On the temporal behaviour in the bending theory of porous thermoelastic plates

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In this paper we consider the bending theory of Mindlin type thermoelastic plates with voids. We study the temporal behaviour of the solution of the boundary-initial value problem of this theory. Assuming that the internal energy density is positive definite, relations describing the asymptotic behaviour of the Cesàro means of various parts of total energy are established. An extension of the results to a large class of thermoelastic materials with voids is given.

1 Introduction

Goodman and Cowin [1] developed a continuum theory of granular materials. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. The volume fraction field represents a kinematical variable assigned to each material particle and it is an additional degree of kinematic freedom. Nunziato and Cowin [2] and Leşan [3] (see also [4]) used the representation proposed by Goodman and Cowin [1] in order to elaborate a theory for the treatment of thermoelastic materials with voids. The basic idea of this theory is to suppose there is a distribution of voids throughout the body.

The theory of thermoelastic materials with voids is a special theory in the framework of the theories of materials with microstructure [5, 6]. The intended applications of this theory are to geological materials and to manufactured porous materials, as well as granular materials. A presentation of this theory can be found in [7].

The theories of thin bodies have been widely developed in the last decades. A special attention has been paid to include some terms in the basic formulation of thin bodies theories in order to reflect the microstructure of materials [8–13].

In the present paper we consider the bending theory of porous thermoelastic plates introduced by Birsan [8]. To study the temporal behaviour of the solution of the boundary–initial value problem we consider the method developed by Chiriţă [14] in classical linear thermoelasticity (see also [15]). In this aim the Cesàro means of various parts of the total energy were introduced. We establish relations describing the asymptotic behaviour of the mean energies. We prove that the mean thermal energy tends to zero as time goes to infinity and the asymptotic equipartition occurs between the Cesàro means of the kinetic and internal energies.

First, the results are obtained under the assumption that the internal energy density is positive definite. Then, we prove that the results hold true for a larger class of materials with voids. In fact, we show that a method used by Chiriţă [19, 20] in the study of spatial behaviour of solutions of some three-dimensional problems can be applied to obtain information about the temporal behaviour in the bending theory of porous thermoelastic plates under milder conditions upon the constitutive coefficients. The method presented here is believed to be successfully used for the study of materials with negative Poisson’s ratio (anti-rubber, dilational materials or auxetic materials) which are most useful in biomechanics [16–18]. Necessary and sufficient conditions characterizing the strong ellipticity of materials with voids were obtained by Chiriţă and Ghiba [21]. These materials were then considered in several works for study of the propagation of progressive waves [21], inhomogeneous waves [22] and seismic waves [23] within the framework of the linear theory of porous media.

We have to outline that, in the linear thermoelasticity of material with voids, Chiriţă and Scalia [24] have studied the asymptotic behaviour of the thermoelastic processes.

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2 Preliminaries. Auxiliary results

We consider a fixed system of rectangular Cartesian axes \( Ox_i \) \((i = 1, 2, 3)\). Throughout this paper Latin indices have the range 1, 2, 3, Greek indices have the range 1, 2. Typical conventions for differential operations are implied such as a superposed dot or comma followed by a subscript in order to denote the partial derivative with respect to time or to the corresponding cartesian coordinate, respectively.

Let us consider the region \( \Sigma \times \left[ -\frac{h}{2}, 0 \right] \) of the physical space \( \mathbb{R}^3 \), where \( \Sigma \) is a domain in \( \mathbb{R}^2 \) whose boundary \( \partial \Sigma \) is a simple \( C^2 \)-curve and \( 0 < h = \text{constant} \ll \text{diam} \Sigma \). We call this region plate with the thickness \( h \).

We assume that \( B \) is the interior of the right cylinder \( \Sigma \times \left[ -\frac{h}{2}, \frac{h}{2} \right] \), and we consider that it is filled by an isotropic and homogeneous thermoelastic material with voids.

Let \( u = (u_1, u_2, u_3) \) be the displacement field, \( \varphi \) the change in volume fraction from the reference volume [1], \( T_0 \) the absolute temperature in the reference state and \( \theta \) the temperature variation from the absolute temperature \( T_0 \) over \( B \). In what follows we will assume that the functions \( u_i, \varphi \) and \( \theta \) are such that:

i) \( u_i \) and \( \varphi \) are of class \( C^2 \) on \( B \times [0, \infty) \);
ii) \( \theta \) is of class \( C^{2,1} \) on \( B \times [0, \infty) \);
iii) \( u_i, \varphi \) and \( \theta \) are of class \( C^1 \) on \( \overline{B} \times [0, \infty) \).

In the bending theory of Mindlin type plates with voids [8, 13] (see also [25–29]), an admissible process is a state of bending on \( B \times [0, \infty) \) provided

\[
\begin{align*}
   u_\alpha &= x_3 v_\alpha(x_1, x_2, t), \\
   u_3 &= w(x_1, x_2, t), \\
   \varphi &= x_3 \psi(x_1, x_2, t), \\
   \theta &= x_3 T(x_1, x_2, t)
\end{align*}
\]

on \( B \times [0, \infty) \).

We consider that the average body forces, the average extrinsic forces, the average moments and the average heat source over \( \left[ -\frac{h}{2}, \frac{h}{2} \right] \) vanish [8]. We assume that the forces, the moments and the heat flux on the faces \( x_3 = \pm \frac{h}{2} \) vanish, too.

The basic system of partial differential equations from the bending theory of thermoelastic plates with voids [8, 13] is the following

\[
\begin{align*}
   M_{\beta\alpha,\beta} - h N_{\alpha} &= \rho I \ddot{v}_\alpha, \\
   N_{\beta,\beta} &= \rho \dot{w}, \\
   H_{\alpha,\alpha} + G - h \Gamma &= \rho \chi I \ddot{v}, \\
   Q_{\alpha,\alpha} - h R &= T_0 \sigma, \quad \text{in} \ \Sigma \times (0, \infty),
\end{align*}
\]

where \( I = \frac{h^3}{12}, \rho \) and \( \chi \) are the bulk mass density and the equilibrated inertia in the reference state.

The quantities \( M_{\beta\alpha}, N_{\alpha}, H_{\alpha}, G, \Gamma, Q_{\alpha}, R, \) and \( \sigma \) are defined [8, 13] by

\[
\begin{align*}
   M_{\beta\alpha} &= \int_{-h/2}^{h/2} x_3 t_{\beta\alpha} dx_3, \quad N_{\alpha} = \frac{1}{h} \int_{-h/2}^{h/2} t_{\alpha\alpha} dx_3, \\
   H_{\alpha} &= \int_{-h/2}^{h/2} x_3 h_{\alpha} dx_3, \quad G = \int_{-h/2}^{h/2} x_3 g dx_3, \quad \Gamma = \frac{1}{h} \int_{-h/2}^{h/2} h_{33} dx_3, \\
   Q_{\alpha} &= \int_{-h/2}^{h/2} x_3 q_{\alpha} dx_3, \quad R = \frac{1}{h} \int_{-h/2}^{h/2} q_{33} dx_3, \quad \sigma = \int_{-h/2}^{h/2} \rho x_3 \eta dx_3.
\end{align*}
\]
where \( t_{ij} \), \( h_i \), \( g \), \( q_c \), and \( \eta \) are quantities from the three dimensional theory [2] which represent the components of the stress tensor, the components of the equilibrated stress vector, the intrinsic equilibrated force, the components of the heat flux vector and the specific entropy, respectively.

Hence, the constitutive equations \([8, 13]\) are given by

\[
\begin{align*}
M_{\alpha\beta} &= I[\lambda \varepsilon_{\gamma\delta_{\alpha\beta}} + 2\mu \varepsilon_{\alpha\beta} + b \psi \delta_{\alpha\beta} - \beta T \delta_{\alpha\beta}], \\
N_{\alpha} &= \mu \gamma_{\alpha}, \\
H_{\beta} &= \alpha I \psi_{,\beta}, \\
\Gamma &= \alpha \psi, \\
G &= -I(b \varepsilon_{\gamma\gamma} + \xi \psi - mT), \\
\sigma &= I(\beta \varepsilon_{\gamma\gamma} + \psi m + aT), \\
Q_{\alpha} &= k I T_{\alpha}, \\
R &= k T, \quad \text{in } \Sigma \times [0, \infty),
\end{align*}
\]

where

\[
\varepsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \quad \gamma_{\alpha} = v_{\alpha} + w_{,\alpha}. \tag{2.5}
\]

The quantities \( \gamma_{\alpha} \) (see e.g. [8, 25]) represent the angles of rotation of the cross–sections \( x_{\alpha} = \text{constant} \) about the middle surface, the shear strains, and \( \lambda, \mu, h, \beta, \alpha, \xi, k, m, \) and \( \alpha \) are constitutive coefficients.

Equations (2.2) can be expressed in terms of \( v_{\alpha}, w, \psi, \) and \( T \). Thus, we obtain

\[
\begin{align*}
I[\mu \Delta v_{\alpha} + (\lambda + \mu)v_{k,\alpha\nu} + b \psi_{,\alpha} - \beta T_{,\alpha}] - \mu h(v_{\alpha} + w_{,\alpha}) &= \rho I \ddot{v}_{\alpha}, \\
\mu \Delta w + \mu w_{,\alpha\nu} &= \rho \ddot{w}, \\
I[\alpha \Delta \psi - b v_{\nu,\alpha} + m T] - (\xi I + \alpha h) \psi &= \rho \chi I \ddot{\psi}, \\
I[k \Delta T - \beta T_0 \dot{v}_{\alpha,\alpha} - m T_0 \dot{\psi}] - k h T &= a IT_0 \ddot{T}, \quad \text{in } \Sigma \times (0, \infty),
\end{align*}
\]

where \( \Delta \) is the Laplace operator.

To the field equations (2.6) we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions

\[
\begin{align*}
v_{\alpha} &= 0, \quad w = 0, \\
\psi &= 0, \quad T = 0, \quad \text{on } \partial \Sigma. \tag{2.7}
\end{align*}
\]

The initial conditions consist in

\[
\begin{align*}
v_{\alpha}(x, 0) &= v_{\alpha}^0(x), \quad \dot{v}_{\alpha}(x, 0) = v_{\alpha}^1(x), \\
w(x, 0) &= w^0(x), \quad \dot{w}(x, 0) = w^1(x), \\
\psi(x, 0) &= \psi^0(x), \quad \dot{\psi}(x, 0) = \psi^1(x), \\
T(x, 0) &= T^0(x), \quad \text{in } \Sigma, \tag{2.8}
\end{align*}
\]

where the functions \( v_{\alpha}^0, v_{\alpha}^1, w^0, w^1, \psi^0, \psi^1, T^0 \) are given continuous functions.

The internal energy density \( W \) per unit area of the middle plane, associated with the kinematic fields \( v_{\alpha}, w \) and with \( \psi \) is defined by

\[
2W = I \left[ \lambda \varepsilon_{\alpha\alpha} \varepsilon_{\beta\beta} + 2\mu \varepsilon_{\alpha\beta} \varepsilon_{\alpha\beta} + \xi \psi^2 + \alpha \psi_{,\beta} \psi_{,\beta} + 2b \psi \varepsilon_{\beta\beta} \right] + h(\mu \gamma_{\alpha} + \alpha \psi^2). \tag{2.9}
\]

Without losing the generality, we can assume that all the quantities involved in this paper are dimensionless. If we do not assume this, in view of linearity of the problem, some new notations which involve the constitutive coefficients can be done in order to obtain a dimensionless form of the equations.

We denote by \( P_0 \), the boundary–initial value problem defined by the relations (2.6)–(2.8).
3 Auxiliary results

Let us consider the following bilinear form

\[ \mathcal{F}(\zeta^{(1)}, \zeta^{(2)}) = \left\{ \begin{array}{l}
\frac{1}{2} \left( \lambda \zeta^{(1)}_{\alpha_{\beta}} \zeta^{(2)}_{\alpha_{\beta}} + 2 \mu \zeta^{(1)}_{\alpha\beta} \zeta^{(2)}_{\alpha\beta} + \xi \phi^{(1)}_{\alpha}(\phi^{(2)}_{\beta} + \alpha \phi^{(1)}_{\alpha} \phi^{(2)}_{\beta} + b(\phi^{(1)}_{\alpha} \phi^{(2)}_{\beta} + \phi^{(2)}_{\alpha} \phi^{(1)}_{\beta})
\right) \\
+ \mu \phi^{(1)}_{\alpha} \phi^{(2)}_{\alpha} + \alpha \phi^{(1)}_{\alpha} \phi^{(2)}_{\alpha}
\end{array} \right\} \]  

(3.1)

for all vectors \( \zeta^{(\alpha)} = \{ \zeta^{(\alpha)}_{11}, \zeta^{(\alpha)}_{12}, \zeta^{(\alpha)}_{21}, \zeta^{(\alpha)}_{22}, \phi^{(\alpha)}_{1}, \phi^{(\alpha)}_{2}, \phi^{(\alpha)}_{1}, \phi^{(\alpha)}_{2}, \phi^{(\alpha)}_{1}, \phi^{(\alpha)}_{2}, \phi^{(\alpha)}_{3} \} \). We associate with the solution of the problem \( P_0 \), the kinetic energy

\[ K(t) = \frac{1}{2} \int_{\Sigma} \rho(I\dot{v}_\alpha(t)v_\alpha(t) + h\dot{u}^2(t) + \chi I\dot{\psi}^2(t))da, \]

(3.2)

the internal energy

\[ U(t) = \int_{\Sigma} \Psi(t)da, \]

(3.3)

the dissipation energy

\[ D(t) = \int_{0}^{t} \frac{k}{T_0} \left[ (I T_{\alpha}(\tau) T_{\alpha}(\tau) + h T^2(\tau)) \right] \int_{0}^{\tau} \int_{\Sigma} \right], \]

(3.4)

the thermal energy

\[ S(t) = \frac{1}{2} \int_{\Sigma} aIT^2(t)da, \]

(3.5)

and the total energy

\[ E(t) = K(t) + U(t) + D(t) + S(t). \]

(3.6)

From the work [8] we have the following energy conservation law

**Lemma 3.1.** Suppose that \( v^{(0)}_\alpha, w^{(0)}, \psi^{(0)} \in W_1(\Sigma), v^{(1)}_\alpha, w^{(1)}, \psi^{(1)} \in W_0(\Sigma) \) and \( T^{(0)} \in W_0(\Sigma) \). Let \( (v_\alpha, w, \psi, T) \) be a solution of the boundary-initial value problem \( P_0 \). Then the following energy conservation holds:

\[ E(t) = E(0), \quad \text{for all} \ t \in [0, \infty). \]

(3.7)

If \( f \) is a continuous function on \( \Sigma \times [0, \infty) \), then we denote by \( \tilde{f} \) the function defined by

\[ \tilde{f}(x, t) = \int_{0}^{t} f(x, \tau) d\tau, \quad (x, t) \in \Sigma \times [0, \infty). \]

(3.8)

Let us introduce the function

\[ J(t) = \frac{1}{2} \int_{\Sigma} \rho(I\dot{v}_\alpha(t)v_\alpha(t) + h\dot{u}^2(t) + \chi I\dot{\psi}^2(t))da + \frac{k}{T_0} \int_{0}^{t} \int_{\Sigma} \left[ hT^2(\tau) + I T_{\alpha}(\tau) T_{\alpha}(\tau) \right] d\tau. \]

(3.9)

In this section we establish some identities of Lagrange-Brun type [30], in terms of the function \( J(\cdot) \), that are essential in investigating the problem of asymptotic partition of total energy.

**Lemma 3.2.** Suppose that \( v^{(0)}_\alpha, w^{(0)}, \psi^{(0)} \in W_1(\Sigma), v^{(1)}_\alpha, w^{(1)}, \psi^{(1)} \in W_0(\Sigma) \) and \( T^{(0)} \in W_0(\Sigma) \). Let \( (v_\alpha, w, \psi, T) \) be a solution of the boundary-initial value problem \( P_0 \). Then the following identity holds:

\[ \frac{dJ}{dt}(t) = \frac{dJ}{dt}(0) + 2 \int_{0}^{t} [K(\tau) - W(\tau)] d\tau + \int_{0}^{t} \int_{\Sigma} \sigma(0)T(\tau)d\tau, \quad \text{for all} \ t \in [0, \infty). \]

(3.10)
In view of (2.2) and (3.1), we have
\[
\zeta = \frac{2}{T_0}(t) + h\Gamma(t)\psi(t) + \sigma(t)T(t).
\]
In view of relations (2.4), we deduce that
\[
H(t) = 2W(t) + aIT^2(t).
\]
Let us remark that the energy equation can be written in the following form
\[
\tilde{Q}_{\alpha,\alpha} - h\tilde{R} = T_0\sigma - T_0\sigma(0).
\]
By using the equations of motion (2.2) and the Eq. (3.13), we obtain
\[
H(t) = [M_{\beta\alpha}(t)\psi_{\alpha}(t) + \psi_{\beta}(t)w(t) + H_{\beta}(t)\psi(t) + \frac{1}{T_0}\tilde{Q}_{\beta}(t)T(t)],_\beta
\]
\[
- \rho(I\dot{\psi}_\alpha(t)\psi_{\alpha}(t) + h\dot{w}(t)w(t) + \chi I\dot{\psi}(t)\psi(t)) - \frac{1}{T_0}\tilde{Q}_{\alpha}(t)T_{\alpha}(t) - \frac{h}{T_0}\tilde{R}(t)T(t) + \sigma(0)T(t).
\]
Thus, in view of the relations (3.12), (3.14), and (2.4), using the divergence theorem we have
\[
\frac{d}{dt} \int_{\Omega} \rho[I\dot{\psi}_\alpha(t)\psi_{\alpha}(t) + h\dot{w}(t)w(t) + \chi I\dot{\psi}(t)\psi(t)] da
\]
\[
+ \int_{\Gamma} \left[ \frac{k}{T_0} IT_{\alpha}(t)T_{\alpha}(t) + \frac{k}{T_0} h\tilde{T}(t)T(t) \right] da
\]
\[
= 2(K(t) - W(t) - S(t)) + \int_{\Sigma} \sigma(0)T(t) da
\]
and the proof is complete. \qed

**Lemma 3.3.** Suppose that \( \psi_\alpha^0, w^0, \psi^0 \in W_1(\Sigma), \psi_\alpha^1, w^1, \psi^1 \in W_0(\Sigma) \) and \( T^0 \in W_0(\Sigma) \). Let \( (\psi_\alpha, w, \psi, T) \) be a solution of the boundary–initial value problem \( P_0 \). Then the following identity holds:
\[
\frac{dJ}{dt}(t) = \frac{1}{2} \left\{ \int_{\Sigma} \left[ \left. \rho[I\dot{\psi}_\alpha(2t)v_\alpha(0) + \dot{v}_\alpha(0)v_\alpha(2t)] + h\rho[\dot{w}(2t)w(0) + \dot{w}(0)w(2t)] + \rho\dot{\psi}(2t)\psi(0) + \dot{\psi}(0)\psi(2t) \right] da \right. \\
+ \int_0^t \int_{\Sigma} \sigma(0)[T(t - \tau) - T(t + \tau)] da d\tau \right\}, \text{ for all } t \in [0, \infty).
\]

**Proof.** Let us define
\[
L(t, \tau) = M_{\beta\alpha}(t)\psi_{\beta\alpha}(\tau) + h\psi_{\beta}(\tau) - G(t)\psi(\tau) + H_{\beta}(t)\psi_{\beta}(\tau) + h\Gamma(t)\psi(\tau) - \sigma(t)T(\tau).
\]
In view of (2.2) and (3.1), we have
\[
\dot{L}(t, \tau) = 2F(\zeta(t), \zeta(\tau)) - \dot{I}[m(T(t)\psi(\tau) + T(\tau)\psi(t)) \\
+ \beta(T(t)\psi_{\beta}(\tau) + T(\tau)\psi_{\beta}(t)) + aT(t)T(\tau)]
\]
where
\[
\zeta(s) = \left\{ \sqrt{T\dot{\psi}_{11}(s)}, \sqrt{T\dot{\psi}_{22}(s)}, \sqrt{T\dot{\psi}_{12}(s)}, \sqrt{T\psi_{21}(s)}, \sqrt{T\psi_{11}(s)}, \sqrt{T\psi_{22}(s)}, \sqrt{T\psi_{12}(s)}, \sqrt{T\psi_{21}(s)} \right\}.
\]
Let us remark that
\[
L(t, \tau) = L(\tau, t).
\]
On the other hand, using the relations (2.2), (2.4), and (3.13), the divergence theorem and the boundary conditions, we deduce that

\[
\int_{\Sigma} L(t, \tau) da = - \int_{\Sigma} \left\{ \rho I \ddot{v}_a(t)v_a(\tau) + \rho h \ddot{w}(t)w(\tau) + \rho \chi \ddot{\psi}(t)\psi(\tau) + k \frac{1}{T_0} \dot{I} \bar{T}_a(t)T(\tau) + k \frac{1}{T_0} h \ddot{\bar{T}}(t)T(\tau) + \sigma(0)T(\tau) \right\} da. \tag{3.20}
\]

Using the above relation we have

\[
\int_{\Sigma} [L(t + \tau, t - \tau) - L(t - \tau, t + \tau)] da \\
= - \int_{\Sigma} \left\{ \rho I \ddot{v}_a(t)v_a(\tau) - \ddot{v}_a(t - \tau)v_a(\tau) + \rho h \ddot{w}(t - \tau)w(t + \tau) - \ddot{w}(t - \tau)w(t + \tau) + \rho \chi \ddot{\psi}(t - \tau)\psi(t + \tau) - \ddot{\psi}(t - \tau)\psi(t + \tau) + k \frac{1}{T_0} \dot{I} \bar{T}_a(t)T(\tau) - \bar{T}_a(t - \tau)T(\tau) + k \frac{1}{T_0} h \ddot{\bar{T}}(t)T(\tau) - \ddot{\bar{T}}(t - \tau)T(t + \tau) + \sigma(0)T(\tau) - T(t + \tau) \right\} da. \tag{3.21}
\]

For any two functions \( f, h \in C^2([0, \infty)) \), the following identities hold

\[
\begin{align*}
\int_0^t \ddot{f}(t + \tau)h(t - \tau) d\tau &= \ddot{f}(2t)h(0) - \ddot{f}(t)h(t) + \int_0^t \ddot{h}(t - \tau) \ddot{f}(t + \tau) d\tau, \\
\int_0^t \dddot{f}(t + \tau)h(t - \tau) d\tau &= -h(0) \ddot{f}(2t) + f(t)h(t) + \int_0^t \dddot{f}(t + \tau)h(t - \tau) d\tau.
\end{align*}
\tag{3.22}
\]

We integrate the relation (3.21) over \([0, t]\) and then we use the relations (3.22). This direct integration leads to the relation (3.16) and the proof is complete. \( \square \)

### 4 Asymptotic partition of energy in the Cesàro sense

In this section we study the asymptotic partition of energy for the solution of the problem \( \mathcal{P}_0 \).

Throughout this section, we suppose that the internal energy density \( \mathcal{W} \) is a positive definite quadratic form in terms of the variables \( \{ \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \gamma_1, \gamma_2, \psi, \psi_1, \psi_2 \} \).

This is true if and only if the constitutive coefficients satisfy the conditions

\[
\mu > 0, \ \alpha > 0, \ \lambda + \mu > 0, \ \xi(\lambda + \mu) > b^2. \tag{4.1}
\]

We have that

\[
\mathcal{F}(\mathbf{e}, \mathbf{e}) = \mathcal{W}, \tag{4.2}
\]

for

\[
\mathbf{e} = \left\{ \sqrt{\mathcal{I}_{\varepsilon_{11}}}, \sqrt{\mathcal{I}_{\varepsilon_{22}}}, \sqrt{\mathcal{I}_{\varepsilon_{12}}}, \sqrt{\mathcal{I}_{\varepsilon_{21}}}, \sqrt{\mathcal{I}_{\psi_1}}, \sqrt{\mathcal{I}_{\psi_2}}, \sqrt{\mathcal{I}_{\psi}}, \sqrt{\mathcal{I}_{\gamma_1}}, \sqrt{\mathcal{I}_{\gamma_2}}, \sqrt{\mathcal{I}_{\psi_1}} \right\}. \tag{4.3}
\]

In view of the assumptions (4.1) we have that there are two positive constants \( \nu_m \) and \( \nu_M \) such that

\[
\nu_m (\zeta \alpha^2 + \phi^2 + \phi_\beta \phi_\beta) + \mu \theta_\alpha \theta_\alpha + \alpha \tau^2 \leq 2\mathcal{F}(\zeta, \zeta) \leq \nu_M (\zeta \alpha^2 + \phi^2 + \phi_\beta \phi_\beta) + \mu \theta_\alpha \theta_\alpha + \alpha \tau^2, \tag{4.4}
\]

for all vectors \( \zeta = \{ \zeta_{11}, \zeta_{22}, \zeta_{12}, \zeta_{21}, \phi_1, \phi_2, \phi, \phi_1, \phi_2, \tau \} \in \mathbb{R}^{10} \).
If \((v_\alpha, w, \psi, T)\) is the solution of the problem \(\mathcal{P}_0\), for the different energies, we introduce the following Cesàro means

\[
\begin{align*}
K_c(t) &= \frac{1}{t} \int_0^t K(s)ds, \\
W_c(t) &= \frac{1}{t} \int_0^t W(s)ds, \\
S_c(t) &= \frac{1}{t} \int_0^t S(s)ds, \\
D_c(t) &= \frac{1}{t} \int_0^t D(s)ds.
\end{align*}
\] (4.5)

The Cesàro mean values of the kinetic and internal energy have been introduced by Day [31] while the Cesàro mean values of the thermal and dissipation energy have been introduced by Chiriţă [14].

Using the identities (3.7), (3.10), and (3.16), we establish the relations that describe the asymptotic behaviour of the mean energies (4.5). The following theorem proves that the mean thermal energy tends to infinity and the asymptotic behaviour of the mean dissipation energy is given by this theorem.

**Theorem 4.1.** Let \((v_\alpha, w, \psi, T)\) be a solution of the boundary–initial value problem \(\mathcal{P}_0\). If the internal energy density is positive definite, then for all choices of the initial data so that \(v_\alpha^1, w^0, \psi^0 \in W_1(\Sigma), v_\alpha^0, w^1, \psi^1 \in W_0(\Sigma)\) and \(T^0 \in W_0(\Sigma)\), we have

\[
\begin{align*}
\lim_{t \to \infty} S_c(t) &= 0, \\
\lim_{t \to \infty} K_c(t) &= \lim_{t \to \infty} W_c(t), \\
\lim_{t \to \infty} D_c(t) &= E(0) - 2 \lim_{t \to \infty} K_c(t) = E(0) - 2 \lim_{t \to \infty} W_c(t).
\end{align*}
\] (4.6)

\[(4.7)\]

\[(4.8)\]

**Proof.** Because \((v_\alpha, w, \psi, T)\) is a solution of the boundary–initial value problem \(\mathcal{P}_0\), from (3.7) we have

\[
K_c(t) + W_c(t) + D_c(t) + S_c(t) = E(0), \text{ for all } t \in (0, \infty).
\] (4.9)

On the other hand, the relations (3.10) and (3.16) imply

\[
K_c(t) - W_c(t) - S_c(t) = -\frac{1}{2t} \frac{dI}{dt}(0) + \frac{1}{4t} \left( \int_\Sigma \rho I [\dot{v}_\alpha(2t)v_\alpha(0) + \dot{v}_\alpha(0)v_\alpha(2t)] \\
+ h\rho [\dot{w}(2t)w(0) + \dot{\psi}(0)w(2t)] + \rho \chi [\dot{\psi}(2t)\psi(0) + \dot{\psi}(0)\psi(2t)] da \\
+ \int_0^t \int_\Sigma \sigma(0)[T(t - \tau) - T(t + \tau)] d\sigma d\tau \right) = -\frac{1}{2t} \int_0^t \int_\Sigma \sigma(0)T(\tau) d\sigma d\tau, \text{ for all } t \in (0, \infty).
\] (4.10)

On the basis of the Poincaré inequality and the relation (3.7), we deduce that

\[
S_c(t) \leq \frac{aT_0}{4T_k} \max \left\{ \frac{C}{T}, \frac{1}{h} \right\} \int_0^t \int_\Sigma \left( \frac{k}{T_0} I T_{,\alpha}(\tau) T_{,\alpha}(\tau) + \frac{k}{T_0} hT(\tau)T(\tau) \right) d\sigma d\tau
\leq \frac{aT_0}{4T_k} \max \left\{ \frac{C}{T}, \frac{1}{h} \right\} E(0), \text{ for all } t \in (0, \infty),
\] (4.11)

where \(C\) is a positive constant.

By letting \(t\) tend to infinity and using that \(v_\alpha^0, w^0, \psi^0 \in W_1(\Sigma), v_\alpha^1, w^1, \psi^1 \in W_0(\Sigma)\), and \(T^0 \in W_0(\Sigma)\) we get the relation (4.6).

Further, by means of the Schwarz inequality and the Poincaré inequality we deduce

\[
\int_\Sigma \rho I \dot{v}_\alpha(0)v_\alpha(2t) da \leq [2K(0)]^{1/2} \left( \int_\Sigma \rho I v_\alpha(2t)v_\alpha(2t) da \right)^{1/2}
\leq [2E(0)]^{1/2} \left( \int_\Sigma v_\alpha,\beta(2t)v_\alpha,\beta(2t) da \right)^{1/2}.
\] (4.12)
Using the divergence theorem and the boundary conditions, we obtain
\[
\int_{\Sigma} \varepsilon_{\alpha\beta}(t) \varepsilon_{\alpha\beta}(t) da \geq \frac{1}{2} \int_{\Sigma} [v_{\alpha\beta}(t)v_{\alpha\beta}(t) + v_{\beta\alpha}(t)v_{\alpha\beta}(t)] da \\
\geq \frac{1}{2} \int_{\Sigma} [v_{\alpha\beta}(t)v_{\alpha\beta}(t) + v_{\alpha\beta}(t)v_{\beta\alpha}(t)] da \\
\geq \frac{1}{2} \int_{\Sigma} v_{\alpha\beta}(t)v_{\alpha\beta}(t) da.
\] (4.13)

Hence, we have
\[
\int_{\Sigma} \rho I \dot{v}_{\alpha}(0)v_{\alpha}(2t) da \leq [2E(0)]^{1/2} (\rho C)^{1/2} \left( \int_{\Sigma} 2I\varepsilon_{\alpha\beta}(2t) \varepsilon_{\alpha\beta}(2t) da \right)^{1/2}.
\] (4.14)

Because the internal energy is positive definite, from the relations (4.4) and (3.7) we obtain
\[
\int_{\Sigma} \rho I \dot{v}_{\alpha}(0)v_{\alpha}(2t) da \leq [2E(0)]^{1/2} \left( \frac{\rho C}{v_m} \right)^{1/2} \left( \int_{\Sigma} 4W(2t) da \right)^{1/2}
\] (4.15)
\[
\leq 2 \left( \frac{2\rho C}{v_m} \right)^{1/2} E(0).
\]

We also have
\[
\int_{\Sigma} \rho I \dot{v}_{\alpha}(2t)v_{\alpha}(0) da \leq [2K(2t)]^{1/2} \left( \int_{\Sigma} \rho I v_{\alpha}(0)v_{\alpha}(0) da \right)^{1/2}
\] (4.16)
\[
\leq [2E(0)]^{1/2} \left( \int_{\Sigma} \rho I v_{\alpha}(0)v_{\alpha}(0) da \right)^{1/2}.
\]

Thus, we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_{\Sigma} \rho I [\dot{v}_{\alpha}(0)v_{\alpha}(2t) + \dot{v}_{\alpha}(2t)v_{\alpha}(0)] da = 0.
\] (4.17)

The Schwarz inequality and the relation (3.7) give us
\[
\int_{\Sigma} \rho h \dot{w}(2t) w(0) da \leq 2E(0)]^{1/2} \left( \int_{\Sigma} \rho h w(0) w(0) da \right)^{1/2}.
\] (4.18)

On the other hand, the Schwarz inequality, the Poincaré inequality and the relation (3.7) give us
\[
\int_{\Sigma} \rho h \dot{w}(0) w(2t) da \leq [2E(0)]^{1/2} (\rho C)^{1/2} \left( \int_{\Sigma} h w_{\alpha\alpha}(2t) w_{\alpha\alpha}(2t) da \right)^{1/2}.
\] (4.19)

We have that
\[
\int_{\Sigma} h \gamma_{\alpha}(t) \gamma_{\alpha}(t) da \geq \int_{\Sigma} h \left( \frac{1}{\delta} - 1 \right) v_{\alpha}(t)v_{\alpha}(t) + (1 - \delta) w_{\alpha}(t)w_{\alpha}(t) da, \text{ for all } 0 < \delta < 1,
\] (4.20)
which imply
\[
\frac{1}{1 - \delta} \int_{\Sigma} h \gamma_{\alpha}(t) \gamma_{\alpha}(t) da + \frac{\delta}{\delta} \int_{\Sigma} h v_{\alpha}(t)v_{\alpha}(t) da \geq \int_{\Sigma} h w_{\alpha}(t)w_{\alpha}(t) da.
\] (4.21)

In view of (4.1), (4.4), and (4.13) we have
\[
\int_{\Sigma} h v_{\alpha}(t)v_{\alpha}(t) da \leq \frac{2hC}{T} \int_{\Sigma} I \varepsilon_{\alpha\beta}(t) \varepsilon_{\alpha\beta}(t) da.
\] (4.22)
Moreover, using (4.4) and (4.1) we have
\[ \int_{\Sigma} h(\gamma(t)\gamma(t)da \leq \frac{1}{\mu} \int_{\Sigma} 2W(t)da. \] (4.23)
Thus, in view of (3.7), (4.22), and (4.23) we have
\[ \int_{\Sigma} h(\gamma(t)\gamma(t)da \leq \frac{2}{\mu} E(0), \]
\[ \int_{\Sigma} h(\gamma(t)\gamma(t)da \leq \frac{4hC}{I_{\nu_{n}}} E(0). \] (4.24)
Using the relations (4.21) and (4.24) we have
\[ \int_{\Sigma} \rho w_{x}w_{x}da \leq 2\rho E(0) \left[ \frac{1}{\mu(1-\delta)} + \frac{2hC}{I_{\nu_{m}} \delta} \right], \text{ for all } 0 < \delta < 1. \] (4.25)
The relations (4.19), (4.21), and (4.24) lead to
\[ \int_{\Sigma} \rho w_{x}w_{x}da \leq 2E(0)(\rho C)^{1/2} \left[ \frac{1}{\mu(1-\delta)} + \frac{2hC}{I_{\nu_{m}} \delta} \right]^{1/2}, \text{ for all } 0 < \delta < 1. \] (4.26)
Moreover, using (3.7), (4.1), and (4.4), we deduce
\[ \int_{\Sigma} \rho w_{x}w_{x}da \leq 2E(0) \right]^{1/2} \left( \int_{\Sigma} \rho x \psi^{2}(da) \right)^{1/2}, \] (4.27)
\[ \int_{\Sigma} \rho x \psi^{2}(da) \right]^{1/2} \left( \int_{\Sigma} \rho x \psi^{2}(2t)da \right)^{1/2} \leq \left( K(0) \frac{\rho x}{ah} \right)^{1/2} \left( \int_{\Sigma} \rho x \psi^{2}(2t)da \right)^{1/2} \leq 2E(0) \left( \frac{\rho x}{ah} \right)^{1/2}. \] (4.28)
Using the inequality (3.7) we get
\[ \int_{t_{0}}^{t} \int_{\Sigma} \sigma(0)[T(t-\tau) - 2T(\tau)]d\sigma d\tau \leq 3t^{1/2} \left[ \frac{T}{2k} \max \left\{ \frac{C}{T}, \frac{1}{h} \right\} \right]^{1/2} \sup_{x \in \Sigma} |\sigma(0)| \] (4.29)
and
\[ \int_{t_{0}}^{t} \int_{\Sigma} \sigma(0)[T(t+\tau)d\sigma d\tau \leq (2t)^{1/2} \left[ \frac{T}{2k} \max \left\{ \frac{C}{T}, \frac{1}{h} \right\} \right]^{1/2} \sup_{x \in \Sigma} |\sigma(0)|. \] (4.30)
The method used above gives us that \( \frac{df}{dt}(0) \) is bounded. We make \( t \to \infty \) in (4.10) and we use the relation (4.11), (4.17), (4.18), (4.26)–(4.30) to obtain the desired relation (4.7). Relation (4.8) is a direct consequence of the relations (4.6), (4.7) and (4.9).

5 Some extensions

In this section we prove that the results established in the previous section hold true for a larger class of materials with voids. We consider the class of materials with voids for which the constitutive coefficients satisfy the following inequalities
\[ \alpha > 0, \mu > 0, \xi > 0, \lambda + 2\mu > 0, \xi \max \left\{ \frac{\lambda + 2\mu}{2}, \frac{2\lambda + 3\mu}{2} \right\} > b^{2}. \] (5.1)
This class of materials is included in the class of strongly elliptic materials [21] and includes the class of materials for which the internal energy density \( \mathcal{V} \) is a positive definite quadratic form.

To this aim we consider the following cases:

i) \( \alpha > 0, \mu > 0, 2\lambda + 3\mu > 0, \xi(2\lambda + 3\mu) > 2b^{2} \),

ii) \( \alpha > 0, \mu > 0, \lambda < 0, \lambda + 2\mu > 0, \xi(\lambda + 2\mu) > 2b^{2} \).
5.1 The case $\alpha > 0, \mu > 0, 2\lambda + 3\mu > 0, \xi(2\lambda + 3\mu) > 2h^2$

For thermoelastic materials with voids characterized by inequalities i) we have the following result:

**Theorem 5.1.** Let $(v_0, w, \psi, T)$ be a solution of the boundary–initial value problem $\mathcal{P}_0$. If the constitutive coefficients satisfy the assumptions i), then for all choices of the initial data so that $v_0^0, w^0, \psi^0 \in W_1(\Sigma), v_0^1, w^1, \psi^1 \in W_0(\Sigma)$, and $T^0 \in W_0(\Sigma)$, the Cesaro mean energies (4.5) have the asymptotic behaviour given by (4.6), (4.7), and (4.8).

**Proof.** In order to treat the temporal behaviour under hypothesis i) upon the constitutive constants of the material, we write the basic equations (2.6) in the form

$$M^{(1)}_{\beta\alpha, \beta} - hN_\alpha = \rho I\ddot{v}_\alpha,$$  \hspace{1cm} (5.2)

where

$$M^{(1)}_{\beta\alpha} = I[\mu v_{\alpha, \beta} + (\lambda + \mu)v_{\alpha, \gamma}\delta_{\alpha\beta} + b\psi\delta_{\alpha\beta} - \beta T\delta_{\alpha\beta}].$$ \hspace{1cm} (5.3)

Let us define

$$2W_1(t) = M^{(1)}_{\beta\alpha}(t)v_{\alpha, \beta}(t) + hN_\alpha(t)\gamma_\alpha(t) + H_\alpha(t)\psi_{\alpha, \beta}(t) - G(t)\psi(t) + h\Gamma(t)\psi(t) + I(\beta v_{\gamma, \gamma}(t) + m\psi(t))T(t).$$ \hspace{1cm} (5.4)

In view of the boundary conditions and using the relations (5.2) and (2.2), it is easy to deduce

$$\int_\Sigma M^{(1)}_{\beta\alpha}(t)v_{\alpha, \beta}(t)da = -\int_\Sigma [hN_\alpha(t) + \rho I\ddot{v}_\alpha(t)]da = \int_\Sigma M_{\beta\alpha}(t)v_{\alpha, \beta}(t)da.$$ \hspace{1cm} (5.5)

Hence, we obtain

$$\int_\Sigma W_1(t)da = \int_\Sigma W(t)da.$$ \hspace{1cm} (5.6)

Let us consider the quadratic forms

$$\mathcal{F}_1 = I[\lambda + 2\mu(v_{1,1}^2 + v_{2,2}^2) + 2(\lambda + \mu)v_{1,2}v_{2,2} + \xi\psi^2 + 2b\psi(v_{1,1} + v_{1,2})],$$

$$\mathcal{F}_2 = \mu(v_{1,2}^2 + v_{2,1}^2),$$

in the variables $v_{1,1}, v_{2,2}, \psi$ and $v_{1,2}, v_{2,1}$, respectively.

In view of the assumptions i), we can say that the quadratic forms $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite quadratic forms.

Clearly, we can find two positive constants $k_m$ and $k_M$ such that

$$k_m I(v_{\alpha, \beta}v_{\alpha, \beta} + \psi^2) \leq \mathcal{F}_1 + \mathcal{F}_2 \leq k_M I(v_{\alpha, \beta}v_{\alpha, \beta} + \psi^2).$$ \hspace{1cm} (5.8)

Moreover, we have

$$2W_1 = \mathcal{F}_1 + \mathcal{F}_2 + h(\mu\gamma_\alpha \gamma_\alpha + \alpha\psi^2).$$ \hspace{1cm} (5.9)

Hence

$$k_m I(v_{\alpha, \beta}v_{\alpha, \beta} + \psi^2) + h(\mu\gamma_\alpha \gamma_\alpha + \alpha\psi^2) \leq 2W_1 \leq k_M I(v_{\alpha, \beta}v_{\alpha, \beta} + \psi^2) + h(\mu\gamma_\alpha \gamma_\alpha + \alpha\psi^2).$$ \hspace{1cm} (5.10)

The Lagrange-Brun inequalities established in Sect. 3 were proved without any assumption upon the constitutive coefficients.

In the proof of Theorem 4.1 we have used the assumption that the internal energy is positive definite in the deduction of the relations (4.15), (4.26), and (4.28). In the following, we show that similar inequalities can be obtained under the conditions i) upon the constitutive coefficients.

To this aim let first remark that the inequalities (4.13) and (4.22) are satisfied without any assumption upon the constitutive coefficients.

Using the inequalities (5.10), (4.13), and (4.24) we deduce

$$\int_\Sigma h\gamma_\alpha(t)\gamma_\alpha(t)da \leq \frac{1}{\mu} \int_\Sigma 2W_1(t)da,$$

$$\int_\Sigma hv_\alpha(t)v_\alpha(t)da \leq \frac{hC}{T} \int_\Sigma \int_\Sigma I\varepsilon_{\alpha\beta}(t)\varepsilon_{\alpha\beta}(t)da \leq \frac{2hC}{Tk_m} \int_\Sigma W_1(t)da.$$ \hspace{1cm} (5.11)
Using the relations (4.21) and (5.6) we have
\[
\int_{\Sigma} \rho k \dot{w}(0) w(2t) da \leq 2E(0)\rho(0)^{1/2} \left[ \frac{1}{\mu(1 - \delta)} + \frac{2hC}{Tk_m \delta} \right]^{1/2}, \quad \text{for all } 0 < \delta < 1.
\] (5.12)

Similar arguments lead to the inequality
\[
\int_{\Sigma} \rho \chi \dot{\psi}(0) \psi(2t) da \leq \left( 2K(0) \frac{\rho X}{\alpha h} \right)^{1/2} \left( \int_{\Sigma} 2W_I(2t) \right)^{1/2}
\leq 2E(0) \left( \frac{\rho X}{\alpha h} \right)^{1/2}.
\] (5.13)

Thus, relations similar with (4.26) and (4.28) hold true under the assumption i).

Moreover, the relations (4.14) and (5.10) lead to the inequality
\[
\int_{\Sigma} \rho I \dot{v}_0(0) v_0(2t) da \leq \left( 2E(0) \right)^{1/2} \left( \int_{\Sigma} 2W_I(2t) \right)^{1/2}.
\] (5.14)

So, in view of (5.6) we have
\[
\int_{\Sigma} \rho I \dot{v}_0(0) v_0(2t) da \leq 2 \left( \frac{\rho C}{K_m} \right)^{1/2} E(0).
\] (5.15)

The conclusions of the present theorem follow from the relations (4.9)–(4.12), (4.16), (4.18), (4.27), (4.29), (4.30), (5.12), (5.13), (5.15) and the boundedness of $\frac{dI}{dt}(0)$. \qed

### 5.2 The case $\alpha > 0$, $\mu > 0$, $\lambda < 0$, $\lambda + 2\mu > 0$, $\xi(\lambda + 2\mu) > 2b^2$.

In this section we give a result which describe the thermal behaviour of the mean energies (4.5) for the thermoelastic materials with voids characterized by the assumptions ii).

**Theorem 5.2.** Let $(v_0, w, \psi, T)$ be a solution of the boundary–initial value problem $P_0$. If the constitutive coefficients satisfy the assumptions ii), then for all choices of the initial data so that $v_0^0, w^0, \psi^0 \in W_1(\Sigma), v_0^1, w^1, \psi^1 \in W_0(\Sigma)$, and $T^0 \in W_0(\Sigma)$, the Césaro mean energies (4.5) have the asymptotic behaviour given by (4.6), (4.7), and (4.8).

**Proof.** In order to treat the temporal behaviour under the hypotheses ii) upon the constitutive constants of the material we write the basic equations (2.6) in the form
\[
M_{\beta\alpha}^{(2)} - h N_\alpha = \rho I \ddot{v}_\alpha,
\] (5.16)

where
\[
M_{\beta\alpha}^{(2)} = I [\mu v_\alpha + (\lambda + \mu) v_\beta + b \psi \delta_\alpha \beta - \beta T \delta_\alpha \beta].
\] (5.17)

We consider the energy
\[
2W_2(t) = M_{\beta\alpha}^{(2)} v_\alpha(t) + h N_\alpha(t) \gamma_\alpha(t) + H_\alpha(t) \psi_\alpha(t) - G(t) \psi(t) + h \Gamma(t) \psi(t)
+ I(\beta v_\gamma \psi(t) + m \psi(t)) \Gamma(t).
\] (5.18)

In view of the boundary conditions and using the relations (5.16) and (2.2), we have
\[
\int_{\Sigma} W_2(t) da = \int_{\Sigma} W(t) da.
\] (5.19)

Let us consider the quadratic form
\[
Q_1 = I[(\lambda + 2\mu)(v_{1,1}^2 + v_{2,2}^2) + \xi \psi^2 + 2b \psi(v_{1,1} + v_{2,2})],
\]
\[
Q_2 = I[(\lambda + 2\mu)(v_{1,2}^2 + v_{2,1}^2) + 2(\lambda + \mu) v_{1,2} v_{2,1}],
\] (5.20)
in the variables \(v_{1,1}, v_{2,2}, \psi\) and \(v_{1,2}, v_{2,1}\), respectively.

We have
\[
2W_2 = Q_1 + Q_2 + h(\mu\gamma_\alpha\gamma_\alpha + \alpha\psi^2).
\] (5.21)

The quadratic forms \(Q_1\) and \(Q_2\) are positive definite quadratic forms if and only if the inequalities ii) are satisfied.

Thus, we can find two positive constants \(\epsilon_m\) and \(\epsilon_M\) such that
\[
\epsilon_m I(w_{\alpha,\beta}w_{\alpha,\beta} + \psi^2) + h(\mu\gamma_\alpha\gamma_\alpha + \alpha\psi^2) \leq 2W_2 \leq \epsilon_M I(w_{\alpha,\beta}w_{\alpha,\beta} + \psi^2) + h(\mu\gamma_\alpha\gamma_\alpha + \alpha\psi^2).
\] (5.22)

In the following, the proof of the present theorem becomes similar as the proof of Theorem 5.1. Using (4.13), (4.21), (4.26), and (5.19) we will have
\[
\int_{\Sigma} \rho \dot{\psi}(0)\psi(2t)dt \leq 2E(0)(\rho C)^{1/2} \left[ \frac{1}{\mu(1-\delta)} + \frac{2hC}{T\epsilon_m\delta} \right]^{1/2}, \text{ for all } 1 < \delta < 1
\]
\[
\int_{\Sigma} \rho I_{\alpha}(0)\psi(2t)dt \leq 2E(0) \left( \frac{\rho \chi}{\alpha I} \right)^{1/2}
\]
\[
\int_{\Sigma} \rho \dot{\psi}(0)\dot{\psi}(2t)dt \leq 2E(0) \left( \frac{\rho \chi}{\alpha I} \right)^{1/2}
\]

Thus, the conclusions of the present theorem follow from the relations (4.9)–(4.12), (4.16), (4.18), (4.27), (4.29), (4.30), (5.23) and the boundedness of \(\frac{dI}{dt}(0)\).

5.3 Concluding result

We have the following main result:

**Theorem 5.3.** Let \((v_{\alpha}, w, \psi, T)\) be a solution of the boundary–initial value problem \(P_0\). If the constitutive coefficients satisfy the assumptions (5.1), then for all choices of the initial data so that \(v_{\alpha}^0, w^0, \psi^0 \in W_1(\Sigma), v_{\alpha}^1, w^1, \psi^1 \in W_0(\Sigma)\) and \(T^0 \in W_0(\Sigma)\), the Césaro mean energies (4.5) have the asymptotic behaviour given by (4.6), (4.7), and (4.8).

**Proof.** The proof is a direct consequence of the Theorems 5.1 and 5.2. To prove this fact, let us remark that

\[
\max \left\{ \frac{\lambda + 2\mu}{2}, \frac{2\lambda + 3\mu}{2} \right\} = \begin{cases} 
\frac{\lambda + 2\mu}{2} & \text{if } \lambda + \mu > 0, \\
\frac{2\lambda + 3\mu}{2} & \text{if } \lambda + \mu > 0.
\end{cases}
\] (5.24)

We split our analysis in two cases: a) \(\lambda + \mu > 0\) and b) \(\lambda + \mu \leq 0\).

a) If \(\lambda + \mu > 0\), then the inequalities (5.1) imply
\[
\alpha > 0, \quad \mu > 0, \quad \xi > 0, \quad \lambda + \mu > 0, \quad \xi(2\lambda + 3\mu) > 2b^2.
\] (5.25)

The class of materials for which the constitutive coefficients satisfy the inequalities (5.25) is included in the class of materials considered in the Sect. 5.1.

In consequence, if \(\lambda + \mu > 0\), then the Césaro mean energies (4.5) have the asymptotic behaviour given by (4.6), (4.7), and (4.8).

b) If \(\lambda + \mu \leq 0\), then the inequalities (5.1) imply
\[
\alpha > 0, \quad \mu > 0, \quad \xi > 0, \quad -\mu \geq \lambda > -2\mu, \quad \xi(\lambda + 2\mu) > 2b^2.
\] (5.26)

In this case, the class of materials for which the constitutive coefficients satisfy the inequalities (5.26) is included in the class of materials considered in the Sect. 5.2.

Hence, if \(\lambda + \mu \leq 0\), then the Césaro mean energies (4.5) have the asymptotic behaviour given by (4.6), (4.7), and (4.8), too.

In conclusion, the Césaro mean energies (4.5) have the asymptotic behaviour given by (4.6), (4.7), and (4.8) for the class of materials with voids characterized by the inequalities (5.1).
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References