 \mathcal{E}(e, \varepsilon_p, \gamma) = \langle \tilde{C}, (\nabla u - P), (\nabla u - P) \rangle + \langle \tilde{H}, P, P \rangle + \langle \tilde{L}, \nabla P, \nabla P \rangle \\
+ 2 \langle \tilde{E}, P, (\nabla u - P) \rangle + 2 \langle \tilde{F}, \nabla P, (\nabla u - P) \rangle + 2 \langle \tilde{G}, \nabla P, P \rangle, \quad (2.11)
$$

where again $u$ is the displacement, $P$ is the micro-distortion and $\tilde{C}, \tilde{H}, \tilde{L}, \tilde{E}, \tilde{F}, \tilde{G}$ are constitutive coefficients.

The relaxed model is a subset of the classical micromorphic model in which we allow the usual elasticity tensors \[8\] to become positive-semidefinite only \[1\]. Moreover, the number of constitutive coefficients is drastically reduced with respect to the classical Mindlin-Eringen micromorphic elasticity model. To be more specific, let us recall that in the elastic free energy from the classical Mindlin-Eringen micromorphic elasticity model the constitutive coefficients are such that

$$
\tilde{C} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \quad \tilde{H} = \text{sym} \tilde{H} \text{sym} : \mathbb{R}^{3 \times 3} \rightarrow \text{Sym}(3), \quad \tilde{L} : \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}^{3 \times 3 \times 3}, \\
\tilde{E} = \tilde{E} \text{sym} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \quad \tilde{F} : \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \quad \tilde{G} = \text{sym} \tilde{G} : \mathbb{R}^{3 \times 3 \times 3} \rightarrow \text{Sym}(3), \quad (2.12)
$$

and the classical independent constitutive variables are

$$
e := \nabla u - P, \quad \varepsilon_p := \text{sym} P, \quad \gamma := \nabla P.
$$

In contrast, for the relaxed micromorphic model we consider a specific form of the constitutive coefficients, namely

$$
\tilde{C} = \text{sym} \tilde{C} \text{sym} : \mathbb{R}^{3 \times 3} \rightarrow \text{Sym}(3), \quad \tilde{H} = \text{sym} \tilde{H} \text{sym} : \mathbb{R}^{3 \times 3} \rightarrow \text{Sym}(3), \quad \tilde{E} = 0, \quad \tilde{F} = 0, \quad \tilde{G} = 0, \quad \tilde{C} : \text{Sym}(3) \rightarrow \text{Sym}(3), \quad \tilde{H} : \text{Sym}(3) \rightarrow \text{Sym}(3). \quad (2.13)
$$

Similarly, we consider a specific form of the tensor $\tilde{L}$ in terms of another tensor $L_c : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ such that

$$
(\tilde{L}, \nabla P, \nabla P)_{\mathbb{R}^{3 \times 3 \times 3}} := (L_c, \text{Curl} P, \text{Curl} P)_{\mathbb{R}^{3 \times 3}}. \quad (2.14)
$$

Let us show how to obtain the tensor $\tilde{L}$ if $L_c$ is given, such that the last identity holds true. Let us first remark that a tensor $\tilde{L} : \mathbb{R}^{3 \times 3 \times 3} \rightarrow \mathbb{R}^{3 \times 3 \times 3}$ is uniquely defined by the fourth order tensors

$$
\tilde{L}_{im} = (\tilde{L}_{i j k m n p})_{im}, \quad \tilde{L}_{im} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \quad (2.15)
$$

and

$$
(\tilde{L}, \nabla P, \nabla P)_{\mathbb{R}^{3 \times 3 \times 3}} := (\tilde{L}_{im} \cdot \nabla P, \nabla P)_{\mathbb{R}^{3 \times 3}}, \quad (2.16)
$$

where Einstein’s summation rule is used. Let $L_c : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ be a given fourth order tensor. We may write this tensor in the form

$$
L_c = (\hat{L}_c^i, \hat{L}_c^j, \hat{L}_c^k), \quad \hat{L}_c^i : \mathbb{R}^3 \rightarrow \mathbb{R}^3. \quad (2.17)
$$

Let us define the tensor $\tilde{L}$ by (2.15) where

$$
\tilde{L}_{im} = \begin{cases} 
2 \text{ skew anti } \hat{L}_c^i \text{ axl skew} & \text{for } i = m \\
0 & \text{for } i \neq m.
\end{cases} \quad (2.18)
$$

Here, for

$$
A = \begin{pmatrix} 
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0 \\
\end{pmatrix} \in \mathfrak{so}(3) \quad (2.19)
$$
we have defined the operators $axl : \mathfrak{so}(3) \to \mathbb{R}^3$ and $\text{anti} : \mathbb{R}^3 \to \mathfrak{so}(3)$ through

$$
axl \begin{pmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{pmatrix} := \begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}, \quad A \cdot v = (axl A) \times v, \quad \forall v \in \mathbb{R}^3, 
$$

(2.20)

$$
A_{ij} = 3 \sum_{k=1}^3 -\epsilon_{ijk}(axl A)_{kj} =: \text{anti}(axl A)_{ij}, \quad (axl A)_{ij} = 3 \sum_{i,j=1}^3 -\frac{1}{2}\epsilon_{ijk} A_{ij},
$$

where $\epsilon_{ijk}$ is the totally antisymmetric third order permutation tensor. For the particular form (2.18) of $\tilde{L}_{im}$, using the formula 2 $axl \nabla P_i = \text{curl} P_i$, we obtain

$$
(\tilde{L}_{im}, \nabla P_m, \nabla P_i)_{\mathbb{R}^{3 \times 3}} = 2 \langle \text{skew anti} \tilde{L}_{im} \rangle axl \nabla P_i, \nabla P_i)_{\mathbb{R}^{3 \times 3}} = 2 \langle \text{anti} \tilde{L}_{im} \rangle axl \nabla P_i, \nabla P_i)_{\mathbb{R}^{3 \times 3}} = 4 \langle \tilde{L}_{im} \rangle axl \nabla P_i, \nabla P_i)_{\mathbb{R}^{3 \times 3}} = \langle \tilde{L}_{im}, \text{curl} P_i, \text{curl} P_i)_{\mathbb{R}^{3 \times 3}}.
$$

(2.21)

Now, we conclude that for a given fourth order tensor $L_c : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}$ the tensor $\tilde{L}$ defined by (2.15) and (2.18) is such that

$$
(\tilde{L}, \nabla P, \nabla P)_{\mathbb{R}^{3 \times 3 \times 3}} = (\tilde{L}_{im}, \nabla P_m, \nabla P_i)_{\mathbb{R}^{3 \times 3 \times 3}} = (\tilde{L}_{im}, \text{curl} P_i, \text{curl} P_i)_{\mathbb{R}^{3 \times 3 \times 3}} = (L_c, \text{Curl} P, \text{Curl} P)_{\mathbb{R}^{3 \times 3 \times 3}}.
$$

(2.22)

Our final comparison of the relaxed model with the classical Mindlin-Eringen free energy [8] is then achieved through observing that

$$
(\tilde{C}, X, X)_{\mathbb{R}^{3 \times 3}} := (C, \text{sym} X, \text{sym} X)_{\mathbb{R}^{3 \times 3}},
$$

(2.23)

and (2.22) define only \textit{positive semi-definite tensors} $\tilde{C}$ and $\tilde{L}$ when $C$ and $L_c$, acting on linear subspaces of $\mathfrak{gl}(3) \cong \mathbb{R}^{3 \times 3}$, are assumed to be strictly positive definite tensors.

Our new set of \textit{independent constitutive variables} for the relaxed micromorphic model is now

$$
\varepsilon_e := \text{sym}(\nabla u - P), \quad \varepsilon_p := \text{sym} P, \quad \alpha := -\text{Curl} P
$$

(2.24)

and the system of partial differential equations is derived from the following free energy

$$
2\mathcal{E}(\varepsilon_e, \varepsilon_p, \alpha) = \langle C, \varepsilon_e, \varepsilon_e \rangle + \langle H, \varepsilon_p, \varepsilon_p \rangle + \langle L_c, \alpha, \alpha \rangle
$$

(2.25)

$$
= \langle C, \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \langle H, \text{sym} P, \text{sym} P \rangle + \langle L_c, \text{Curl} P, \text{Curl} P \rangle,
$$

elastic energy microstrain self-energy dislocation energy

$$
\sigma = D_e \mathcal{E}(\varepsilon_e, \varepsilon_p, \alpha) \in \text{Sym}(3), \quad s = D_p \mathcal{E}(\varepsilon_e, \varepsilon_p, \alpha) \in \text{Sym}(3), \quad m = D_\alpha \mathcal{E}(\varepsilon_e, \varepsilon_p, \alpha) \in \mathbb{R}^{3 \times 3},
$$

where $C : \Omega \to L(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$, $L_c : \Omega \to L(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ and $H : \Omega \to L(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ are fourth order positive definite elasticity tensors, and functions of class $C^1(\Omega)$.

For the rest of the paper we assume that the constitutive tensors have the following symmetries\footnote{Minor symmetries mean $C_{ijrs} = C_{rjis}$ and $H_{ijrs} = H_{rjis}$, while major symmetry asks $C_{ijrs} = C_{rjisz}$. In other words, the first set of minor symmetries $C_{ijrs} = C_{rjis}$ implies $C : \mathbb{R}^{3 \times 3} \to \text{Sym}(3)$, while from the second set of minor symmetries $C_{ijrs} = C_{rjisz}$ it follows $C : \text{Sym}(3) \to \mathbb{R}^{3 \times 3}$. Hence, together, the minor symmetries imply $C : \text{Sym}(3) \to \text{Sym}(3)$. The major symmetries $C_{ijrs} = C_{rjisz}$ are enough to have $C(X, X) = (X, X)$ for all $X \in \mathbb{R}^{3 \times 3}$.} (2.26)

$$
C_{ijrs} = C_{jirs} = C_{jirs}, \quad H_{ijrs} = H_{rjsi} = H_{rjsi} \quad \text{(minor+ major symmetries)}, \quad (L_c)_{ijrs} = (L_c)_{rjsi} \quad \text{(only major symmetries)}.
$$

We assume that the fourth order elasticity tensors $C$, $L_c$ and $H$ are positive definite. Then, there are positive numbers $c_M$, $c_m$ (the maximum and minimum elastic moduli for $C$), $(L_c)_M$, $(L_c)_m$ (the maximum and minimum moduli for $L_c$) and $h_M$, $h_m$ (the maximum and minimum moduli for $H$) such that

$$
\begin{align*}
2c_m ||X||^2 &\le (C, X, X) \le 2c_M ||X||^2 \quad \text{for all } X \in \text{Sym}(3), \\
2(L_c)_m ||X||^2 &\le (L_c, X, X) \le 2(L_c)_M ||X||^2 \quad \text{for all } X \in \mathbb{R}^{3 \times 3}, \\
2h_m ||X||^2 &\le (H, X, X) \le 2h_M ||X||^2 \quad \text{for all } X \in \text{Sym}(3).
\end{align*}
$$

(2.27)

Further we assume, without loss of generality, that $c_M$, $c_m$, $(L_c)_M$, $(L_c)_m$, $h_M$ and $h_m$ are constants.
Remark 2.1 Since \( P \) is determined in \( H(\text{Curl};\Omega) \) in our relaxed model the only possible description of boundary values is in terms of tangential traces \( P\mathbf{\kappa} \). This follows from the standard theory of the \( H(\text{Curl};\Omega) \)-space [34].

In the absence of time dependence, the basic dynamic equations given in [1] reduce in the static case to the following system of partial differential equations (the Euler-Lagrange equations corresponding to (2.25))

\[
\begin{align*}
0 &= \text{Div}[C, \text{sym}(\nabla u - P)] + f, \quad \text{balance of forces}, \quad (2.28) \\
0 &= -\text{Curl}[[\mathbf{L}, \text{Curl} P] + C \text{sym}(\nabla u - P) - \mathbb{H} \text{sym} P + M, \quad \text{balance of moment stresses}, \\
\end{align*}
\]

in \( \Omega \). Consistently with our previous remarks, we consider the weaker (compared to the classical) boundary conditions

\[
u(x) = 0 \quad \text{and the tangential condition} \quad P(x) . \mathbf{\tau} = 0 \quad x \in \partial \Omega, \quad (2.29)
\]

for all tangential vectors \( \mathbf{\tau} \) at \( \partial \Omega \). In the following we suppose that the body loads satisfy the following regularity conditions

\[
f, M \in L^2(\Omega). \quad (2.30)
\]

Since our new approach, in marked contrast to classical asymmetric micromorphic models, features a symmetric Cauchy stress tensor \( \sigma = C \text{sym}(\nabla u - P) \), the linear Cosserat approach ([48, 49, 50, 51, 52]: \( \mu_c > 0 \)) is excluded here (see [53, 54] for further discussions).

In contrast with the classical 7+11 parameters of the isotropic Mindlin and Eringen model [6, 7], we have altogether only seven parameters \( \mu_c, h_c, \lambda_h, \alpha_1, \alpha_2, \alpha_3 \). For isotropic materials, our system specializes to

\[
\begin{align*}
0 &= \text{Div} \sigma + f, \quad 0 = -\text{Curl} m + \sigma - s + M \quad \text{in} \quad \Omega, \quad (2.31)
\end{align*}
\]

where

\[
\begin{align*}
\sigma &= 2\mu_c \text{sym}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P) \mathbf{\mathbf{I}}, \\
m &= \alpha_1 \text{dev sym Curl} P + \alpha_2 \text{skew Curl} P + \alpha_3 \text{tr(Curl} P) \mathbf{\mathbf{I}}, \\
s &= 2\mu_h \text{sym} P + \lambda_h \text{tr}(P) \mathbf{\mathbf{I}}.
\end{align*}
\]

Thus, for isotropic elastic materials we obtain the complete system of linear partial differential equations in terms of the kinematical unknowns \( u \) and \( P \)

\[
\begin{align*}
0 &= \text{Div}[2\mu_c \text{sym}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P) \mathbf{\mathbf{I}}] + f, \quad (2.33) \\
0 &= -\text{Curl}[[\alpha_1 \text{dev sym Curl} P + \alpha_2 \text{skew Curl} P + \lambda_h \text{tr}(P) \mathbf{\mathbf{I}}] \\
&+ 2\mu_c \text{sym}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P) \mathbf{\mathbf{I}} - 2\mu_h \text{sym} P - \lambda_h \text{tr}(P) \mathbf{\mathbf{I}} + M \quad \text{in} \quad \Omega.
\end{align*}
\]

In this model, the asymmetric parts of \( P \), which are not suppressed, are entirely due only to moment stresses and applied body moments. In this sense, the macroscopic and microscopic scales are fully separated.

The positive definiteness required for the tensors \( C, \mathbb{H} \) and \( \mathbf{L} \) implies for an isotropic material the following restriction upon the parameters \( \mu_c, \lambda_c, h, \alpha_1, \alpha_2 \) and \( \alpha_3 \)

\[
\mu_c > 0, \quad 2\mu_c + 3\lambda_c > 0, \quad \mu_h > 0, \quad 2\mu_h + 3\lambda_h > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0. \quad (2.34)
\]

Therefore, positive definiteness for our isotropic model does not involve extra nonlinear side conditions between constitutive coefficients [8, 55]. For the mathematical treatment of the linear relaxed model there arises the need for new integral type inequalities which we have presented in the previous subsection. Using the new results established by Neff, Pauly and Witsch [35, 36, 37] and by Bauer, Neff, Pauly and Starke [38, 39] we are now able to manage also energies depending on the dislocation energy and having symmetric Cauchy stresses.

If, in order to describe the mechanical behavior of a wider range of microstructured materials, we add the anti-symmetric term \( 2\mu_c \text{skew}(\nabla u - P) \) in the expression of the Cauchy stress tensor \( \sigma \), where \( \mu_c \geq 0 \) is the Cosserat couple modulus, then our analysis works also for \( \mu_c \geq 0 \). The model in which \( \mu_c > 0 \) is the isotropic Eringen-Claus model for dislocation dynamics [45, 56, 3] and it is, in fact, derived from the following free energy

\[
E(\varepsilon, \varepsilon_p, \alpha) = \mu_c \|\text{sym}(\nabla u - P)\|^2 + \mu_c \|\text{skew}(\nabla u - P)\|^2 + \frac{\lambda_c}{2} \|\text{tr}(\nabla u - P)\|^2 + \mu_h \|\text{sym} P\|^2 + \frac{\lambda_h}{2} \|\text{tr}(P)\|^2 \\
+ \frac{\alpha_1}{2} \|\text{dev sym Curl} P\|^2 + \frac{\alpha_2}{2} \|\text{skew Curl} P\|^2 + \frac{\alpha_3}{2} \|\text{tr}(\text{Curl} P)\|^2. \quad (2.35)
\]
For $\mu_c > 0$ and if the other inequalities (2.34) are satisfied, the existence and uniqueness follow along the well known classical lines. There is no need for any new integral inequality. To the sake of simplicity, we only present in this paper well-posedness results for the relaxed model $\mu_c = 0$. These results, however, still hold for $\mu_c > 0$ and can be easily generalized with some additional calculations.

3 Existence of the solution

In this section we establish an existence theorem for the solution of the boundary value problem $(P)$ defined by (2.28) and (2.29). To this aim, we will rewrite the boundary value problem $(P)$ in a weak form in a Hilbert space. The suitable Hilbert space for the equilibrium problem in the relaxed model is

$$\mathcal{X} = \{ w = (u, P) \mid u \in H^1_0(\Omega), \quad P \in H_0(\text{Curl}; \Omega) \}. $$

(3.1)

According to Theorem 2.2, on $\mathcal{X}$ we have the following norm

$$|||w|||_\mathcal{X} = \left( \|u\|^2_{H^1_0(\Omega)} + |||P|||^2 \right)^{\frac{1}{2}}, $$(3.2)

which is equivalent with the usual norm on $\mathcal{X}$

$$\|w\|_\mathcal{X} = \left( \|u\|^2_{H^1_0(\Omega)} + \|P\|^2_{H(\text{Curl}; \Omega)} \right)^{\frac{1}{2}}. $$

(3.3)

On $\mathcal{X}$ we define the bilinear form

$$(w_1, w_2) = \int_\Omega \left( \langle C \cdot \text{sym}(\nabla u_1 - P_1), \text{sym}(\nabla u_2 - P_2) \rangle + \langle H \cdot \text{sym} P_1, \text{sym} P_2 \rangle + \langle L_c \cdot \text{Curl} P_1, \text{Curl} P_2 \rangle \right) dv,$$

where $w_1 = (u_1, P_1) \in \mathcal{X}$ and $w_2 = (u_2, P_2) \in \mathcal{X}$. From [1] we have a first algebraic estimate:

**Lemma 3.1** If $C$ and $H$ satisfy the relations (2.26)$_{1,2}$ and (2.27)$_{1,3}$, then there is a positive constant $a_1$ such that

$$a_1(\|\text{sym} \nabla u\|^2 + \|\text{sym} P\|^2) \leq \langle C \cdot \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \langle H \cdot \text{sym} P, \text{sym} P \rangle$$

for all $u \in H^1(\Omega)$ and $P \in H(\text{Curl}; \Omega)$.

Let us define the linear operator $l : \mathcal{X} \to \mathbb{R}$, describing the influence of external loads,

$$l(\tilde{w}) = \int_{\Omega} (\langle f, \tilde{u} \rangle + \langle M, \tilde{P} \rangle) dv \quad \text{for all} \quad \tilde{w} \in \mathcal{X}. $$

(3.5)

Using a similar calculus as in [57] we are able to give a weak formulation of the problem $(P)$. We say that $w$ is a weak solution of the problem $(P)$ if and only if

$$(w, \tilde{w}) = l(\tilde{w}) \quad \text{for all} \quad \tilde{w} \in \mathcal{X}. $$

(3.6)

Regarding the existence of a weak solution $(u, P)$ of the problem $(P)$ we establish the following result:

**Theorem 3.2** Assume that

i) the constitutive coefficients satisfy the symmetry relations (2.26) and the inequalities (2.27);

ii) the loads satisfy the regularity conditions (2.30).

Then there exists one and only one solution of the problem (3.6).
The Lax-Milgram theorem used in the proof of the previous theorem also offers a continuous dependence result on the loads $f, M$. Moreover, the weak solution $w$ minimizes the corresponding energy functional $\frac{1}{2}(w, w) - l(w)$ on $\mathcal{X}$.

## 4 Static problem for a further relaxed model

In [1] a further relaxed model is proposed, where the constitutive coefficients of the general expression of the energy (2.11) are chosen such that

\[
\tilde{C} = \text{sym} C : \mathbb{R}^{3 \times 3} \rightarrow \text{Sym}(3), \quad \tilde{H} = \text{dev sym} \mathbb{H} \text{ dev sym} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3},
\]

\[
\tilde{E} = 0, \quad \tilde{P} = 0, \quad \tilde{C} = 0, \quad \mathcal{L}_c = \text{dev} \mathcal{L} \text{ dev} : \mathbb{R}^{3 \times 3} \rightarrow \mathfrak{sl}(3),
\]

\[
\mathcal{C} : \text{Sym}(3) \rightarrow \text{Sym}(3), \quad \mathcal{H} : \mathfrak{sl}(3) \cap \text{Sym}(3) \rightarrow \mathfrak{sl}(3), \quad \mathcal{L} : \mathfrak{sl}(3) \rightarrow \mathfrak{sl}(3).
\]
This model considers an even weaker energy expression, i.e. it depends only on the set of independent constitutive variables
\[
\varepsilon_e = \text{sym}(\nabla u - P), \quad \text{dev} \varepsilon_p = \text{dev sym} P, \quad \text{dev} \alpha = - \text{dev Curl} P. \tag{4.2}
\]
In this model, it is neither implied that \( P \) remains symmetric, nor that \( P \) is trace-free, but only the trace free symmetric part of the micro-distortion \( P \) and the trace-free part of the micro-dislocation tensor \( \alpha \) contribute to the stored energy. The model in its general anisotropic form is then:
\[
\begin{align*}
0 &= \text{Div}[\mathcal{C} \cdot \text{sym}(\nabla u - P)] + f, \\
0 &= -\text{Curl}[\text{dev}[\mathcal{L} \cdot \text{Curl} P] + \mathcal{C} \cdot \text{sym}(\nabla u - P) - \mathbb{H} \cdot \text{dev sym} P + M \quad \text{in} \quad \Omega.
\end{align*}
\tag{4.3}
\]
In the isotropic case the model turns into
\[
\begin{align*}
0 &= \text{Div}[2\mu_e \text{sym}(\nabla u - P) + \lambda_e \text{tr}(\nabla u - P) \cdot \mathbb{1}] + f, \\
0 &= -\text{Curl}[\alpha_1 \text{dev sym} \text{Curl} P + \alpha_2 \text{skew Curl} P] \\
&\quad + 2\mu_e \text{sym}(\nabla u - P) + \lambda_e \text{tr}(\nabla u - P) \cdot \mathbb{1} - 2\mu_h \text{dev sym} P + M \quad \text{in} \quad \Omega.
\end{align*}
\tag{4.4}
\]
To the system of partial differential equations of this model we adjoin the weaker boundary conditions
\[
\begin{align*}
u(x) &= 0, \quad P(x), \tau(x) = 0 \quad x \in \partial \Omega. \tag{4.5}
\end{align*}
\]
We remark again that \( P \) is not trace-free in this formulation and no projection is performed. We denote the symmetric part of the micro-distortion \( P \) by \( \text{sym}(P) \), and the trace-free part of the micro-dislocation tensor \( \alpha \) is not trace-free in this formulation and no projection is performed. We denote the symmetric part of the micro-distortion \( P \) by \( \text{sym}(P) \), and the trace-free part of the micro-dislocation tensor \( \alpha \) is not trace-free in this formulation and no projection is performed. We denote the symmetric part of the micro-distortion \( P \) by \( \text{sym}(P) \), and the trace-free part of the micro-dislocation tensor \( \alpha \) is not trace-free in this formulation and no projection is performed.

Further on, we study the existence of the solution of the problem \((\tilde{P})\). Since the method is similar with that used in Section 3 we only point out the differences which arise for our modified problem. We consider the same Hilbert space \( X \) as defined in Section 3 and we define the following bilinear form
\[
((w_1, w_2)) = \int_{\Omega} \left( \langle \mathcal{C} \cdot \text{sym}(\nabla u_1 - P_1), \text{sym}(\nabla u_2 - P_2) \rangle + \langle \mathbb{H} \cdot \text{dev sym} P_1, \text{dev sym} P_2 \rangle + \langle \mathcal{L} \cdot \text{dev Curl} P_1, \text{dev Curl} P_2 \rangle \right) dv,
\]
where \( w_1 = (u_1, P_1) \in X \) and \( w_2 = (u_2, P_2) \in X \). We say that \( w \) is a weak solution of the problem \((\tilde{P})\) if and only if
\[
((w, \tilde{w})) = l(\tilde{w}), \quad \text{for all} \quad \tilde{w} \in X, \tag{4.7}
\]
where \( l \) is defined by (3.5). In order to prove the existence of a weak solution of the problem (4.7), let us recall that, using Theorem 2.4, in [57] the following lemma was proved:

**Lemma 4.1** Assume that \( \mathcal{C} \) and \( \mathbb{H} \) satisfy the conditions (2.27), then the following estimate holds true
\[
a_2 \left( \| \nabla u \|^2_{L^2(\Omega)} + \| \text{dev sym} P \|^2_{L^2(\Omega)} \right) \leq \int_{\Omega} \left( \langle \mathcal{C} \cdot \text{sym}(\nabla u - P), \text{sym}(\nabla u - P) \rangle + \langle \mathbb{H} \cdot \text{dev sym} P, \text{dev sym} P \rangle \right) dv,
\]
for all \( u \in H_0^1(\Omega) \) and \( P \in H(\text{Curl}; \Omega) \), where \( a_2 \) is a positive constant.

Let us remark that in view of the above Lemma, there is a positive constant \( a_3 \) such that
\[
a_3 \left( \| \nabla u \|^2_{L^2(\Omega)} + \| \text{dev sym} P \|^2_{L^2(\Omega)} + \| \text{dev Curl} P \|^2_{L^2(\Omega)} \right) \leq ((w, w)), \tag{4.8}
\]
where \( w = (u, v, K, P) \in X \). In other words, using Corollary 2.3 we have
\[
C \| w \|_X \leq ((w, w)), \quad \text{for all} \quad w = (u, v, K, P) \in X, \tag{4.9}
\]
where \( C \) is a positive constant. Hence \((\cdot, \cdot))\) is coercive. Moreover, the bilinear form \((\cdot, \cdot))\) is bounded, i.e. there is a positive constant \( C \) such that
\[
((w, \tilde{w})) \leq C \| w \|_X \| \tilde{w} \|_X, \quad \text{for all} \quad w, \tilde{w} \in X. \tag{4.10}
\]
Hence, we are able to formulate the following existence result:

**Theorem 4.2** Assume that

i) the constitutive coefficients satisfy the symmetry relations (2.26) and the inequalities (2.27);

ii) the loads satisfy the regularity conditions (2.30).

Then there exists one and only one solution of the problem (4.7). Moreover, the weak solution \( w \) minimizes the energy functional \( \frac{1}{2}((w, w)) - l(w) \) on \( X \) and we have the continuous dependence of the weak solution upon the loads \( f, M \).

5 Gauge theory of dislocations

In this subsection we explain how we can construct a gauge theory of dislocations and which are the relations of the constructed theory with the relaxed theory of micromorphic elastic materials, the Mindlin-Eringen/Claus-Eringen theory, and with other models of dislocations. Here we assume smooth functions, if not otherwise stated.

5.1 Ground states in the gauge theory of dislocations

For dislocation gauge theory, the gauge group is the three-dimensional translation group \( T(3) \). First of all, we may postulate a local (or soft) translation transformation for the displacement \( u \) as gauge transformation

\[
 u^* = u + \tau ,
\]

(5.1)

where \( \tau \) is a space-dependent (or local) translation vector. The transformation (5.1) represents the generalization of a rigid body translation with \( \tau = \text{const} \). Of course, the displacement gradient is not invariant under a local translational transformation

\[
 \nabla u^* = \nabla u + \nabla \tau ,
\]

(5.2)

due to the second term. We require from the corresponding energy density to stay invariant under the internal transformation of the displacement field. Therefore, we have to describe the deformation using constitutive variables which are invariant under internal transformations. This justifies, in the gauge theory of dislocations, the introduction of the micro-distortion tensor \( P \). In gauge theory of dislocation, the micro-distortion tensor \( P \) is called the translational gauge potential. A starting assumption in the gauge theory of dislocation is that the micro-distortion tensor \( P \) (the translational gauge potential) possesses the following inhomogeneous transformation law with respect to the translation group:

\[
 P^* = P + \nabla \tau ,
\]

(5.3)

This assumption solves the invariance problem of the displacement gradient under local translation transformation. However, together with the general invariance assumption this precludes the presence of the microstress tensor \( s \) (i.e. \( \mathbb{H} = 0 \)). Since \( P \) is a gauge potential, it transforms inhomogeneously. Then the gauge potential couples to the displacement field \( u \) by the \( T(3) \)-gauge-covariant derivative

\[
 D^* u := \nabla u - P = e ,
\]

(5.4)

i.e. the elastic distortion (relative distortion) from Mindlin-Eringen theory [8] (see [1]). It is clear that the so called \( T(3) \)-gauge-covariant derivative is not a derivative in the common meaning. Thus, we have redefined the elastic distortion \( e \) by means of the gauge-covariant derivative in terms of the displacement gradient (total distortion) and the plastic distortion. We may call \( e \) the incompatible elastic distortion. Now \( e \) is gauge-invariant under local \( T(3) \)-transformations

\[
 e^* = e .
\]

(5.5)
In $T(3)$-gauge theory, the Curl of the gauge potential gives rise to an additional physical state quantity, the *translational field strength* (the micro-dislocation density tensor), $\alpha$, defined by

$$\alpha = -\text{Curl} \, P,$$  \hspace{1cm} (5.6)

or in terms of $e$

$$\alpha = \text{Curl} \, e.$$  \hspace{1cm} (5.7)

Thus, the translational field strength gives in a natural way the *dislocation density tensor* as state quantity. Since $\alpha$ is a state quantity, it has to be gauge-invariant:

$$\alpha^* = \alpha.$$  \hspace{1cm} (5.8)

In addition, it must fulfill the so called *translational Bianchi identity*

$$\text{Div} \, \alpha = 0,$$  \hspace{1cm} (5.9)

pointing out that dislocations cannot end inside the body. Therefore, the physical state quantities (the set of constitutive variables) in the dislocation gauge theory are

$$e = \nabla u - P, \quad \alpha = \text{Curl} \, e = -\text{Curl} \, P \quad \text{(but not only $P$ itself)}.$$  \hspace{1cm} (5.10)

Now some other field theoretical remarks are in order. As pointed out by Lazar [59] it is remarkable that the gauge-field theoretical structure of the dislocation gauge theory may be understood using a *Higgs mechanism* in the translational gauge theory. In the translational Higgs mechanism the displacement field $u$ plays the physical role of a Nambu-Goldstone field giving the Proca tensor field $e$, which is a physical state quantity, a “mass”. Using an affine gauge approach [60] it turns out that the gauge potential $P$ has the geometrical meaning of the translational part of the generalized affine connection and $\alpha$ is the translational part of the affine curvature. A systematic investigation of conservation and balance laws in dislocation gauge theory using Lie-point symmetries has been carried out by Lazar and Anastassiadis [41]. An important result was a straightforward definition and physical interpretation of the Peach-Koehler force analogous to the Lorentz force in electrodynamics since there is a lot of confusion about the physical nature of the Peach-Koehler force in the literature.

The strain energy density of the dislocation gauge theory is given by

$$2 \tilde{E}(e, \alpha) = \langle \hat{\sigma}, e \rangle + \langle m, \alpha \rangle - 2 \langle \hat{\sigma}_0, e \rangle,$$  \hspace{1cm} (5.11)

where $\hat{\sigma}$ denotes the *force-stress tensor* from the Mindlin-Eringen theory and $m$ is the so-called *pseudomoment stress tensor* [42], i.e. the moment stress tensor from the relaxed theory of micromorphic elastic materials [1] (see also [45]). The stress $\hat{\sigma}_0$ plays the role of a nucleation field giving the Proca tensor field $e$, which is related to the body moment tensor in the Eringen-Claus model [45]. Thus, force-stress is the specific response to elastic distortion and pseudomoment stress (with the dimension of a moment stress tensor) is the specific response to dislocations. In general, this yields: $\hat{\sigma} = \sigma + \text{skew} \hat{\sigma}$, where $\sigma = \text{sym} \hat{\sigma}$ is the Cauchy-stress from the relaxed theory of micromorphic elastic materials (2.32) (see also [1]). The idea of a static dislocation gauge theory is to use three terms in the strain energy density (5.11). The first term contains the elastic distortion field $e$. Another one proportional to the dislocation density tensor $\alpha$ having the meaning of dislocation energy density and a term containing a background stress tensor $\hat{\sigma}_0$, which is needed for self-equilibrating of the dislocations. No constitutive equations are proposed for $\hat{\sigma}_0$ which is considered to be known. Using the calculus of variations, the following field equations are derived when body forces $f$ are present

$$0 = \text{Div} \hat{\sigma} + f,$$  \hspace{1cm} balance of forces  \hspace{1cm} (5.12)

$$0 = -\text{Curl} \, m + \hat{\sigma} - \hat{\sigma}_0,$$  \hspace{1cm} balance of dislocation stresses.

In the balance of dislocation stresses, it can be seen that the dislocation fields are driven by an effective stress $\hat{\sigma} - \hat{\sigma}_0$. In order to make the presentation of the models of this paper uniform, we may say that the anisotropic

---

2Here, a state quantity is by definition a gauge invariant object.
constitutive relations of this model\(^3\) are obtained (see e.g. \([61]\)) by assuming that the constitutive coefficients of the general expression of the energy \((2.11)\) are such that
\[
\widetilde{C} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}, \quad \widetilde{H} = 0, \quad \widetilde{E} = 0, \quad \widetilde{G} = 0,
\]
(5.13)
the tensor \(\widetilde{L} : \mathbb{R}^{3 \times 3 \times 3} \to \mathbb{R}^{3 \times 3 \times 3}\) has the specific form given by \((2.17), (2.15)\) and \((2.18)\) in terms of another tensor \(\mathbb{L}_c : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}\), while a specific form of the tensor \(\widetilde{F} : \mathbb{R}^{3 \times 3 \times 3} \to \mathbb{R}^{3 \times 3}\) is constructed in the following. The tensor \(\widetilde{F}\) is uniquely defined by the tensors
\[
\widetilde{F}_{im} = (\widetilde{F}_{ijmnp})_{im}, \quad \widetilde{F}_{im} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^3.
\]
(5.14)
Let the tensor \(\mathbb{B} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}\) be given
\[
\mathbb{B} = (\mathbb{B}^1, \mathbb{B}^2, \mathbb{B}^3), \quad \mathbb{B}^i : \mathbb{R}^3 \to \mathbb{R}^3.
\]
(5.15)
We define a specific tensor \(\tilde{F}\) by (5.14) where
\[
\tilde{F}_{im} = \begin{cases} 2 \mathbb{B}^i \text{axl skew} & \text{for } i = m \\ 0 & \text{for } i \neq m, \end{cases}
\]
(5.16)
For this particular form (5.16) of the tensor \(\tilde{F}\) we obtain
\[
\langle \tilde{F}, \nabla P, \nabla u - P \rangle_{\mathbb{R}^{3 \times 3}} = \langle \mathbb{F}_{im}, \nabla P_m, \nabla u_i - P_i \rangle_{\mathbb{R}^3} = 2 (\mathbb{B}^i \text{ axl skew } P_i, \nabla u_i - P_i)_{\mathbb{R}^3}
\]
(5.17)
\[
= \langle \mathbb{B}^i, \text{curl } P_i, \nabla u_i - P_i \rangle_{\mathbb{R}^3} = \langle \mathbb{C}, \text{Curl } P, \nabla u - P \rangle_{\mathbb{R}^{3 \times 3}}.
\]
Therefore, the constitutive equations are
\[
\tilde{\sigma} = \tilde{C} \cdot e + \mathbb{B} \cdot \alpha, \quad m = \mathbb{B}^* \cdot e + \mathbb{L}_c \cdot \alpha .
\]
(5.18)
Dimensionally, \([\mathbb{L}_c] = \ell [\mathbb{L}] = \ell^2 [\mathbb{C}],\) where \(\ell\) is a material length-scale parameter and, therefore, they have the dimensions: \([\mathbb{L}_c] = \text{force}, [\mathbb{L}] = \text{force/length}, \) and \([\mathbb{C}] = \text{force/length}^2\). Moreover, it is assumed that the material tensors satisfy the following major symmetries
\[
\mathbb{C}_{ijkl} = \mathbb{C}_{klij}, \quad (\mathbb{L}_c)_{ijkl} = (\mathbb{L}_c)_{klij} .
\]
(5.19)
Here, the tensor \(\mathbb{H}\) is absent since the term \(\langle \mathbb{H}, \text{sym } P, \text{sym } P \rangle\) is not translation gauge invariant. This leads to absent specific micro-stress. However, the stress \(\tilde{\sigma}_0\) is a self-equilibrating stress and incorporates the external body moment tensor \(M\) from the Eringen-Claus model \([45]\). Substituting the constitutive relations (5.18) into (5.12), we obtain
\[
0 = \text{Div}[\mathbb{C} \cdot e + \mathbb{B} \cdot \alpha] + f ,
\]
(5.20)
\[
\tilde{\sigma}^0 = - \text{Curl}[\mathbb{B}^* \cdot e + \mathbb{L}_c \cdot \alpha] + \mathbb{C} \cdot e + \mathbb{B} \cdot \alpha \quad \text{in } \Omega .
\]
On the other hand, if we substitute Eqs. (5.10) into (5.20), we obtain the complete system of linear partial differential equations in terms of the kinematical fields \(u\) and \(P\) in the framework of dislocation gauge theory
\[
0 = \text{Div}[\mathbb{C} \cdot (\nabla u - P) + \mathbb{B} \cdot (\text{Curl} (\nabla u - P))] + f ,
\]
(5.21)
\[
\tilde{\sigma}^0 = \text{Curl}[\mathbb{L}_c, (\text{Curl} (\nabla u - P)) - \mathbb{B}^* \cdot (\nabla u - P)] + \mathbb{C} \cdot (\nabla u - P) + \mathbb{B} \cdot (\text{Curl} (\nabla u - P)) \quad \text{in } \Omega .
\]
Let us remark that by applying on both sides of equation (5.21) the Div-operator, we deduce that the statically admissible background field \(\tilde{\sigma}^0\) has to satisfy
\[
\text{Div } \tilde{\sigma}^0 + f = 0 \quad \text{in } \Omega .
\]
(5.22)
\(^3\)In the relaxed micromorphic model \([1]\), for simplicity we have omitted the mixed terms. Another reason to omit the mixed terms was that for centro-symmetric materials these terms are absent and for arbitrary anisotropic materials they would induce nonzero force-stress \(\sigma\) for zero elastic distortion \(e = \nabla u - P = 0\). Moreover, we have shown \([1]\) how our energy without any mixed terms leads, in principle, to complete equations for the Cosserat model, the microstretch model and the microvoids model in dislocation format.
Moreover, in view of (5.22), it follows that (5.21) results from (5.21)₂. Therefore, the equations are not sufficient to find the fields \( u \) and \( P \) individually, but only the elastic distortion \( e = \nabla u - P \) can be determined. Thus, in terms of the elastic distortion \( e \), the independent equations of the gauge theory of dislocations are
\[
\hat{\sigma}^0 = \text{Curl}[\mathbb{I}_c, (\text{Curl} e) - \tilde{\mathbb{B}}^*, e] + \tilde{\mathbb{C}}. e + \tilde{\mathbb{B}}. (\text{Curl} e) \quad \text{in} \quad \Omega,
\]
(5.23)
where \( \hat{\sigma}_0 \) is a solution of the problem
\[
0 = \text{Div} \hat{\sigma}_0 + f \quad \text{in} \quad \Omega, \quad \hat{\sigma}_0.n = 0 \quad \text{on} \quad \partial\Omega.
\]
(5.24)
To the system of partial differential equations of this model we adjoin the weaker tangential boundary conditions
\[
e. \tau = 0 \quad \text{on} \quad \partial\Omega.
\]
(5.25)
The isotropic constitutive relations are given by [42]
\[
\hat{\sigma} = 2 \mu_e \text{sym} e + 2 \mu_c \text{skew} e + \lambda_c \text{tr}(e) \cdot \mathbb{I},
\]
(5.26)
and the positive semi-definiteness required for the tensors \( \tilde{\mathbb{C}}, \tilde{\mathbb{B}} \) and \( \mathbb{I}_c \) implies for isotropic materials the following restriction upon the parameters \( \mu_e, \lambda_c, \mu_c, \alpha_1, \alpha_2 \) and \( \alpha_3 \)
\[
\mu_c \geq 0, \quad 2\mu_e + 3\lambda_c \geq 0, \quad \mu_c \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_3 \geq 0.
\]
(5.27)
If we put \( \mu_c = 0 \) in (5.26)₁, the force-stress tensor becomes symmetric \( \hat{\sigma} = \sigma \). The static field equations used by Lazar and Anastassiadis [42] in the isotropic gauge theory of dislocations read
\[
\hat{\sigma}^0 = \text{Curl}[\alpha_1 \text{dev} \text{sym} (\text{Curl} e) + \alpha_2 \text{skew} (\text{Curl} e) + \alpha_3 \text{tr}(\text{Curl} e) \cdot \mathbb{I}]
\]
(5.28)
\[
+ 2\mu_e \text{sym} e + 2\mu_c \text{skew} e + \lambda_c \text{tr}(e) \cdot \mathbb{I},
\]
or equivalently in terms of the displacement vector \( u \) and plastic distortion tensor \( P \)
\[
\hat{\sigma}^0 = -\text{Curl}[\alpha_1 \text{dev} \text{sym} (\text{Curl} P) + \alpha_2 \text{skew} (\text{Curl} P) + \alpha_3 \text{tr}(\text{Curl} P) \cdot \mathbb{I}]
\]
(5.29)
\[
+ 2\mu_e \text{sym}(\nabla u - P) + 2\mu_c \text{skew}(\nabla u - P) + \lambda_c \text{tr}(\nabla u - P) \cdot \mathbb{I},
\]
where the coefficients \( \alpha_1, \alpha_2, \alpha_3 \) correspond to \( a_1, a_2, \frac{a_3}{3} \) from Lazar’s original notations (see, e.g., [42]).

5.2 Existence and uniqueness in the gauge theory of dislocations

We assume in the following that \( \hat{\sigma}^0 \in L^2(\Omega) \) is known and we study the existence of a solution \( e \) of the boundary value problem \( (\mathcal{P}_G) \) of the gauge theory of dislocation, defined by the equations (5.23) and the boundary conditions (5.25). Let us consider the following energy
\[
(\tilde{\mathbb{C}}. e, e) + 2(\tilde{\mathbb{B}}, \text{Curl} e, e) + (\mathbb{I}_c, \text{Curl} e, \text{Curl} e).
\]
(5.30)
In order to give a weak formulation of the boundary value problem of the gauge theory, we deduce
\[
\langle e, \text{Curl} (\mathbb{I}_c, \text{Curl} e) \rangle = \text{div} [(\mathbb{I}_c, \text{Curl} e) \times e_i] + (\mathbb{I}_c, \text{Curl} e, \text{Curl} e),
\]
(5.31)
\[
\langle e, \text{Curl} (\tilde{\mathbb{B}}^*, e) \rangle = \text{div} [(\tilde{\mathbb{B}}^*, e) \times e_i] + (\tilde{\mathbb{B}}^*, e, \text{Curl} e),
\]
where \( e_i \) are the rows of the tensor \( e \in \mathbb{R}^{3 \times 3} \) and Einstein’s summation rule is used.

We consider the same Hilbert space \( H_0(\text{Curl}; \Omega) \) and we define the following bilinear form
\[
[e, \bar{e}] = \int_{\Omega} \left( (\tilde{\mathbb{C}}, e, \bar{e}) + (\tilde{\mathbb{B}}, (\text{Curl} e), \bar{e}) + (\tilde{\mathbb{B}}^*, (\text{Curl} e), \bar{e}) + (\mathbb{I}_c, \text{Curl} e, \text{Curl} \bar{e}) \right) dv,
\]
(5.32)
where \( e, \bar{e} \in H_0(\text{Curl}; \Omega) \). Let us define the linear operator \( l : H_0(\text{Curl}; \Omega) \rightarrow \mathbb{R} \)
\[
\tilde{l}(\bar{e}) = \int_{\Omega} (\hat{\sigma}_0, \bar{e}) dv \quad \text{for all} \quad \bar{e} \in \mathcal{X}.
\]
(5.33)
We say that \( e \) is a weak solution of the following boundary values problem \( (\mathcal{P}_G) \) if and only if
\[
[e, \bar{e}] = \tilde{l}(\bar{e}) \quad \text{for all} \quad \bar{e} \in H_0(\text{Curl}; \Omega).
\]
(5.34)
Theorem 5.1 Assume that

i) the constitutive coefficients \(^4\) satisfy the symmetry relations (5.19);

ii) there are the positive constants \(c_1, c_2\) such that

\[
c_1 \int_{\Omega} (\|\text{sym} e\|^2 + \|\text{Curl} e\|^2) \, dv \leq \int_{\Omega} \tilde{E}(e) \, dv \leq c_2 \int_{\Omega} (\|e\|^2 + \|\text{Curl} e\|^2) \, dv \quad \forall e \in H_0(\text{Curl}; \Omega);
\]

iii) the statically admissible background field \(\sigma^0\) satisfies the regularity condition \(\sigma^0 \in L^2(\Omega)\).

Then there exists one and only one solution \(e\) of the problem (5.34).

Proof. It is simple to prove that the Cauchy-Schwarz inequality and the hypothesis ii) lead to the boundedness of \([\cdot, \cdot]\). Besides this, from hypothesis ii) and using Theorem 2.2, there are the positive constants \(C_1, C_2 > 0\) such that

\[
[e, e] \geq C_1 \int_{\Omega} (\|\text{sym} e\|^2 + \|\text{Curl} e\|^2) \, dv \geq C_2 \|e\|^2_{H_0(\text{Curl}; \Omega)}, \tag{5.35}
\]

for all \(e \in H_0(\text{Curl}; \Omega)\), i.e. \([\cdot, \cdot]\) is coercive. Finally, the Schwarz inequality implies that the linear operator \(\hat{l}(\cdot)\) is bounded. By the Lax-Milgram theorem it follows that (5.34) has one and only one solution. \(\square\)

Remark 5.1 The Lax-Milgram theorem used in the proof of the previous theorem also offers a continuous dependence result on the loads \(f\). Moreover, the weak solution \(e\) minimizes the energy functional \(\frac{1}{2}[e, e] - \hat{l}(e)\) on \(H_0(\text{Curl}; \Omega)\).

Remark 5.2 The uniqueness of the solution \(e = \nabla u - P\) does not imply that \(u\) and \(P\) are uniquely determined.

5.3 The gauge theory of dislocations for isotropic materials

For isotropic constitutive relations, Lazar [40] and Lazar and Anastassiadis [42] have decomposed the dislocation density tensor \(\alpha\) into its SO(3)-irreducible pieces (see (2.1)), called “the axitor”, “the tentor” and “the trator” parts, i.e.

\[
\alpha = \underbrace{\text{dev sym} \alpha}_{\alpha^{(1)}; \text{"tendor"}} + \underbrace{\text{skew} \alpha}_{\alpha^{(2)}; \text{"trator"}} + \underbrace{\frac{1}{3} \text{tr}(\alpha) \cdot \mathbb{1}}_{\alpha^{(3)}; \text{"axitor"}}, \tag{5.36}
\]

In general, the dislocation density tensor reads in matrix-form

\[
\alpha = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}, \tag{5.37}
\]

The indices \(i\) and \(j\) of \(\alpha_{ij}\) determine the orientation of the Burgers vector and the dislocation line, respectively. Therefore, the diagonal components describe screw dislocations and the off-diagonal components describe edge dislocations. Substituting of (5.37) into (5.36), the \textit{axitor} reads

\[
\alpha^{(3)} = \frac{\alpha_{11} + \alpha_{22} + \alpha_{33}}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.38}
\]

\(^4\)There are no explicit, separate assumption upon the tensors \(\hat{C}, \hat{B}, \hat{L}_{\kappa}\). In the admissible case \(\mathbb{B} = 0\), the positive definiteness of the internal energy \(\hat{E}\) is equivalent with the positive definiteness of the \(\hat{C}, \hat{L}_{\kappa}\). The existence results hold also true for \(\mathbb{B} = 0\).

\(^5\)This condition is weaker than the positive definiteness of the energy \(\hat{E}(e)\) in terms of \(e\) and \(\text{Curl} e\) and shows that the existence result may also work for zero Cosserat couple modulus \(\mu_c = 0\).
describing the sum of all possible screw dislocations, the \textit{trator} is given by

\[
\alpha^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & \alpha_{12} - \alpha_{21} & \alpha_{13} - \alpha_{31} \\ \alpha_{21} - \alpha_{12} & 0 & \alpha_{23} - \alpha_{32} \\ \alpha_{31} - \alpha_{13} & \alpha_{32} - \alpha_{23} & 0 \end{pmatrix},
\]

(5.39)
describing “skew-symmetric” edge dislocations with the property \(\alpha^{(2)} = -(\alpha^{(2)})^T\), and the \textit{tentor} possesses the form

\[
\alpha^{(1)} = \frac{1}{2} \begin{pmatrix} 2\alpha_{11} & \alpha_{12} + \alpha_{21} & \alpha_{13} + \alpha_{31} \\ \alpha_{21} + \alpha_{12} & 2\alpha_{22} & \alpha_{23} + \alpha_{32} \\ \alpha_{31} + \alpha_{13} & \alpha_{32} + \alpha_{23} & 2\alpha_{33} \end{pmatrix} - \frac{\alpha_{11} + \alpha_{22} + \alpha_{33}}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(5.40)
describing “symmetric” edge dislocations with the property \(\alpha^{(3)} = (\alpha^{(3)})^T\) and also single screw dislocations.

In order to appreciate the structure of the dislocation density tensor \(\alpha\), we outline some particular cases: screw dislocations, edge dislocations, anti-plane strain of a screw dislocation and a plane strain problem of edge dislocations. For the three-dimensional elastoplastic dislocation problem, \textit{screw dislocations} correspond to

\[
P = \begin{pmatrix} 0 & P_{12} & P_{13} \\ P_{21} & 0 & P_{23} \\ P_{31} & P_{32} & 0 \end{pmatrix} \quad \Rightarrow \quad \alpha = -\text{Curl} \ P = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix},
\]

(5.41)
while \textit{edge dislocations} correspond to

\[
P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \quad \Rightarrow \quad \alpha = -\text{Curl} \ P = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 0 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 0 \end{pmatrix},
\]

(5.42)
Both situations are connected with the displacement vector \(u = (u_1, u_2, u_3)\), in the sense that \(\nabla u\) and \(P\) have a similar structure.

In [42] various non-singular special solutions to (5.28) for screw and edge dislocations with dislocation line in \(x_3\)-direction were constructed. A solution of a screw dislocation in a functionally graded material within the gauge theory of dislocations was given in [62]. The anti-plane strain of a screw dislocation corresponds to

\[
P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ P_{31}(x_1, x_2) & P_{32}(x_1, x_2) & 0 \end{pmatrix} \quad \Rightarrow \quad \alpha = -\text{Curl} \ P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{33}(x_1, x_2) \end{pmatrix},
\]

(5.43)
which is connected with the following displacement vector

\[
u = \begin{pmatrix} 0, 0, u_3(x_1, x_2) \end{pmatrix}^T.
\]

The plane strain problem of edge dislocations corresponds to

\[
P = \begin{pmatrix} P_{11}(x_1, x_2) & P_{12}(x_1, x_2) & 0 \\ P_{21}(x_1, x_2) & P_{22}(x_1, x_2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \alpha = -\text{Curl} \ P = \begin{pmatrix} 0 & 0 & \alpha_{13}(x_1, x_2) \\ 0 & 0 & \alpha_{23}(x_1, x_2) \\ 0 & 0 & 0 \end{pmatrix},
\]

(5.45)
and the corresponding displacement vector reads

\[
u = \begin{pmatrix} u_1(x_1, x_2), u_2(x_1, x_2), 0 \end{pmatrix}^T.
\]

(5.46)

For a screw dislocation the tentor and the axitor give a non-zero contribution while for an edge dislocation the tentor and the trator give a non-zero contribution. Such gauge theoretical solutions can be physically meaningful (e.g., regularization of the stress and strain singularities, natural dislocation core spreading making redundant the artificial cut-off radius, and appearance of characteristic length scale parameters).

In the variational formulation, the dislocation model can be seen as an elastic (reversible) description of a material, which may respond to external loads by an elastic distortion field \(e\) which is not anymore a gradient (incompatible). This is not yet an irreversible plasticity formulation, since elasticity does not change the state of the body (by definition).
5.4 Special cases of the gauge model of dislocations

Since the presented gauge model of dislocations [42] (where the microstress is not taken into account) with six material parameters, \( \mu_e, \lambda_e, \mu_c, \alpha_1, \alpha_2, \alpha_3 \), is a general gauge model for dislocations in a linear isotropic medium, it contains some interesting special cases based on particular assumptions on the material moduli. Special cases are:

- **Force stresses are symmetric**: \( \mu_c = 0 \). This case is a particular case \((H = 0\), i.e. no specific microstress\) of the relaxed model.

- **Force stresses are symmetric and no axitor**: \( \mu_c = 0, \alpha_3 = 0 \). For this we retrieve a particular case \((H = 0\) of the further relaxed model.

- **Edelen gauge model of dislocations** [63, 64, 65] with the following choice:
  \[
  \alpha_1 = \alpha_2, \quad \alpha_1 = 3 \alpha_3, \quad \mu_c = 0.
  \]  
  (5.47)

- **Popov-Kröner gauge model of dislocations** [66] with the following choice:
  \[
  \alpha_1 = \frac{3 \mu_e (2d)^2}{24}, \quad \alpha_2 = \frac{\mu_e (2d)^2}{24} \frac{3 + \nu}{1 - \nu}, \quad \alpha_3 = 0, \quad \mu_c = 0,
  \]  
  (5.48)

  where \( d \) is a characteristic mesoscopic length, and therefore
  \[
  \alpha_2 = \frac{(3 + \nu) \alpha_1}{3(1 - \nu)}.
  \]  
  (5.49)

- **Einstein choice** [67, 40]
  \[
  \alpha_1 = -\alpha_2, \quad \alpha_1 = -6 \alpha_3, \quad \mu_c = 0.
  \]  
  (5.50)

  It is called the “Einstein choice” since, with this choice, the dislocation energy, \( \frac{1}{2} \langle m, \alpha \rangle \), is equivalent (up to a boundary term) to the three-dimensional Einstein-Hilbert Lagrangian (e.g. [60, 40]). Further comments regarding the Einstein choice are included in Subsection 5.6.

- **Strain gradient-like choice** [42]
  \[
  \alpha_2 = \frac{1 + \nu}{1 - \nu} \alpha_1, \quad \alpha_1 = -6 \alpha_3, \quad \mu_c = 0.
  \]  
  (5.51)

  The interesting feature of this choice is that the solutions of the stress fields of screw and edge dislocations given in [68, 69] in the framework of strain gradient elasticity can be reproduced.

However, the Einstein choice (5.50) and the choice (5.51) do not satisfy the positivity condition (5.27).

In the following two subsections we complete the comparison of the gauge theory of dislocation presented in this paper with some other existing models through the fundamental solution of force stresses and discussing the Einstein choice of the constitutive coefficients.

5.5 Fundamental solution of force stresses and characteristic lengths

In general, a characteristic length is an important dimension that defines the scale of a physical system. For instance, in gradient elasticity the characteristic lengths are in the range of the lattice parameters, that is in the order \( \ell \sim 10^{-10} \) m (see, e.g., [70]). Therefore, such a theory can be used for understanding the nano-mechanical phenomena at such length scales. In all generalized elasticity theories (e.g., micropolar elasticity, gradient elasticity) where the material tensors have different dimensions, characteristic length scales appear (e.g., [8, 71, 6]). Thus, the existing characteristic length scales are given in terms of the material tensors with different dimensions. Their structure can be seen directly in characteristic field equations and they appear explicitly in the construction of the Green tensors (fundamental solutions) of the field equations (e.g., [8, 71, 6]). All fundamental solutions of linear generalized elasticity contain the corresponding characteristic length scales as parameters.
Following Lazar and Anastassiadis [42], we give the Green tensor, which is the fundamental solution of equation (5.28), in terms of force stresses \( \hat{\sigma} \). If the Cosserat couple modulus \( \mu_c > 0 \) [72, 73, 74], we have the inverse constitutive relation for \( e \)

\[
e = \frac{\mu_c + \mu_e}{4\mu_e \mu_c} \hat{\sigma} + \frac{\mu_c - \mu_e}{4\mu_e \mu_c} (\hat{\sigma})^T - \frac{\nu}{2\mu_e (1 + \nu)} \text{tr}(\hat{\sigma}) \cdot \mathbb{I},
\]

(5.52)

where the Poisson's ratio \( \nu \) is expressed in terms of the Lamé coefficients \( \lambda_e \) and \( \mu_e \)

\[
\nu = \frac{\lambda_e}{2(\lambda_e + \mu_e)}, \quad \lambda_e = \frac{2\mu_e \nu}{1 - 2\nu}.
\]

(5.53)

Using the force equilibrium condition \( \text{Div} \hat{\sigma} = 0 \) for vanishing forces and (5.52), we write the equation (5.28) only in terms of \( \sigma \)

\[
\square_G \hat{\sigma} = \hat{\sigma}^0,
\]

(5.54)

where \( \square_G : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3} \) is a differential matrix operator defined by

\[
(\square_G \hat{\sigma})_{ij} = \frac{1}{4\mu_e \mu_c} \left[ (c_1 - c_2 + 2c_3) \frac{2 \mu_e \nu}{1 + \nu} - 2c_3 \mu_c \right] (\delta_{ij} \Delta \hat{\sigma}_{ii} - \partial_i \partial_j \hat{\sigma}_{ii}) - [c_1(\mu_e + \mu_c) - c_2(\mu_e - \mu_c)] \Delta \hat{\sigma}_{ij}
\]

\[
+ [c_1(\mu_e - \mu_c) - c_2(\mu_e + \mu_c)] (\partial_j \partial_k \hat{\sigma}_{ki} - \Delta \hat{\sigma}_{ji}) + [2c_2 \mu_e - c_3(\mu_e + \mu_c)] \partial_i \partial_j \hat{\sigma}_{ki} + 4\mu_e \mu_c \hat{\sigma}_{ij},
\]

(5.55)

and

\[
\partial_i = \frac{\partial}{\partial x_i}, \quad \Delta = \frac{\partial^2}{\partial x_i}, \quad c_1 := \frac{1}{3}(2\alpha_1 + 3\alpha_3), \quad c_2 := \frac{1}{3}(3\alpha_3 - \alpha_1), \quad c_3 := \frac{1}{2}(\alpha_2 - \alpha_1).
\]

(5.56)

A fundamental solution of the equation (5.54) is a matrix field \( \Sigma \in \mathbb{R}^{3 \times 3} \) which satisfies the condition [75]

\[
\square_G \hat{\Sigma} = \delta(x) \cdot L \quad \forall x \in \mathbb{R}^3,
\]

(5.57)

where \( \delta(\cdot) \) is the Dirac delta and \( L \in \mathbb{R}^{3 \times 3} \) has constant components. Hence, the fundamental solution is the solution of equation (5.54) corresponding to a “point body pseudo-moment” or given Dirac-delta point stress of magnitude \( L \). Eventually, the fundamental solution can be written in the following form

\[
\hat{\Sigma} = G \cdot L,
\]

(5.58)

where \( G : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3} \) is a fourth order tensor called the Green tensor of Eq. (5.54). In this way, the three-dimensional Green tensor of Eq. (5.54) is given by (see [42])

\[
G_{ijkl} = \frac{1}{8\pi} \left\{ \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \frac{e^{-r/\ell_1}}{\ell_1^2 r} + \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right) \frac{e^{-r/\ell_4}}{\ell_4^2 r} - \left( \delta_{ij} \Delta - \partial_i \partial_j \right) \delta_{kl} \left[ \frac{1}{r} \left( e^{-r/\ell_1} - e^{-r/\ell_4} \right) \right] \right\},
\]

(5.59)

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \) and

\[
\ell_1^2 = \frac{\alpha_1}{2 \mu_e}, \quad \ell_2^2 = \frac{(1 - \nu) \alpha_2}{2 \mu_e (1 + \nu)}, \quad \ell_3^2 = \frac{(\mu_e + \mu_c)(\alpha_1 + \alpha_2)}{8 \mu_e \mu_c}, \quad \ell_4^2 = \frac{\alpha_1 + 6\alpha_3}{6 \mu_c}.
\]

(5.60)

The solution (5.58) with (5.64) represents the three-dimensional force stress field of a point stress. Therefore, we solved the (force) stress problem of a concentrated body pseudo-moment (or concentrated background stress \( \hat{\sigma}^0 \)) for vanishing body forces in an unbounded material, provided the invertibility formula (5.52) holds true. Using the convolution theorem, we obtain the particular solution of the force stress tensor \( \hat{\sigma} \) as convolution of the Green tensor with \( \hat{\sigma}^0 \)

\[
\hat{\sigma}_{ij} = G_{ijkl} \ast \hat{\sigma}^0_{kl},
\]

(5.61)
where $\ast$ denotes the convolution. With the help of this equation, three-dimensional problems can be solved for any given $\bar{\sigma}^0$ or any body pseudo-moment tensor$^6$. From Eq. (5.60) it can be seen that in the Lazar-Anastassiadiis dislocation model four characteristic lengths can be defined in terms of the six material parameters of an isotropic material, $\mu_e, \lambda_e, \mu_c, \alpha_1, \alpha_2, \alpha_3$. In addition, the lengths $\ell_1, \ell_2$ and $\ell_3$ fulfill the following relation

$$
\ell_3^2 = \frac{\mu_e + \mu_c}{4\mu_e} \left( \ell_1^2 + \frac{1 + \nu}{1 - \nu} \ell_2^2 \right).
$$

(5.62)

Thus, four characteristic length scales exist in the static and isotropic gauge theory of dislocations. The characteristic length $\ell_1$ depends on $\mu_e$ and is similar in the form to the internal length in the couple stress theory [5]. Moreover, it can be seen that $\ell_2$ depends on the Poisson’s ratio $\nu$. Therefore, the length $\ell_2$ is the characteristic length of dilatation. Because the characteristic lengths $\ell_3$ and $\ell_4$ depend on $\mu_c$ they look like the two characteristic lengths of micropolar elasticity, namely the characteristic lengths for bending and torsion (see, e.g., [71]). The parameter of the axitor $\alpha_3$ gives only a contribution to the characteristic length $\ell_4$. For $\mu_c \rightarrow \infty$, only the bending length, which is the characteristic length in the theory of couple stresses, survives

$$
\lim_{\mu_c \rightarrow \infty} \ell_3^2 = \frac{\alpha_1 + \alpha_2}{8 \mu_e}, \quad \lim_{\mu_c \rightarrow \infty} \ell_4^2 = 0.
$$

(5.63)

On the other hand, if $\mu_c \rightarrow 0$, $\mu_c \geq 0$, then $\ell_3$ and $\ell_4$ diverge. Moreover, for $\mu_c \rightarrow 0$, $\mu_c \geq 0$

$$
G_{ijkl} \rightarrow \frac{1}{8\pi} \left\{ (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) \frac{e^{-r/\ell_1}}{\ell_1 r} - (\delta_{ij}\Delta - \partial_i \partial_j) \delta_{kl} \left[ \frac{1}{r} (e^{-r/\ell_1} - e^{-r/\ell_2}) \right] 
- (\delta_{ij}\partial_i + \delta_{jl}\partial_j) \delta_{kh} \left[ \frac{1}{r} (e^{-r/\ell_1} - 1) \right] \right\}.
$$

(5.64)

For symmetric force stresses, the only relevant characteristic lengths are $\ell_1$ and $\ell_2$ which are given in terms of the material parameters $\alpha_1, \alpha_2, \mu_c$ and $\nu$. For that reason the axitor might be neglected in the case of symmetric force stress model like in the further relaxed model. However, for $\mu_c \rightarrow 0$ and $\bar{\sigma}$ symmetric the constitutive equation (5.26) is not invertible. Only the symmetric elastic strain sym $e$ may be determined as function of $\bar{\sigma}$. Therefore, Lazar’s approach towards special solutions via Green’s function needs finally the invertibility of the force stress $\sigma$ as function of $e$. This is only possible for $\mu_c > 0$. Nevertheless, our theorem 5.1 provides existence and uniqueness for $\mu_c = 0$ in bounded domains.

In the Edelen choice (5.47), the characteristic lengths (5.60) become

$$
\ell_1^2 = \frac{\alpha_1}{2 \mu_e}, \quad \ell_2^2 = \frac{(1 - \nu) \alpha_1}{2 \mu_e (1 + \nu)}, \quad \ell_3^2 \text{ not defined, } \ell_4^2 \text{ not defined}.
$$

(5.65)

The lengths $\ell_1$ and $\ell_2$ coincide with the lengths $M^{-1}$ and $N^{-1}$ introduced by Kadić and Edelen [64] and Edelen and Lagoudas [65] in the dislocation gauge theory. Due to $\mu_c \rightarrow 0$, $\mu_c \geq 0$ (symmetric force stresses), $\ell_3$ and $\ell_4$ formally do not exist.

For the so-called Popov-Kröner choice (5.48) and (5.49), the characteristic lengths (5.60) become

$$
\ell_1^2 = \frac{\alpha_1}{2 \mu_c}, \quad \ell_2^2 = \frac{(3 + \nu) \alpha_1}{6 \mu_c (1 + \nu)}, \quad \ell_3^2 \text{ not defined, } \ell_4^2 \text{ not defined}.
$$

(5.66)

Again due to $\mu_c = 0$ (symmetric force stresses), $\ell_3$ and $\ell_4$ formally do not exist.

For the so-called Einstein choice (5.50), the characteristic lengths (5.60) reduce to

$$
\ell_1^2 = \frac{\alpha_1}{2 \mu_c}, \quad \ell_2^2 = -\frac{(1 - \nu) \alpha_1}{2 \mu_c (1 + \nu)}, \quad \ell_3^2 = 0, \quad \ell_4^2 = 0.
$$

(5.67)

Thus, only two characteristic lengths survive. But the length $\ell_2$ is now imaginary. These lengths $\ell_1$ and $\ell_2$ agree with the lengths $M^{-1}$ and $N^{-1}$ used by Malyshhev [67].

For the strain gradient-like choice (5.51), the characteristic lengths (5.60) modify to

$$
\ell_1^2 = \ell_2^2 = \frac{\alpha_1}{2 \mu_e}, \quad \ell_3^2 \text{ not defined, } \ell_4^2 \text{ not defined}.
$$

(5.68)

---

$^6$Only for an unbounded domain and $\mu_c > 0$. 
Here, $\ell_1^2 = \ell_2^2$ reproduces the characteristic length of gradient elasticity theory of Helmholtz type (see [69]). Since $\mu_c \rightarrow 0$, $\mu_c \geq 0$ (symmetric force stresses), $\ell_3$ and $\ell_4$ are not defined.

Therefore, in contrast with the Teisseyre’s model [76] (see the Subsection 5.6), the dislocation model proposed by Lazar and Anastassiads [42, 77] is more general since it allows asymmetric stresses. As we have seen, Edelen [64, 65, 78], Malyshev [67] and Lazar [40] discussed and used some conditions upon the constitutive coefficients in order to obtain a model for dislocations with symmetric force stress. Malyshev [67] and Lazar [40] used the so-called Einstein choice in three dimensions. One important difference between the asymmetric and the symmetric gauge theoretical models of dislocations is that the asymmetric model possesses four characteristic length scale parameters while a symmetric model (e.g. Einstein choice) has only two characteristic length scale parameters [42].

5.6 On the Einstein choice

In order to write the equations (5.28) in terms of the divergence operator (see Eqs. (1)-(4) and (36)-(38) from [76] and Eq. (3.39) from [56]) the following moment stress tensor has been introduced

$$\Lambda_{pkl} = \alpha_1 \epsilon_{prn}(\epsilon_{kpn}\delta_{kl} - \epsilon_{kln}\delta_{pl}) + \alpha_2 \epsilon_{pkn}\alpha_{ln} + \alpha_3 (\epsilon_{pln}\alpha_{kn} - \epsilon_{kln}\alpha_{pn}) \tag{5.69}.$$ 

In a previous paper [1] we have identified the constitutive coefficients of the dislocation energy in the Eringen-Claus model [45, 56, 3] with the coefficients in our isotropic case, namely

$$\alpha_1 = a_2 - a_3, \quad \alpha_2 = a_2 - a_3 - 2a_1, \quad \alpha_3 = \frac{2a_3 + a_2}{3}. \tag{5.70}$$

Imposing the additional assumption that the moments of rotations have to vanish, Teisseyre [79] also requires that the corresponding differences between the stress moment tensor components and body couples appearing in the equation vanish. This is the reason why Teisseyre [79] assumed that

$$\Lambda_{pkl,p} = \Lambda_{pklt}, \quad M_{lk} = M_{kl}. \tag{5.71}$$

In order to satisfy (5.71), Teisseyre considered the following sufficient condition

$$a_2 = -a_3, \quad a_1 = -2a_3. \tag{5.72}$$

In terms of our notations and in the dislocation gauge theory such a condition reads [40, 42]

$$\alpha_2 = -\alpha_1, \quad \alpha_3 = -\frac{\alpha_1}{6}, \tag{5.73}$$

and this is called the Einstein choice in three dimensions [40].

Moreover, using the assumption (5.71), it is natural to introduce the tensor

$$m_{kl} = \frac{1}{2} \epsilon_{klm} \Lambda_{mln}, \tag{5.74}$$

and further, written in terms of the operator Curl, it turns into

$$m = a_3 \text{tr} (\text{Curl } P) \cdot \mathbf{1} + 2a_1 \text{ skew } \text{Curl } P + (a_2 - a_3) (\text{Curl } P)^T. \tag{5.75}$$

In terms of $m$, the equation (5.28) may be rewritten as

$$\sigma^0 = -\text{Curl} (m^T) + 2\mu_e \text{ sym}(\nabla u - P) + 2\mu_e \text{ skew}(\nabla u - P) + \lambda_e \text{ tr}(\nabla u - P) \cdot \mathbf{1}. \tag{5.76}$$

In other words (see [1]), equation (5.71) demands that $m$ is such that

$$\text{Curl}(m^T) \in \text{Sym}(3). \tag{5.77}$$

Hence, in view of (5.75), the previous constraint (5.77) means that

$$\text{Curl}[a_1 \text{ dev } \text{sym } \text{Curl } P + a_2 \text{ skew } \text{Curl } P + a_3 \text{ tr}(\text{Curl } P) \cdot \mathbf{1}] \in \text{Sym}(3), \tag{5.78}$$

The above symmetry condition was studied in [1] and the following result has been established:

---

7In fact the condition $a_2 = a_1 + a_3$ is necessary and sufficient to satisfy (5.71) if $P \in \text{Sym}(3)$. In addition, in another paper [79], Teisseyre assumed that $a_3 = 0$ which removes the effects of the micro-dislocation tensor $\alpha = -\text{Curl } P$ completely.
Remark 5.3

i) If \( \alpha_1 = -6 \alpha_3 \) and \( \alpha_2 = 6 \alpha_3 \), then
\[
\text{Curl}\{ \alpha_1 \text{ dev sym Curl } P + \alpha_2 \text{ skew Curl } P + \alpha_3 \text{ tr(Curl } P) \cdot 1 \} \in \text{Sym}(3) \quad \forall \ P \in \mathbb{R}^{3x3}.
\] (5.79)

\[
i) \text{Given } P \in \text{Sym}(3), \text{ then we have}
\[
\text{Curl}\{ \alpha_1 \text{ dev sym Curl } P + \alpha_2 \text{ skew Curl } P + \alpha_3 \text{ tr(Curl } P) \cdot 1 \} \in \text{Sym}(3)
\] (5.80)

\[
\text{if and only if } \alpha_1 = -\alpha_2.
\]

Thus, the Einstein choice (5.73) implies that
\[
\text{Curl}[m^T] \in \text{Sym}(3) \quad \text{for all } \ P \in \mathbb{R}^{3x3}.
\] (5.81)

It is obvious that the Einstein choice (5.73) violates the conditions (5.27) and (2.34) (see also [42]). The conditions (5.73) were used by Malyshev [67] and Lazar [40] in order to investigate dislocations with symmetric force stress. Using the Einstein choice (5.73), the constitutive relation (5.26) reduce to [40]
\[
m = \alpha_1 \kappa,
\] (5.82)

where
\[
\kappa = \alpha^T - \frac{1}{2} \text{tr}(\alpha) \cdot 1
\] (5.83)

is the well-known Nye tensor (e.g. [80, 81]). The inverse is given by
\[
\alpha = \kappa^T - \text{tr}(\kappa) \cdot 1.
\] (5.84)

Then the balance of dislocation stresses (5.12) reads
"\[
\bar{\sigma}^0 = \bar{\sigma} - \alpha_1 \text{ inc (sym } e),
\] (5.85)

where
\[
\text{inc}(\cdot) = \text{Curl}((\text{Curl } \cdot)^T)
\]
denotes the incompatibility operation, which is defined as the Curl from the right and the Curl from the left acting on a tensor of rank two [82]. The tensor inc (sym } e) is equivalent to the (linearized) three-dimensional Einstein tensor. Eq. (5.85) may be decomposed into the symmetric and the skew-symmetric parts
\[
\text{sym} \bar{\sigma}^0 = \bar{\sigma} - \alpha_1 \text{ inc (sym } e)
\]
\[
\text{skew} \bar{\sigma}^0 = \text{skew } \bar{\sigma}.
\] (5.86)

Hence, if \( \mu_c = 0 \), then the Einstein choice (5.73) implies that \( \bar{\sigma}^0 \in \text{Sym}(3) \). Using the Einstein choice and \( \mu_c = 0 \), the gauge theoretical dislocation model possesses only symmetric force stresses and no moment stresses and is described by Eq. (5.86).

Remark 5.4 There are no general existence result for the minimization problem corresponding to the Einstein choice, since in this case the internal density energy is not positive definite.

---

8In the Lazar’s original notations the conditions (5.73) becomes \( a_2 = -a_1 \) and \( a_3 = -\frac{a_1}{2} \)
6 Conclusion

The first purpose of this paper is to prove that the equilibrium problem in two relaxed models of micromorphic elastic materials have unique solutions in a suitable Hilbert space. To this aim, for each model, we use some new specific inequalities [35, 36, 37, 38]. The novelty of the considered models in comparison with other existing models was discussed in details in [1, 4, 83, 57]. The second purpose of this paper is to identify more similarities/differences between the considered relaxed models and dislocation gauge theory. We consider some specific problems and some specific solutions and we discuss some assumption made in the constructions of different models. We also give a novel existence result in the dislocation gauge theory. Moreover, we point out how each considered model may be obtained starting from the general form of the energy of micromorphic elastic solids through some specific form of the general constitutive coefficients when allowing for only positive semi-definite tensors.

Acknowledgements

Ionel-Dumitrel Ghiba acknowledges support from the Romanian National Authority for Scientific Research (CNCS-UEFISCDI), Project No. PN-II-ID-PCE-2011-3-0521. Markus Lazar gratefully acknowledges the grants from the Deutsche Forschungsgemeinschaft (Grant Nos. La1974/2-2, La1974/3-1).

References

P. Neff, D. Pauly, and K.J. Witsch. Poincaré meets Korn via Maxwell: Extending Korn's first inequality to incompatible


ESAIM: COCV


P. Neff. Existence of minimizers for a finite-strain micromorphic elastic solid.


V. Girault and P.A. Raviart. Journal of Biomechanical Engineering

J.F.C. Yang and R.S. Lakes. Transient study of couple stress in compact bone: torsion,.

Phil. Mag.


