On the deformation of transversely isotropic porous elastic circular cylinder

I.-D. GHIBA

“Octav Mayer” Mathematics Institute
Romanian Academy of Science, Iaşi Branch
Bd. Carol I, nr. 8, 700506-Iaşi, Romania
e-mail: ghiba_dumitrel@yahoo.com

In this paper we study the deformation of right circular cylinders filled by a linear transversely isotropic porous material. We construct a solution of the relaxed Saint–Venant’s problem using the results established for anisotropic porous cylinders. Firstly, we decompose the relaxed Saint–Venant’s problem into two problems: extension–bending–torsion problem and flexure problem. Then, for each of them we give the exact expressions of the solutions.

Key words: transversely isotropic porous materials; relaxed Saint–Venant’s problem; circular cylinder.
Mathematics Subject Classifications (2000): 74E10, 74G05, 74G75, 74G99.

Copyright © 2009 by IPPT PAN

1. Introduction

The theory of elastic materials with voids is a special case of that for materials with microstructure. In this paper we consider the theory of materials with voids introduced by Cowin and Nunziato [1]. The basic idea of this theory is to suppose that there is a continuous distribution of voids throughout the elastic body. In this theory, the bulk density is written as the product of two fields: the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom and it was employed previously by Goodman and Cowin [2] to develop a continuum theory of granular materials. The first investigations in the theory of thermoelastic materials with voids are due to Nunziato and Cowin [3] and Ieșan [4]. The intended applications of the theory are geological materials and manufactured porous materials. A presentation of this theory can be found in [5, 6].

Saint–Venant’s problem consists of determining the equilibrium of an elastic cylinder loaded by surface forces distributed over its plane ends. In the relaxed Saint–Venant’s problem the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. Using the method introduced by Toupin [7] in classical elasticity, Batra and
Yang [8] have proved that these changes of the ends conditions produce negligible errors, except possibly near the ends.

In this paper we consider the relaxed Saint–Venant’s problem for right circular cylinders made of a transversely isotropic homogeneous elastic material with voids. An elastic material is a transversely isotropic material [9] if at each point there is a principal direction and an infinite number of principal directions in the plane normal to the first direction. The case of transversely isotropic materials is an important branch of applied mathematics and engineering science. In [10] Ding et al. have presented the methods to study different types of problems which arise in the theory of transversely isotropic elastic materials. Besides the well-known applications of this type of material in the mechanics of rocks [11]–[16], the transversely isotropic materials are very useful in many branches of biology [17]–[21]. The recent studies of fiber-reinforced composites [22, 23] and the modern technologies also encourage the study of transversely isotropic materials.

For the treatment of the deformation of a right circular cylinder filled with a transversely isotropic porous material, we use the results established by Ghiba [24]. These results are established using the method described by Ieşan in the books [25, 26]. This method gives a possibility to reduce the Saint–Venant’s problem to some generalized plane strain problems. In fact, in the paper [24], two classes of semi-inverse solutions were described in the set of solutions of Saint–Venant’s problem that may be expressed in terms of solutions of some generalized plane strain problems. We use these classes obtained in the anisotropic case to solve the tension, bending, torsion and flexure problems of transversely isotropic porous elastic circular cylinders.

We outline that a study of Saint–Venant’s problem for homogeneous and isotropic porous elastic cylinders has been presented by Dell’Isola and Batra [27]. The semi-inverse method used in the present paper has been employed to study the Saint–Venant’s problem for different types of materials in the papers [28]–[33].

2. Formulation of the problem

We consider a right circular cylinder of length $L$ and radius $a$, occupied by an homogeneous porous, transversely isotropic elastic material. We denote by $B$ the interior of cylinder, by $\partial B$ the boundary of $B$ and by $D \subset \mathbb{R}^2$ the interior of the bounded cross-section. As shown in Fig. 1, we choose a rectangular Cartesian system $Ox_1x_2x_3$ so that the $Ox_3$-axis is parallel to the generator of the cylinder and $O$ is the center of one of its ends. The lateral boundary of the cylinder is $\Pi = \partial D \times (0, L)$ and $D_0$ and $D_L$ are, respectively, the cross-sections located at $x_3 = 0$ and $x_3 = L$. 
The Latin subscripts and superscripts are understood to range over the integers 1, 2, 3, unless we specify else, whereas Greek subscripts and superscripts are confined to the range 1, 2; summation over repeated subscripts is implied and comma followed by a subscript denotes partial derivative with respect to the corresponding Cartesian coordinate; where no confusion may occur, we suppress the dependence upon the spatial variables.

Let \( \mathbf{u} \) be the displacement field over \( B \) and \( \varphi \) the volume distribution function [2]. We denote by \( \mathbf{U} \) the four-dimensional vector \((u_i, \varphi)\). The linear strain measure \( e_{ij} \) is given by

\[
e_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).
\]

The components of the stress tensor, the components of the equilibrated stress vector and the intrinsic equilibrated body force for anisotropic porous material [1] are

\[
t_{ij}(\mathbf{U}) = C_{ijrs}e_{rs} + B_{ij}\varphi + D_{ijr}\varphi_r,
\]

\[
h_i(\mathbf{U}) = D_{rsi}e_{rs} + d_i\varphi + A_{ij}\varphi_j,
\]

\[
g(\mathbf{U}) = -B_{ij}e_{ij} - \xi\varphi - d_i\varphi_i,
\]

where \( C_{ijrs}, B_{ij}, A_{ij}, D_{ijk}, d_i \) and \( \xi \) are the constitutive coefficients which satisfy the symmetry relations

\[
C_{ijrs} = C_{rsij} = C_{jirs}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \quad D_{ijk} = D_{jik}.
\]

We suppose that the axis \( Ox_3 \) is an axis of elastic symmetry and the planes normal to this axis are planes of isotropy. In the case of transverse isotropy, the
mechanical response of the body remains unaffected due to arbitrary rotations from the direction of $Ox_3$ and due to reflections from the planes perpendicular to this direction. Thus, the symmetry group is $[34]$

\begin{equation}
(2.4) \quad \mathcal{G} = \left\{ \begin{bmatrix}
\pm \cos \theta & -\sin \theta & 0 \\
\pm \sin \theta & \cos \theta & 0 \\
0 & 0 & \pm 1
\end{bmatrix}; \; \theta \in [0, 2\pi) \right\}.
\end{equation}

For this class of materials we have only ten non-zero independent constitutive coefficients

\begin{equation}
(2.5) \quad c_{ij} \equiv C_{iijj}, \quad i,j \in \{1, 2, 3\} \text{ (not summed)}, \quad c_{11} = c_{22}, \quad c_{13} = c_{23},
\end{equation}

\begin{align*}
&c_{41} \equiv C_{2323} = C_{1313}, \quad b_1 \equiv B_{11} = B_{22}, \quad b_3 \equiv B_{33}, \\
&a_1 \equiv A_{11} = A_{22}, \quad a_3 \equiv A_{33} \text{ and } \xi.
\end{align*}

We note that in the case of isotropic materials with voids, the number of independent constitutive coefficients is five [1].

The constitutive equations (2.2), in the case of transversely isotropic materials are reduced to

\begin{align*}
t_{11}(\mathbf{U}) &= c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + b_1\varphi, \\
t_{22}(\mathbf{U}) &= c_{12}e_{11} + c_{22}e_{22} + c_{13}e_{33} + b_1\varphi, \\
t_{33}(\mathbf{U}) &= c_{13}e_{11} + c_{13}e_{22} + c_{33}e_{33} + b_3\varphi, \\
t_{12}(\mathbf{U}) &= (c_{11} - c_{12})e_{12}, \\
t_{13}(\mathbf{U}) &= 2c_{44}e_{13}, \\
t_{23}(\mathbf{U}) &= 2c_{44}e_{23}, \\
h_1(\mathbf{U}) &= a_1\varphi, \\
h_2(\mathbf{U}) &= a_1\varphi, \\
h_3(\mathbf{U}) &= a_3\varphi, \\
g(\mathbf{U}) &= -b_1(e_{11} + e_{22}) - b_3e_{33} + \xi\varphi.
\end{align*}

\begin{equation}
(2.6)
\end{equation}

The surface force and the equilibrated stress at a regular point of $\partial B$, are given by

\begin{equation}
(2.7) \quad t_i(\mathbf{U}) = t_{ij}(\mathbf{U})n_j, \quad h(\mathbf{U}) = h_j(\mathbf{U})n_j,
\end{equation}

respectively, where $n_j$ are the components of the outward unit normal to $\partial B$. 
The equilibrium equations, in the absence of the body force and the extrinsic equilibrated body force, are
\begin{equation}
(2.8) \quad t_{ij,j} = 0, \quad h_{ii} + g = 0 \quad \text{in } B.
\end{equation}

Throughout this paper we assume that the internal energy density
\begin{equation}
(2.9) \quad W(U) = \frac{1}{2} c_{11} e_{11}^2 + c_{12} e_{11} e_{22} + c_{13} e_{11} e_{33} + \frac{1}{2} c_{11} e_{22}^2 + c_{13} e_{22} e_{33} + \frac{1}{2} c_{33} e_{33}^2 + 2 c_{44} e_{23}^2 + 2 c_{44} e_{13}^2 + (c_{11} - c_{12}) e_{12}^2 + b_1 e_{11} \varphi + b_1 e_{22} \varphi + b_3 e_{33} \varphi + \frac{1}{2} \xi \varphi^2 + \frac{1}{2} a_1 \varphi_{,1} \varphi_{,1} + \frac{1}{2} a_1 \varphi_{,2} \varphi_{,2} + \frac{1}{2} a_3 \varphi_{,3} \varphi_{,3}
\end{equation}
is positive definite quadratic in terms of $e_{ij}$, $\varphi$ and $\varphi_{,i}$. This is true if and only if
\begin{equation}
(2.10) \quad c_{11} > 0, \quad c_{11} > c_{12} > -c_{11}, \quad (c_{11} + c_{12}) c_{33} > 2c_{13}^2, \quad c_{44} > 0, \quad a_1 > 0, \quad a_3 > 0, \quad \xi [-2c_{13}^2 + (c_{11} + c_{12}) c_{33}] > b_3^2 (c_{11} + c_{12}) - 4b_1 b_3 c_{13} + 2b_1^2 c_{33}.
\end{equation}

The cylinder is assumed to be free from lateral loading, so that the conditions on the lateral surface are
\begin{equation}
(2.11) \quad t_i = 0, \quad h = 0 \quad \text{on } \Pi.
\end{equation}

We consider the loading at the end $D_0$ to be statically equivalent to the given force $\mathbf{R}$ and the given moment $\mathbf{M}$. Then, for $x_3 = 0$, we have the conditions
\begin{equation}
(2.12) \quad \int_D t_{3i}(U) da = -R_i, \quad \int_D \varepsilon_{ijk} x_j t_{3k}(U) da = -M_i.
\end{equation}

From the existence results [5], for the equilibrium, we must have similar conditions at the end $D_L$.

The relaxed Saint–Venant’s problem for $B$ consists in determination of the displacement field $\mathbf{u}$ and the voluminal distribution function $\varphi$ on $B$, solution of the equilibrium equations (2.8), which satisfy the requirements (2.11) and (2.12).

We decompose the relaxed Saint–Venant’s problem (P), into the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ characterized by

$(\mathcal{P}_1)$ (tension–bending–torsion): $R_a = 0$,

$(\mathcal{P}_2)$ (flexure): $R_3 = M_3 = 0$.

In this paper we study the Saint–Venant’s problem reducing the above problems to some generalized plane problems.
By the state of generalized plane strain for the interior of the cross-section domain, \( D \subset \mathbb{R}^2 \), of the considered cylinder, we mean the state in which the displacement field \( w \) and the voluminal distribution \( \psi \) depend only on \( x_1 \) and \( x_2 \):

\[
(2.13) \quad w_i = w_i(x_1, x_2), \quad \psi = \psi(x_1, x_2), \quad (x_1, x_2) \in D.
\]

In this case, the components of the stress tensor, the components of the equilibrated stress vector and the intrinsic equilibrated body force are functions of \( x_1 \) and \( x_2 \).

For a state of generalized plane strain \( W = (w_i(x_1, x_2), \psi(x_1, x_2)), (x_1, x_2) \in D \), we define the operators

\[
S_1(W) = c_{11}w_{1,11} + c_{12}w_{2,21} + \frac{1}{2}(c_{11} - c_{12})(w_{1,22} + w_{2,12}) + b_1\psi_1,
\]

\[
S_2(W) = c_{11}w_{2,22} + c_{12}w_{1,12} + \frac{1}{2}(c_{11} - c_{12})(w_{2,11} + w_{1,21}) + b_1\psi_2,
\]

\[
S_3(W) = w_{3,\alpha\alpha},
\]

\[
C(W) = a_1\psi_{,\alpha} - b_1w_{\alpha,\alpha} - \xi\psi,
\]

\[
H_1(W) = (c_{11}w_{1,1} + c_{12}w_{2,2} + b_1\psi)n_1 + \frac{1}{2}(c_{11} - c_{12})(w_{1,2} + w_{2,1})n_2,
\]

\[
H_2(W) = \frac{1}{2}(c_{11} - c_{12})(w_{2,1} + w_{1,2})n_1 + (c_{11}w_{2,2} + c_{12}w_{1,1} + b_1\psi)n_2,
\]

\[
H_3(W) = w_{3,\alpha}n_\alpha,
\]

\[
D(W) = a_1\psi_{,\alpha}n_\alpha.
\]

(2.14)

In what follows, we construct a solution of the problems \((P_1)\) and \((P_2)\) using the semi-inverse method \([25, 26]\).

3. Construction of solution of the problem \((P_1)\)

Let us consider \( W^{(s)} = (w^{(s)}, \psi^{(s)}), s = 1, 2, 3 \) solutions of the problems characterized by the equations

\[
(3.1) \quad S_i(W^{(s)}) + f_i^{(s)} = 0, \quad C(W^{(s)}) + \epsilon^{(s)} = 0 \quad \text{in } D,
\]

and the boundary conditions

\[
(3.2) \quad H_i(W^{(s)}) = \tilde{T}_i^{(s)}, \quad D(W^{(s)}) = \tilde{H}^{(s)} \quad \text{on } \partial D,
\]
where
\[
\begin{align*}
  f^{(\gamma)}_\alpha &= c_{13} \delta_{\alpha \gamma}, \\
  f^{(3)}_3 &= 0, \\
  f^{(3)}_i &= 0, \\
  \ell^{(\gamma)} &= -b_3 x_\gamma, \\
  \ell^{(3)} &= -b_3, \\
  \tilde{T}^{(\gamma)}_\alpha &= -c_{13} x_\gamma n_\alpha, \\
  \tilde{T}^{(3)}_3 &= 0, \\
  \tilde{T}^{(3)}_\alpha &= -c_{13} n_\alpha, \\
  \tilde{H}^{(i)} &= 0.
\end{align*}
\]

According to the existence results presented in [5], these generalized plane strain problems have solutions.

In view of the results established in [24], we find a solution of the problem \((P_1)\) to be
\[
\text{U}^I = \sum_{s=1}^{4} a_s \text{U}^{(s)},
\]
where the vectors \(\text{U}^{(s)} = (\text{u}^{(s)}, \psi^{(s)})\), \(s = 1, 2, 3, 4\) are defined by
\[
\begin{align*}
  u^{(\beta)}_\alpha &= \frac{1}{2} x^2 \delta_{\alpha \beta} + w^{(\beta)}_\alpha (x_1, x_2), \\
  u^{(3)}_\alpha &= u^{(3)}_\alpha (x_1, x_2), \\
  u^{(3)}_3 &= x^3 + w^{(3)}_3 (x_1, x_2), \\
  u^{(4)}_\alpha &= \varepsilon^{3}_\alpha \beta x_\beta x_3, \\
  u^{(4)}_3 &= 0, \\
  \varphi^{(i)} &= \psi^{(i)}, \quad i = 1, 2, 3, \\
  \varphi^{(4)} &= 0.
\end{align*}
\]
and the unknown constants \(a_s\), \(s = 1, 2, 3, 4\), are solutions of the following algebraic system:
\[
\begin{align*}
  \sum_{s=1}^{4} a_s D_{\alpha s} &= \varepsilon^{3}_\alpha \beta M_\beta, \\
  \sum_{s=1}^{4} a_s D_{3s} &= -R_3, \\
  \sum_{s=1}^{4} a_s D_{4s} &= -M_3.
\end{align*}
\]

with
\[
\begin{align*}
  D_{3s} &= \int_D t^{33} (\text{U}^{(s)}) \ da, \\
  D_{\beta s} &= \int_D x_\beta t^{33} (\text{U}^{(s)}) \ da, \\
  D_{4s} &= \int_D \varepsilon^{3}_\alpha \beta x_\alpha t^{33} (\text{U}^{(s)}) \ da.
\end{align*}
\]
Because the internal energy is positive definite, we can prove [24] that

\begin{equation}
\det(D_{rs}) \neq 0,
\end{equation}

so that the system (3.6) uniquely determines the constants $a_s$, $s = 1, 2, 3, 4$.

In what follows, we solve the three problems defined by relations (3.1)–(3.3).

First, it is easy to see that $W^{(3)}$ defined by

\begin{equation}
w^{(3)}_1 = -\nu_1 x_1, \quad w^{(3)}_2 = -\nu_1 x_2, \quad \psi^{(3)} = -\nu_2, \quad w^{(3)}_3 = 0,
\end{equation}

with

\begin{equation}
\nu_1 = \frac{c_{13} \xi - b_1 b_3}{(c_{11} + c_{12}) \xi - 2b_1}, \quad \nu_2 = \frac{(c_{11} + c_{12}) b_3 - 2c_{13} b_1}{(c_{11} + c_{12}) \xi - 2b_1^2},
\end{equation}

is a solution of the third problem.

Next, we search a solution $W^{(1)}$ of the first problem in the form

\begin{equation}
w^{(1)}_1 = v^{(1)}_1 - \frac{1}{2} \nu_1 (x_1^2 - x_2^2),
\end{equation}

\begin{equation}
w^{(1)}_2 = v^{(1)}_2 - \nu_1 x_1 x_2,
\end{equation}

\begin{equation}
w^{(1)}_3 = 0,
\end{equation}

\begin{equation}
\psi^{(1)} = \phi^{(1)} - \nu_2 x_1,
\end{equation}

where $V^{(1)} = (v^{(1)}_1, v^{(1)}_2, \phi^{(1)})$ is a solution of the problem defined by the equations

\begin{equation}
\mathcal{S}_\alpha(V^{(1)}) = 0, \quad \mathcal{C}(V^{(1)}) = 0 \text{ in } D,
\end{equation}

and the boundary conditions

\begin{equation}
\mathcal{H}_\alpha(V^{(1)}) = 0, \quad \mathcal{D}(V^{(1)}) = a_1 \nu_2 n_1 \text{ on } \partial D.
\end{equation}

To solve the above problem we use the method presented by Ieşan and Nappa in [29]. Thus, we rewrite this problem in the polar coordinates $(r, \theta)$. We denote by $u$ and $v$ the components of the vector $V^{(1)} = (v^{(1)}_1, v^{(1)}_2)$ in polar coordinates. Because the properties of materials are invariants of rotations about
On the deformation of transversely isotropic porous elastic . . .

9

the axis $Ox_3$, the constitutive equations in polar coordinates are

$$
t_{rr}(V^{(1)}) = c_{11}e_{rr}(V^{(1)}) + c_{12}e_{r\theta}(V^{(1)}) + b_1\phi^{(1)},$$

$$
t_{\theta\theta}(V^{(1)}) = c_{12}e_{rr}(V^{(1)}) + c_{11}e_{r\theta}(V^{(1)}) + b_1\phi^{(1)},$$

$$
t_{r\theta}(V^{(1)}) = (c_{11} - c_{12})e_{r\theta}(V^{(1)}),$$

(3.14)

$$
h_r = a_1 \frac{\partial \phi^{(1)}}{\partial r}, \quad h_r = a_1 \frac{\partial \phi^{(1)}}{\partial r},$$

$$
h_\theta = a_1 \frac{1}{r} \frac{\partial \phi^{(1)}}{\partial \theta},$$

$$
g = -b_1 \left[ \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} \right] - \xi\phi^{(1)},$$

where

(3.15) \quad \varepsilon_{rr} = \frac{\partial u}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial v}{\partial \theta} + u \right), \quad \varepsilon_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{1}{r} v \right).$

The equilibrium equations become

$$
\frac{\partial t_{rr}}{\partial r} + \frac{1}{r} \frac{\partial t_{r\theta}}{\partial \theta} + \frac{1}{r} (t_{rr} - t_{\theta\theta}) = 0,
$$

(3.16)

$$
\frac{\partial t_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial t_{\theta\theta}}{\partial \theta} + \frac{2}{r} t_{r\theta} = 0,
$$

$$
\frac{1}{r} \frac{\partial}{\partial r} (rh_r) + \frac{1}{r} \frac{\partial h_\theta}{\partial \theta} + g = 0,
$$

The boundary conditions (3.2) become

(3.17) \quad t_{rr} = 0, \quad t_{r\theta} = 0, \quad h_r = \frac{1}{a} a_1 \nu_2 \cos \theta.

Let us introduce the quantities

(3.18) \quad c_1 = \frac{1}{2} \left( 1 - \frac{c_{12}}{c_{11}} \right), \quad c_2 = \frac{b_1}{c_{11}}.$

As in [29], we search a solution of the above problem in the following form:

(3.19) \quad u(r, \theta) = U^{(1)}(r) \cos \theta, \quad v(r, \theta) = V^{(1)}(r) \sin \theta, \quad \phi(r, \theta) = \Psi^{(1)}(r) \cos \theta,$

where $U^{(1)}, V^{(1)}$ and $\Psi^{(1)}$ are solutions of the following system of differential equation:
\[ r^2 \frac{d^2 U^{(1)}}{dr^2} + r \frac{dU^{(1)}}{dr} - (1 + c_1)U^{(1)} + r(1 - c_1)\frac{dV^{(1)}}{dr} - (1 + c_1)V^{(1)} + c_2 r^2 \frac{d\Psi^{(1)}}{dr} = 0, \]

(3.20) \[ c_1 \left( r^2 \frac{d^2 V^{(1)}}{dr^2} + r \frac{dV^{(1)}}{dr} \right) - (1 + c_1)V^{(1)} - r(1 - c_1)\frac{dU^{(1)}}{dr} - (1 + c_1)U^{(1)} - c_2 r \Psi^{(1)} = 0, \]

\[ a_1 \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\Psi^{(1)}}{dr} \right) - \frac{1}{r^2} \Psi^{(1)} \right] - b_1 \left[ \frac{1}{r} \frac{d}{dr} \left( rU^{(1)} \right) + \frac{1}{r} V^{(1)} \right] - \xi \Psi^{(1)} = 0. \]

It is easy to see that, in view of (2.10), we have

(3.21) \[ \xi - \frac{2b_1^2}{c_{11} + c_{12}} > \frac{(b_3(c_{11} + c_{12}) - 2b_1c_{13})^2}{(c_{11} + c_{12})(-2c_{13}^{2} + (c_{11} + c_{12})c_{33})} > 0. \]

For this type of system, Ieşan and Nappa [29] give the following solution:

\[ U^{(1)} = A_1 + Q_1 A_2 r^2 - \frac{c_2}{2p} A_3[I_0(pr) + I_2(pr)], \]

(3.22) \[ V^{(1)} = -A_1 - Q_2 A_2 r^2 + \frac{c_2}{2p} A_3[I_0(pr) - I_2(pr)], \]

\[ \Psi^{(1)} = A_3 I_1(pr) + \frac{8b_1 c_1}{a(1 - 3c_1)p^2} A_2 r, \]

where \( I_n \) is the modified Bessel functions of order \( n \) and

\[ p^2 = \frac{\xi}{a_1} - \frac{b_1^2}{c_{11} a_1}, \]

(3.23) \[ Q_1 = \frac{1}{1 - 3c_1} \left( 1 - 3c_1 - \frac{3c_1 c_2 b_1}{a_1 p^2} \right), \]

\[ Q_2 = \frac{1}{1 - 3c_1} \left( 3 - 3c_1 - \frac{c_1 c_2 b_1}{a_1 p^2} \right). \]

We note that in view of relation (3.21) and (2.10), it follows that the real number \( p \) is well-defined.

From the boundary conditions (3.17), the unknown constants \( A_2 \) and \( A_1 \) are

(3.24) \[ A_2 = -\frac{2c_1 b_1 I_2(pa) \nu_2 p^2 a_1 (1 - 3c_1)}{\Gamma a^2 p^4 (1 - 3c_1) a_1 I'_1(pa) - 16c_1^2 b_1^2 a_2(pa)}, \]

\[ A_3 = \frac{1}{a p I'_1(pa)} \left( \nu_2 - \frac{8b_1 c_1}{a_1 p^2 (1 - 3c_1)} A_2 \right), \]
where

\[ \Gamma = (2c_{11} + c_{12})Q_1 - c_{12}Q_2 + \frac{8b_2^2c_1}{a_1(1 - 3c_1)p^2}. \]

From the above discussion we can conclude that

\[ w_1^{(1)} = \left( U^{(1)}(r) - \frac{1}{2} \nu_1 r^2 \right) \cos^2 \theta - \left( V^{(1)}(r) - \frac{1}{2} \nu_1 r^2 \right) \sin^2 \theta, \]

\[ w_2^{(1)} = (U^{(1)}(r) + V^{(1)}(r) - r^2) \sin \theta \cos \theta, \]

\[ w_3^{(1)} = 0, \]

\[ \psi^{(1)} = (\Psi^{(1)}(r) - r) \cos \theta \]

is a solution of the first problem defined by (3.1)–(3.3).

Similarly, we can find the solution of the second problem to be

\[ w_1^{(2)} = (U^{(1)}(r) + V^{(1)}(r) - r^2) \sin \theta \cos \theta, \]

\[ w_2^{(2)} = \left( U^{(1)}(r) - \frac{1}{2} \nu_1 r^2 \right) \sin^2 \theta - \left( V^{(1)}(r) - \frac{1}{2} \nu_1 r^2 \right) \cos^2 \theta, \]

\[ w_3^{(2)} = 0, \]

\[ \psi^{(2)} = (\Psi^{(1)}(r) - r) \sin \theta. \]

From relations (3.5), (3.7), (3.9), (3.26) and (3.27) we obtain the components of the matrix \( D_{ij} \) \( 4 \times 4 \)

\[ D_{11} = D_{22} = J, \quad D_{33} = E\pi a^2, \quad D_{44} = \frac{c_{13}\pi a^4}{2}, \]

\[ D_{12} = D_{21} = D_{3\beta} = D_{\beta 3} = D_{4\beta} = D_{\beta 4} = D_{43} = D_{34} = 0, \]

where

\[ J = \frac{\pi}{4}a^4Q + \frac{\pi a}{p^2}(b_3 - c_{13}c_2)A_3(apI_0(pa) - 2I_1(pa)), \]

\[ Q = E + c_{13}(3Q_1 - Q_2)A_2 + \frac{8b_1b_3c_1}{a_1p^2(1 - 3c_1)}A_2, \]

\[ E = -2\nu_1c_{13} - b_3\nu_2 + c_{33}. \]

From the algebraic systems (3.6) we find the unknown constants \( a_s \) to be

\[ a_1 = \frac{M_2}{J}, \quad a_2 = -\frac{M_1}{J}, \quad a_3 = -\frac{R_3}{E\pi a^2}, \quad a_4 = -\frac{M_3}{c_{13}\pi a^4}. \]

With these, we have a complete expression of the solution of the problem \((P_1)\).

In view of the constitutive equations (2.6) we can observe that the solution constructed in this section corresponds to the null-equilibrated stress vector’s values on the ends of the cylinder.
4. Solution of the problem ($P_2$)

In this section we construct a solution of the problem ($P_2$). The form proposed for this solution is suggested by the results presented in [24]. The general type of solution proposed in [24] for anisotropic case and for arbitrary cross-section of the cylinder have a complex form. Using the results established in the previous Section, we propose the following simplified expression for a solution, $U^{II} = (u^{II}, \varphi^{II})$, of the problem ($P_2$):

$$u^{II}_\alpha = -\frac{1}{6} b_\alpha x_3^3 + \sum_{\beta=1,2} x_3 b_\beta u^{(\beta)}_\alpha,$$

$$u^{II}_3 = \frac{1}{2} b_\rho x_\rho x_3^2 + w^*_3,$$

$$\varphi^{II} = \sum_{\beta=1,2} x_3 b_\beta \varphi^{(\beta)},$$

where $W^{(s)} = (w^{(s)}, \psi^{(s)})$ are solutions of the generalized plane strain problem defined in the previous section, $b_\beta$ are unknown constants which will be determined, while the function $w^*_3$ is a solution of the following problem:

$$\Delta w^*_3 = -\sum_{\beta=1,2} b_\beta t^{33}(U^{(\beta)}) - c_{44} \sum_{\beta=1,2} b_\beta w^{(\beta)}_{\alpha,\alpha},$$

$$w^*_{3,\alpha} n_\alpha = -c_{44} \sum_{\beta=1,2} b_\beta w^{(\beta)}_{\alpha} n_\alpha,$$

where $\Delta = \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2}$ is the Laplace operator in two dimensions.

We can observe that, in view of (3.7), (3.28), the necessary condition for the existence of solution of the above Neumann-type problem holds.

With the help of relations (3.4), (3.26) we can rewrite this problem in the polar coordinates $(r, \theta)$

$$\Delta w^*_3 = (Mr + NI_1(pr))(b_1 \cos \theta + b_2 \sin \theta),$$

$$\frac{\partial w^*_3}{\partial r} = -(U^{(1)}(a) - \frac{1}{2} \nu_1 a^2)(b_1 \cos \theta + b_2 \sin \theta),$$

where

$$M = -\left\{ \left( 1 + \frac{c_{13}}{c_{44}} \right) [(3Q_1 - Q_2)A_2 - 2\nu_1] \right. + \frac{8b_1 c_1}{a_1 (1 - 3c_1) c_{44} p^2} A_2 + \frac{c_{13}}{c_{44} c_{33}} + \frac{b_3}{c_{44}} \right\},$$

$$N = \left[ \left( 1 + \frac{c_{13}}{c_{44}} \right) c_3 - \frac{b_3}{c_{44}} \right] A_3.$$
In [24], Ghiba gives the expression of solution of this type of problem in the form

\[ w_3^* = W(r)(b_1 \cos \theta + b_2 \sin \theta), \]  

with

\[ W(r) = M \frac{r^3}{8} + \frac{N}{p^2} I_1(pr) \left[ \frac{3}{4} Ma^2 + 2N \frac{I_1(pa)}{p^2} + 2(U^{(1)}(a) - \frac{1}{2} \nu_1a^2) \right] r. \]  

On the other hand, in view of the end conditions characteristic for the flexure problem (see also Remark 4.2 from the paper [24]), the unknown constants \( b_\alpha \) must satisfy the equations

\[ \sum_{\beta=1,2} b_\beta D_{\alpha\beta} = -R_\alpha. \]  

Thus, we obtain

\[ b_1 = -\frac{R_1}{J}, \quad b_2 = -\frac{R_2}{J}. \]  

The solution of the problem \((P_2)\) corresponds to the following equilibrated stress on the ends of cylinder:

\[ h = \frac{a_1}{J}(\phi - \nu_2 r)(R_1 \cos \theta + R_2 \sin \theta) \text{ on } D_0, \]

\[ h = -\frac{a_1}{J}(\phi - \nu_2 r)(R_1 \cos \theta + R_2 \sin \theta) \text{ on } D_L \]

and we can observe that the resultant flux of porosity vanishes on the ends of cylinder.

5. Conclusion

In the present paper we study the relaxed Saint–Venant’s problem for circular cylinders filled with a transversely isotropic elastic porous material. The solution of the tension–bending–torsion problem is given by the relations (3.4), (3.9), (3.26), (3.27), (3.30) and the solution of the flexure problem is given by the relations (4.1) and (4.8). From the linearity of the problem, we note that the relaxed Saint–Venant’s problem has a solution of the form

\[ U = U^I + U^{II}. \]

Thus, the relaxed Saint–Venant’s problem for an elastic elastic transversely isotropic porous circular cylinder is completely solved. As a particular case we can retrieve the solution obtained for isotropic porous materials [24].
Acknowledgment

The author acknowledges the support of the Romanian Ministry of Education and Research, CNCSIS Grant code ID–401, Contract No. 15/28.09.2007. The author is grateful to the Referee for several helpful observations concerning this work.

References


Received April 30, 2009; revised version July 7, 2009.