On the fundamental solutions for micropolar fluid–fluid mixtures under steady state vibrations✩

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Abstract

This paper deals with the theory of mixtures which have as constituents two micropolar incompressible fluids. First, using a specific algorithm, a Galerkin type representation of solution is given for the linearized two-dimensional dynamical problem. Then, the steady-state vibration problem is considered and uniqueness theorems are established for both bounded and unbounded domains. Finally, the Galerkin type representation is used to construct the fundamental solution for two-dimensional steady-state vibration problem.

Keywords: micropolar fluid–fluid interaction, uniqueness theorem, steady state vibrations, fundamental solution

1. Introduction

In [1], Eringen has developed a continuum theory for geophysical fluids. He assumed that these fluids are in fact mixtures consisting of two micropolar fluids. In the micropolar fluid theory [2, 3, 4], the rotational degrees of freedom play a central role, so the fluids can exhibit couple stresses and non-symmetric stress tensor. Thus, we have six degrees of freedom, instead of three degrees of freedom considered in classical fluid mechanics. In the recent years, many papers are dedicated to include some additional terms in the basic formulation of the theory of mixtures in order to reflect the microstructure of the constituents [5, 6].

In the classical theory of geophysical fluids, the Navier-Stokes equations are usually considered. But, the Navier-Stokes equations consider a single pure fluid and ignore the effects of intrinsic rotations and the couple stresses. Kirwan and Chang [7] have outlined the influence of couple stresses in the micropolar Ekman problem. The mathematical model proposed by Eringen [1] is based on the mixture theory of micropolar fluids and

✩This work was supported by CNCSIS-UEFISCSU, project PN II-RU TE code 184, no. 86/30.07.2010.

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can be successfully applied in modeling of some natural phenomena, e.g. tornadoes, hurricanes and avalanches. In the atmosphere, air together with rain, snow or sand may be considered as mixtures consisting of two micropolar fluids.

In the present paper we consider the isothermal theory of homogeneous mixtures consisting of two micropolar incompressible viscous fluids as introduced by Eringen [1]. The main objective of the present work is to construct a fundamental solution for the two–dimensional problem governing the steady–state vibration of a micropolar fluid-fluid mixture. Because the theory take into account many effects, the the equations obtained by Eringen [1] represent a complex system of partial differential equations. Our first goal is to obtain a Galerkin type representation of solution for the dynamical two dimensional problem.

In various boundary-value problems from continuum mechanics it is important to give a representation of the general solution to the field equations in terms of elementary (harmonic, biharmonic etc.) functions. For this, we use the method introduced by Moisil [8]. In this algorithm to every differential operator it is associated a polynomial function and to every differential matrix operator a matrix which has as components polynomial functions. Thus, to a system of partial differential equations it is associated a system of algebraic equations. Solving the later, we can easy construct a representation formula for the former. Since the equations of the theory considered in this paper is very complex, to implement this algorithm we use the mathematical software Wolfram MATHEMATICA 7.0.1. To avoid any input mistakes, we validate the result by direct computations.

Then, we give a Galerkin type representation in order to construct a fundamental solution for the two–dimensional problem governing the motion of a micropolar fluid-fluid mixture in the case of steady–state vibrations. For both bounded and unbounded domains we deduce some reciprocity theorems and we study the uniqueness of solutions.

We note that the fundamental solution obtained here represents the initial point in the introduction of potential method for this theory. As in [9, 10] we may use the properties of the fundamental solutions to prove existence results for interior and exterior problems.

In the framework of the theory of mixtures, fundamental solutions for static or steady-state oscillation problems have been constructed in [11]–[16]. However, any of these studies do not consider mixtures of micropolar fluids. On the other hand, in the theory of micropolar fluids, we recall that Ramkisson and Majumdar [17, 18] obtained the fundamental solution for slow, steady-state motion of micropolar fluids, in planar and three dimensional space, while Easwaran and Majumdar [19] gave the fundamental solutions for the two-dimensional slow flow of micropolar fluids in the time dependent problem. We also note that in the theory of pure fluids with microstructure the proposed problems have been studied by Dragoş [20] and Nappa [21]. Moreover, Olmstead and Majumdar [22] gave the fundamental solution for a Oseen flow in the two dimensional case. These results have been completed by Shu and Lee [23].
2. Basic Equations. Auxiliary results

We consider a chemically inert mixture consisting of two micropolar viscous fluids $B^{(1)}$ and $B^{(2)}$. The motions of the mixture are studied with respect to a fixed system of rectangular Cartesian axes $Ox_k$ ($k = 1, 2, 3$). We shall employ the usual summation and differentiation conventions, that is: Latin subscripts, unless otherwise specified, are understood to range over the integers $(1, 2, 3)$ and the Greek subscripts are confined to the range $(1, 2)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate.

The superscripts $a = 1, 2$ are related to the components of the mixtures and summation over repeated subscripts is not used for these indices.

Considering incompressible micropolar viscous fluids and supposing that the flow of the mixture is one slow, then the basic equations governing the motion of the mixture are [1]

\[
\begin{align*}
\varepsilon_{r,r}^{(1)} &= 0, \quad \varepsilon_{r,r}^{(2)} = 0, \\
-\pi_{r,r}^{(1)} + (\mu_1 + k_1)\nu_{r,jj}^{(1)} + k_1\varepsilon_{r,jk}v_{k,j}^{(1)} + \sigma\nu_{r,jj}^{(2)} + \sigma\varepsilon_{r,jk}v_{k,j}^{(2)} - \xi(\nu_{r}^{(1)} - \nu_{r}^{(2)}) + F_{r}^{(1)} = \rho^{(1)} \frac{\partial}{\partial t} \nu_{r}^{(1)}, \\
(\alpha_1 + \beta_1)\nu_{r,jj}^{(1)} + \gamma_1\nu_{r,jj}^{(1)} + k_1(\varepsilon_{r,jk}v_{k,j}^{(1)} - 2\nu_{r}^{(1)}) + \tau\nu_{r,jj}^{(2)} + \sigma(\varepsilon_{r,jk}v_{k,j}^{(2)} - 2\nu_{r}^{(2)}) - \omega(\nu_{r}^{(1)} - \nu_{r}^{(2)}) + L_{r}^{(1)} = \rho^{(1)} j^{(1)} \frac{\partial}{\partial t} \nu_{r}^{(1)}, \\
-\pi_{r,r}^{(2)} + (\mu_2 + k_2)\nu_{r,jj}^{(2)} + k_2\varepsilon_{r,jk}v_{k,j}^{(2)} + \sigma\nu_{r,jj}^{(1)} + \sigma\varepsilon_{r,jk}v_{k,j}^{(1)} + \xi(\nu_{r}^{(1)} - \nu_{r}^{(2)}) + F_{r}^{(2)} = \rho^{(2)} \frac{\partial}{\partial t} \nu_{r}^{(2)}, \\
(\alpha_2 + \beta_2)\nu_{r,jj}^{(2)} + \gamma_2\nu_{r,jj}^{(2)} + k_2(\varepsilon_{r,jk}v_{k,j}^{(2)} - 2\nu_{r}^{(2)}) + \tau\nu_{r,jj}^{(1)} + \sigma(\varepsilon_{r,jk}v_{k,j}^{(1)} - 2\nu_{r}^{(1)}) + \omega(\nu_{r}^{(1)} - \nu_{r}^{(2)}) + L_{r}^{(2)} = \rho^{(2)} j^{(2)} \frac{\partial}{\partial t} \nu_{r}^{(2)},
\end{align*}
\]

where $\rho^{(a)}$ denotes the mass density of the constituent $B^{(a)}$, $\nu_{r}^{(a)}$ is the velocity of $B^{(a)}$; $\nu_{r}^{(a)}$ is the microrotation rate of $B^{(a)}$; $\mu^{(a)}, k^{(a)}, \alpha^{(a)}, \beta^{(a)}, \gamma^{(a)}, \sigma, \tau$ are micropolar fluid viscosities; $\xi$ is the momentum generation coefficient due to the velocity difference; $\omega$ is the momentum generation coefficient due to the difference in gyration; $F_{r}^{(1)}, F_{r}^{(2)}$ are the body forces; $L_{r}^{(1)}, L_{r}^{(2)}$ are the body couples; $j^{(a)}$ is the microinertia density of $B^{(a)}$ and $\varepsilon_{r,jk}$ is the alternating tensor.

In this paper we are interested in two–dimensional motions of the mixture. Thus, we suppose that we have the following body loads

\[
\begin{align*}
F_{a}^{(a)} &= F_{a}^{(a)}(x_1, x_2), \quad F_{3}^{(a)} = 0, \\
L_{a}^{(a)} &= 0, \quad L_{3}^{(a)} = L_{3}^{(a)}(x_1, x_2), \quad (x_1, x_2) \in \Omega,
\end{align*}
\]

where $\Omega$ is fixed two dimensional domain.
Corresponding to these loads, we seek the solution of the system (2.1) in the form

\[ v^{(a)}_\beta = v^{(a)}_\beta(x_1, x_2), \quad v^{(a)}_\gamma = 0, \quad \nu^{(a)}_\beta = 0, \quad \nu^{(a)}_\gamma = 0, \quad (x_1, x_2) \in \Omega. \]  

(2.3)

Let us define the following differential operators

\[
\Gamma_1(\Lambda_1, \Lambda_2) \equiv (\mu_1 + k_1)\Lambda_1 - \xi - \rho^{(1)}\Lambda_2, \\
\Gamma_2(\Lambda_1, \Lambda_2) \equiv (\mu_2 + k_2)\Lambda_1 - \xi - \rho^{(2)}\Lambda_2, \\
\Theta_1(\Lambda_1, \Lambda_2) \equiv \gamma_1\Lambda_1 - 2k_1 - \omega - \rho^{(1)}j^{(1)}\Lambda_2, \\
\Theta_2(\Lambda_1, \Lambda_2) \equiv \gamma_2\Lambda_1 - 2k_2 - \omega - \rho^{(2)}j^{(2)}\Lambda_2.
\]

(2.4)

Then, the substitution of (2.2) and (2.3) into (2.1) yields the equations

\[
v^{(1)}_{\alpha,\alpha} = 0, \quad v^{(2)}_{\alpha,\alpha} = 0, \\
\frac{\partial v^{(1)}_{\alpha}}{\partial t} + Q_1 v^{(1)}_{\alpha} + k_1\varepsilon_{3\alpha\beta}\nu^{(1)}_{\beta} + Q_3 v^{(2)}_{\alpha} + \sigma\varepsilon_{3\alpha\beta}\nu^{(2)}_{\beta} = -F^{(1)}_{\alpha}, \\
P_1\nu^{(1)}_{\alpha} + k_1\varepsilon_{3\alpha\beta}\nu^{(1)}_{\beta,\alpha} + P_3\nu^{(2)}_{\alpha} + \sigma\varepsilon_{3\alpha\beta}\nu^{(2)}_{\beta,\alpha} = -L^{(1)}, \\
\frac{\partial v^{(2)}_{\alpha}}{\partial t} + Q_2 v^{(2)}_{\alpha} + k_2\varepsilon_{3\alpha\beta}\nu^{(2)}_{\beta} + Q_3 v^{(1)}_{\alpha} + \sigma\varepsilon_{3\alpha\beta}\nu^{(1)}_{\beta} = -F^{(2)}, \\
P_2\nu^{(2)}_{\alpha} + k_2\varepsilon_{3\alpha\beta}\nu^{(2)}_{\beta,\alpha} + P_3\nu^{(1)}_{\alpha} + \sigma\varepsilon_{3\alpha\beta}\nu^{(1)}_{\beta,\alpha} = -L^{(2)},
\]

(2.5)

where

\[
Q_1 \equiv \Gamma_1(\Delta, \frac{\partial}{\partial t}), \quad Q_2 \equiv \Gamma_2(\Delta, \frac{\partial}{\partial t}), \\
P_1 \equiv \Theta_1(\Delta, \frac{\partial}{\partial t}), \quad P_2 \equiv \Theta_1(\Delta, \frac{\partial}{\partial t}), \\
Q_3 \equiv \sigma\Delta + \xi, \quad P_3 \equiv \tau\Delta - 2\sigma + \omega,
\]

(2.6)

and \(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\) is the Laplace operator.

Further, we will use the following bilinear forms

\[
\mathcal{W}_1 \left( (\xi^{(1)}_{\alpha\beta}, \xi^{(2)}_{\alpha\beta}), (\eta^{(1)}_{\alpha\beta}, \eta^{(2)}_{\alpha\beta}) \right) = \mu_1\xi^{(1)}_{\alpha\beta}\eta^{(1)}_{\alpha\beta} + (\mu_1 + k_1)\xi^{(1)}_{\alpha\beta}\eta^{(1)}_{\alpha\beta} + \mu_2\xi^{(2)}_{\alpha\beta}\eta^{(2)}_{\alpha\beta} + (\mu_2 + k_2)\xi^{(2)}_{\alpha\beta}\eta^{(2)}_{\alpha\beta} + \sigma(\xi^{(1)}_{\alpha\beta}\eta^{(2)}_{\alpha\beta} + \xi^{(2)}_{\alpha\beta}\eta^{(1)}_{\alpha\beta}),
\]

(2.7)

and the corresponding quadratic forms

\[
\mathcal{W}_1 \left( s^{(1)}_{\alpha\beta}, s^{(2)}_{\alpha\beta} \right) \equiv \mathcal{W}_1 \left( (s^{(1)}_{\alpha\beta}, s^{(2)}_{\alpha\beta}), (s^{(1)}_{\alpha\beta}, s^{(2)}_{\alpha\beta}) \right)
\]

and

\[
\mathcal{W}_2 \left( \xi^{(1)}_{\alpha}, \xi^{(2)}_{\alpha} \right) \equiv \mathcal{W}_2 \left( (\xi^{(1)}_{\alpha}, \xi^{(2)}_{\alpha}), (\xi^{(1)}_{\alpha}, \xi^{(2)}_{\alpha}) \right).
\]
The quadratic form $W_2(\xi^{(1)}, \xi^{(2)})$ is positive definite if and only if
\[ \gamma_1 > 0, \quad \gamma_1 \gamma_2 - \tau^2 > 0. \tag{2.8} \]

The eigenvalues of the matrix associated with the quadratic form $W_1(\xi^{(1)}, \xi^{(2)})$ are the following
\[ \lambda_{1,2} = \frac{1}{2} \left( k_1 + k_2 \pm \sqrt{4\sigma^2 + (k_1 - k_2)^2} \right), \]
\[ \lambda_{3,4} = \frac{1}{2} \left( k_1 + k_2 + 2(\mu_1 + \mu_2) \pm \sqrt{4\sigma^2 + (k_1 - k_2 + 2\mu_1 - 2\mu_2)^2} \right). \tag{2.9} \]

We can see that the eigenvalues $\lambda_{1,2}$ are strictly positive real numbers if and only if
\[ k_1 > 0, \quad k_1k_2 - \sigma^2 > 0. \tag{2.10} \]

On the other hand, the eigenvalues $\lambda_{3,4}$ are strictly positive real numbers if and only if
\[ (k_1 + 2\mu_1)(k_2 + 2\mu_2) - \sigma^2 > 0, \quad k_1 + 2\mu_1 > 0 \tag{2.11}. \]

If we apply directly the Sylvester’s criterion for the matrix associated with the positive definite quadratic form $W_1(\xi^{(1)}, \xi^{(2)})$ we also deduce other conditions on the constitutive coefficients. For example, we have the inequality
\[ (k_1 + \mu_1)(k_2 + \mu_2) - \sigma^2 > 0, \tag{2.12} \]
which will be utilized later.

In this paper we suppose that the dissipation potential
\[ \Phi = W_1(\xi^{(1)}, \xi^{(2)}) + W_2(\nu^{(1)}, \nu^{(2)}) + \xi(v^{(1)} - v^{(2)}) + \varpi(\nu^{(1)} - \nu^{(2)})(\nu^{(1)} - \nu^{(2)}) \tag{2.13} \]
is a positive defined quadratic form in terms of $a_{\alpha\beta}^{(a)}(v, \nu) = v^{(a)}_{\beta,\alpha} + \varepsilon_{\beta,\alpha}^{(a)}, \nu^{(a)}_{\alpha}, (v^{(1)}_{\alpha} - v^{(2)}_{\alpha})$ and $(\nu^{(1)} - \nu^{(2)})$.

This means that
\[ \xi > 0, \quad \varpi > 0 \tag{2.14} \]
and the relations (2.8), (2.10) and (2.11) hold true.

3. **A solution of the field equations**

In this section we establish a Galerkin type representation of the solution of the field equations (2.5).
With the help of the operators introduced in the previous section, we define the following differential operators:

\[
\begin{align*}
D_1 &\equiv -P_3^2Q_2 + \sigma^2P_2\Delta - 2\sigma k_2P_3\Delta + P_1(P_2Q_2 + k_1^2\Delta), \\
D_{21} &\equiv P_3(-\sigma Q_1 + k_1Q_3) - \sigma(P_1Q_2 + \sigma k_1\Delta) + k_2(P_1Q_1 + k_1^2\Delta), \\
D_{22} &\equiv P_3(-\sigma Q_2 + k_2Q_3) - \sigma(P_2Q_3 + \sigma k_2\Delta) + k_1(P_2Q_2 + k_1^2\Delta), \\
D_3 &\equiv -P_3^2Q_3 + \sigma k_1P_3\Delta - (\sigma^2 + k_1k_2)P_3\Delta + P_1(P_2Q_3 + \sigma k_2\Delta), \\
D_{41} &\equiv \sigma P_3Q_3 + P_1(\sigma Q_2 - k_2Q_3) + \sigma^3\Delta - k_1(P_3Q_2 + \sigma k_2\Delta), \\
D_{42} &\equiv \sigma P_3Q_3 + P_2(\sigma Q_1 - k_1Q_3) + \sigma^3\Delta - k_2(P_3Q_1 + \sigma k_1\Delta), \\
D_5 &\equiv -P_3^2Q_1 + k_1^2P_2\Delta - 2\sigma k_1P_3\Delta + P_1(P_2Q_1 + \sigma^2\Delta),
\end{align*}
\]

We will also use the differential operators:

\[
\begin{align*}
\Box_1 &\equiv P_2(Q_1Q_2 - Q_3^2) + [\sigma^2Q_2 + k_2(k_2Q_1 - 2\sigma Q_3)]\Delta, \\
\Box_2 &\equiv P_1(Q_1Q_2 - Q_3^2) + [\sigma^2Q_1 + k_1(k_1Q_2 - 2\sigma Q_3)]\Delta, \\
\Box_3 &\equiv P_3(Q_1Q_2 - Q_3^2) + [(\sigma^2 + k_1k_2)Q_3 - \sigma(k_2Q_1 + k_1Q_2)]\Delta, \\
\Box &\equiv P_3\Box_3 + P_2\Box_1 + P_1\Box_2 - P_2P_1(Q_1Q_2 - Q_3^2) \\
&\quad + P_3[-\sigma(k_2Q_1 + k_1Q_2) + (\sigma^2 + k_1k_2)Q_2]\Delta + (\sigma^2 - k_1k_2)^2\Delta.
\end{align*}
\]

In the above quantities and in the following, if \( \mathcal{D}_n \), \( n = 1, 2, \ldots, m \) are differential operators and \( G \) is, for example, a \( C^\infty(\mathbb{R}^2) \) function, we use the notation

\[ \mathcal{D}_1\mathcal{D}_2\ldots\mathcal{D}_mG \equiv \mathcal{D}_1(\mathcal{D}_2(\ldots(\mathcal{D}_m(G))\ldots)). \]

Using the method introduced by Moisil [8] we construct a representation of Galerkin type for the solution of the dynamical problem. To obtain this representation, we use the mathematical software Wolfram MATHEMATICA 7.0.1. Because the calculus are very complex, we give directly the expressions of our representation and then, to avoid the input mistake, we directly verify, without using any software, if our solution is a good one.

**Theorem 3.1.** Let

\[
\begin{align*}
v^{(1)}_{\alpha} &= \Delta D_1G^{(1)}_{\alpha} - D_1G^{(1)}_{\beta,\alpha\beta} - \epsilon_{\alpha\beta\gamma}D_{22}H^{(1)}_{\beta,\gamma} - \\
&\quad - \Delta D_3G^{(2)}_{\beta,\alpha\beta} + D_3G^{(2)}_{\beta,\alpha\beta} - \epsilon_{\alpha\beta\gamma}D_{41}H^{(2)}_{\beta,\gamma} + P^{(1)}_{\alpha}, \\
\nu^{(1)} &= \epsilon_{\alpha\beta\gamma}\Delta D_{22}G^{(1)}_{\alpha,\beta\gamma} + \Box_1H^{(1)} + \epsilon_{\alpha\beta\gamma}\Delta D_{42}G^{(2)}_{\alpha,\beta\gamma} + \Box_3H^{(2)}, \\
v^{(2)}_{\alpha} &= -\Delta D_3G^{(1)}_{\alpha} + D_3G^{(1)}_{\beta,\alpha\beta} - \epsilon_{\alpha\beta\gamma}D_{42}H^{(1)}_{\beta,\gamma} + \\
&\quad + \Delta D_5G^{(2)}_{\alpha} - D_5G^{(2)}_{\beta,\alpha\beta} - \epsilon_{\alpha\beta\gamma}D_{21}H^{(2)}_{\beta,\gamma} + P^{(2)}_{\alpha},
\end{align*}
\]
\[
\begin{align*}
\nu^{(2)} &= \varepsilon_{\alpha\beta\gamma} \Delta D_{41} G_{\alpha\beta}^{(1)} + \Box_3 H^{(1)} + \varepsilon_{\alpha\beta\gamma} \Delta D_{21} G_{\alpha\beta}^{(2)} + \Box_2 H^{(2)}, \\
\pi^{(1)} &= -\Box G_{\alpha\gamma}^{(1)} + Q_1 P^{(1)} + Q_3 P^{(2)}, \\
\pi^{(2)} &= -\Box G_{\alpha\gamma}^{(2)} + Q_3 P^{(1)} + Q_2 P^{(2)},
\end{align*}
\]

where \(G_{\alpha}^{(a)}, H^{(a)}\) and \(P^{(a)}\) are solutions of the partial differential equations
\[
\Delta \Box G_{\alpha}^{(a)} = -F_{\alpha}^{(a)}, \quad \Box H^{(a)} = -L^{(a)}, \quad \Delta P^{(a)} = 0. \tag{3.4}
\]

Then \(v_{\alpha}^{(a)}, \nu^{(a)}\) and \(\pi^{(a)}\) satisfy the equations (2.5).

**Proof.** It is easy to see that we have
\[
\varepsilon_{\alpha\beta\gamma} \varepsilon_{3\rho\tau} = \delta_{\alpha\rho} \delta_{\beta\tau} - \delta_{\alpha\gamma} \delta_{\beta\rho}, \tag{3.5}
\]
and
\[
\varepsilon_{\alpha\beta\gamma} A_{\alpha\beta} = 0, \tag{3.6}
\]
for every \(A \in C^2(\mathbb{R}^2)\).

To prove that the functions \(v_{\alpha}^{(a)}\) satisfy the incompressibility condition (2.5)_{1,2}, we use the relations (3.4) and (3.6) with \(A = H^{(a)}\).

Let us prove the relation (2.5)_3. Thus, from (3.3) and (3.5), we obtain
\[
\begin{align*}
-\pi^{(1)}_{\alpha} + Q_1 v^{(1)}_{\alpha} + k_1 \varepsilon_{3\alpha\beta} v^{(2)}_{\beta} + Q_3 v^{(2)}_{\alpha} + \sigma \varepsilon_{3\alpha\beta} v^{(2)}_{\beta} &= \\
= \Delta (Q_1 D_1 + k_1 \Delta D_{22} - Q_3 D_3 + \sigma \Delta D_{41}) G_{\alpha}^{(1)} + \\
+(-Q_1 D_1 - k_1 \Delta D_{22} + Q_3 D_3 - \sigma \Delta D_{41} + \Box) G_{\beta,\alpha}^{(1)} + \\
+\Delta (-Q_1 D_3 + k_1 \Delta D_{42} + Q_3 D_5 + \sigma \Delta D_{21}) G_{\alpha}^{(2)} + \\
+(-Q_1 D_3 + k_1 \Delta D_{42} + Q_3 D_5 + \sigma \Delta D_{21}) G_{\beta,\alpha}^{(2)} - \\
-\varepsilon_{\alpha\beta\gamma} (-Q_1 D_{22} + k_1 \Box - Q_3 D_{42} + \sigma \Box_3) H^{(1)} - \\
-\varepsilon_{\alpha\beta\gamma} (-Q_1 D_{41} + k_1 \Box - Q_3 D_{21} + \sigma \Box_2) H^{(2)}.
\end{align*}
\]

But, direct calculus give us the following identities
\[
\begin{align*}
Q_1 D_1 + k_1 \Delta D_{22} - Q_3 D_3 + \sigma \Delta D_{41} &= \Box, \\
k_1 \Delta D_{42} + Q_3 D_5 + \sigma \Delta D_{21} &= Q_1 D_3, \\
k_1 \Box_1 + \sigma \Box_3 &= Q_1 D_{22} + Q_3 D_{42}, \\
k_1 \Box_3 + \sigma \Box_2 &= Q_1 D_{41} + Q_3 D_{21}.
\end{align*}
\]

In view of the relations (3.8), (3.7) and (3.3)_1, we can conclude that
\[
-\pi^{(1)}_{\alpha} + Q_1 v^{(1)}_{\alpha} + k_1 \varepsilon_{3\alpha\beta} v^{(1)}_{\beta} + Q_3 v^{(2)}_{\alpha} + \sigma \varepsilon_{3\alpha\beta} v^{(2)}_{\beta} = -F^{(1)}_{\alpha}.
\]
In a similar manner, using the relations (3.6), (3.5) and the identities
\[ P_1 D_{22} - k_1 D_1 + P_3 D_{41} + \sigma D_3 = 0, \]
\[ P_1 D_{42} + k_1 D_3 + P_3 D_{21} - \sigma D_5 = 0, \]
\[ P_1 \Box_1 + P_3 \Box_3 + k_1 \Delta D_{22} + \sigma \Delta D_{42} = \Box, \]
\[ P_1 \Box_3 + k_1 \Delta D_{41} + P_3 \Box_2 + \sigma \Delta D_{21} = 0, \]
\[ Q_3 D_1 + \sigma \Delta D_{22} - Q_2 D_3 + k_2 \Delta D_{41} = 0, \]
\[ -Q_3 D_3 + \sigma \Delta D_{42} + Q_2 D_5 + k_2 \Delta D_{21} = \Box, \]
\[ Q_3 \Box_1 + P_3 \Box_3 + k_1 \Delta D_{22} + \sigma \Delta D_{42} = \Box, \]
\[ Q_3 \Box_3 + k_1 \Delta D_{41} + P_3 \Box_2 + \sigma \Delta D_{21} = 0, \]
\[ \sigma D_1 - P_3 D_{22} - k_2 D_3 - P_2 D_{41} = 0, \]
\[ \sigma D_3 + P_3 D_{42} - k_2 D_5 + P_2 D_{21} = 0, \]
\[ \sigma \Delta D_{22} + P_3 \Box_1 + k_2 \Delta D_{42} + P_3 \Box_3 = 0 \]
\[ \sigma \Delta D_{41} + P_3 \Box_2 + k_2 \Delta D_{21} + P_2 \Box_2 = \Box. \]

we prove that the functions \( v^{(a)}_\alpha, \nu^{(a)} \) and \( \pi^{(a)} \) are solutions of the basic equations (2.5)_{4,5,6} and the proof is complete.

In the following sections we will use this Galerkin representation in order to study the wave solutions and to obtain fundamental solutions for steady–state vibrations.

4. Two dimensional steady–state vibrations

In this section we study the problem of steady–state vibrations in micropolar fluid–fluid mixtures.

We suppose that
\[ F_\beta = \text{Re} \left[ F_\beta^*(x_1, x_2)e^{-i\omega t} \right], \]
\[ L^{(a)} = \text{Re} \left[ L^{(a)}(x_1, x_2)e^{-i\omega t} \right], \quad (x_1, x_2) \in \Omega, \quad i = \sqrt{-1} \]

and we search the solution \((v_1^{(1)}, v_2^{(1)}, \nu^{(1)}, v_1^{(2)}, v_2^{(2)}, \nu^{(2)}, \pi^{(1)}, \pi^{(2)})\) of the system (2.5) in the following form
\[ v^{(a)}_\alpha = \text{Re} \left[ v^{(a)}_\alpha^*(x_1, x_2)e^{-i\omega t} \right], \]
\[ \nu^{(a)} = \text{Re} \left[ \nu^{(a)}(x_1, x_2)e^{-i\omega t} \right], \]
\[ \pi^{(a)} = \text{Re} \left[ \pi^{(a)}(x_1, x_2)e^{-i\omega t} \right], \quad (x_1, x_2) \in \Omega, \quad i = \sqrt{-1} \]

where \( \omega > 0 \) is the frequency of vibration.

Let us introduce the differential operators:
the asymptotic relations of the type

\[ Q_1^* \equiv \Gamma_1(\Delta, -i\omega), \quad Q_2^* \equiv \Gamma_2(\Delta, -i\omega), \]
\[ P_1^* \equiv \Theta_1(\Delta, -i\omega), \quad P_2^* \equiv \Theta_1(\Delta, -i\omega). \tag{4.3} \]

We denote by \( D \left( \frac{\partial}{\partial x} \right) \) the \( 8 \times 8 \) matrix having the components

\[
D_{\alpha;\beta} = Q_1^* \delta_{\alpha\beta}, \quad D_{\alpha;3} = -D_{3;\alpha} = k_1 \varepsilon_{\alpha\beta} \frac{\partial}{\partial x_\beta},
\]
\[
D_{\alpha;3+3} = D_{\beta+3;\alpha} = Q_2^\ast \delta_{\alpha\beta}, \quad D_{\alpha;6} = -D_{6;\alpha} = \sigma \varepsilon_{\alpha\beta} \frac{\partial}{\partial x_\beta},
\]
\[
D_{3;3} = P_1^* \quad D_{3,3+3} = -D_{3+3;3} = -\sigma \varepsilon_{\alpha\beta} \frac{\partial}{\partial x_\beta}, \quad D_{3,6} = D_{6;3} = P_3, \tag{4.4}
\]
\[
D_{\alpha+3,3} = Q_2^* \delta_{\alpha\beta}, \quad D_{\alpha+3,6} = -D_{6;\alpha+3} = k_2 \varepsilon_{\alpha\beta} \frac{\partial}{\partial x_\beta},
\]
\[
D_{6,6} = P_2^* \quad D_{\alpha;7} = D_{\alpha;3+8} = -D_{7;\alpha+3} = -D_{8;\alpha+3} = -\frac{\partial}{\partial x_\alpha},
\]
\[
D_{\alpha;8} = D_{8;\alpha} = D_{7;3} = 0 = D_{3;7} = D_{3;8} = D_{8;3} = D_{\alpha+3;7} = D_{7;\alpha+3} = 0,
\]
\[
D_{6;7} = D_{7;6} = D_{7;7} = D_{6;8} = D_{8;6} = D_{7;8} = D_{8;7} = D_{8;8} = 0.
\]

Thus, the unknown amplitude \( \mathbf{U} = (v_1^{s(1)}, v_2^{s(1)}, \nu^{s(1)}, v_1^{s(2)}, v_2^{s(2)}, \nu^{s(2)}, \pi^{s(1)}, \pi^{s(2)}) \) is solution of the following system of equations

\[
D \left( \frac{\partial}{\partial x} \right) \mathbf{U} = -\mathbf{F}, \tag{4.5}
\]

where \( \mathbf{F} = (F_1^{s(1)}, F_2^{s(1)}, L^{s(1)}, F_1^{s(2)}, F_2^{s(2)}, L^{s(2)}, 0, 0). \)

### 4.1. Reciprocity and uniqueness theorems

In this subsection we establish a reciprocity theorem and we study the uniqueness of solution. Let us consider a finite domain \( \Omega^+ \subset \mathbb{R}^2 \), bounded by the Lyapunov curve \( \partial \Omega, \Omega^\ast = \Omega^+ \cup \partial \Omega, \) and \( \Omega^- = \mathbb{R}^2 \setminus \overline{\Omega}^\ast \) is the exterior of \( \Omega^+ \). We use the notation \( \Omega^- = \Omega^- \cap S(0, R) \), where \( S(0, R) \) is the disk with its center at \( 0 \) and radius \( R \).

The eight–dimensional vector function \( \mathbf{U} = (v_1^{s(1)}, v_2^{s(1)}, \nu^{s(1)}, v_1^{s(2)}, v_2^{s(2)}, \nu^{s(2)}, \pi^{s(1)}, \pi^{s(2)}) \) is called regular vector in \( \Omega^+ \) if \( \upsilon^{s(a)}(r), \nu^{s(a)}(r), \pi^{s(a)}(r) \in C^1(\overline{\Omega^+}) \) and \( \pi^{s(a)}(r) \in C^1(\overline{\Omega^+}) \cap C^2(\Omega^+) \).

If \( \mathbf{U} \) is defined in \( \Omega^- \), then \( \mathbf{U} \) is called regular vector in \( \Omega^- \) if \( \upsilon^{s(a)}(r), \nu^{s(a)}(r) \in C^1(\overline{\Omega^-}) \) and \( \pi^{s(a)}(r) \in C^1(\overline{\Omega^-}) \cap C^2(\Omega^-) \), and \( \upsilon^{s(a)}(r), \nu^{s(a)}(r), \pi^{s(a)}(r) \in L^2(\partial \Omega^-) \) for every \( R > 0 \) and satisfies the asymptotic relations of the type

\[
v_{\alpha}^{s(a)} = O(r^{-1}), \quad v_{\alpha\beta}^{s(a)} = o(r^{-1}),
\]
\[
\nu^{s(a)} = o(r^{-1}), \quad \nu_{\alpha}^{s(a)} = O(r^{-1}), \quad \pi^{s(a)} = o(r^{-1}), \tag{4.6}
\]
where $r^2 = x_3 x_3$.

We define the following problems:

**Problem $(P_1)$**: to find the solution of the system (4.5) which is a regular vector in $\Omega^+$, satisfying the boundary conditions

$$v^{*}(a)^{\alpha} = g^{(a)}_{\alpha}, \quad \nu^{*}(a)^{\alpha} = \phi^{(a)}_{\alpha} \text{ on } \partial \Omega,$$

where $g^{(a)}_{\alpha}$ and $\phi^{(a)}_{\alpha}$ are given.

**Problem $(P_2)$**: to find the solution of the system (4.5) which is a regular vector in $\Omega^-$, satisfying the boundary conditions

$$v^{*}(a)^{\alpha} = g^{(a)}_{\alpha}, \quad \nu^{*}(a)^{\alpha} = \phi^{(a)}_{\alpha} \text{ on } \partial \Omega,$$

where $g^{(a)}_{\alpha}$ and $\phi^{(a)}_{\alpha}$ are given.

Let us introduce the matricial differential operator

$$H \left( \frac{\partial}{\partial n}, n(x) \right) = \left| H_{mn} \left( \frac{\partial}{\partial x}, n(x) \right) \right|_{6 \times 8},$$

where

$$H_{\alpha;\beta} = (\mu_1 + k_1) \delta_{\alpha\beta} \frac{\partial}{\partial x} + \mu_1 n_{\beta} \frac{\partial}{\partial x_{\alpha}}, \quad H_{\alpha;3} = k_1 \varepsilon_{3\alpha\beta} n_{\beta};$$

$$H_{\alpha;3+ \beta} = \sigma \delta_{\alpha\beta} \frac{\partial}{\partial n}, \quad H_{\alpha;6} = \sigma \varepsilon_{3\alpha\beta} n_{\beta}, \quad H_{\alpha;7} = -n_{\alpha},$$

$$H_{\alpha;8} = H_{3;\alpha} = 0, \quad H_{3;3} = \gamma_1 \frac{\partial}{\partial n}, \quad H_{3;3+3} = 0, \quad H_{3;6} = \tau \frac{\partial}{\partial n},$$

$$H_{3;7} = H_{3;8} = 0, \quad H_{\alpha+3;\beta} = \sigma \delta_{\alpha\beta} \frac{\partial}{\partial n},$$

$$H_{\alpha+3;3} = \sigma \varepsilon_{3\alpha\beta} n_{\beta}, \quad H_{\alpha+3;3+3} = (\mu_2 + k_2) \delta_{\alpha\beta} \frac{\partial}{\partial n} + \mu_2 n_{\beta} \frac{\partial}{\partial x_{\alpha}}$$

$$H_{\alpha+3;3} = k_2 \varepsilon_{3\alpha\beta} n_{\beta}, \quad H_{\alpha+3;7} = 0, \quad H_{\alpha+3;8} = -n_{\alpha}, \quad H_{6;6} = 0,$$

$$H_{6;3} = \tau \frac{\partial}{\partial n}, \quad H_{6;3+3} = 0, \quad H_{6;6} = \gamma_2 \frac{\partial}{\partial n}, \quad H_{6;7} = H_{6;8} = 0,$$

and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x_{\alpha}} n_{\alpha}.$$

In the following, we denote by $\hat{V}$ the vector obtained from a vector $V$ by cutting the last two components.

**Theorem 4.1.** If $U = (u_{1}^{(1)}, u_{1}^{*}(1), \psi_{1}^{*(1)}, u_{2}^{(2)}, u_{2}^{*}(2), \psi_{2}^{*(2)}, \nu_{1}^{*(1)}, \nu_{1}^{*(2)}, \nu_{2}^{*(2)}, \pi_{1}^{*(1)}, \pi_{2}^{*(2)})$ and $V = (v_{1}^{(1)}, v_{2}^{(1)}, \nu_{1}^{*(1)}, v_{1}^{*(1)}, \nu_{1}^{*(2)}, v_{2}^{*(2)}, \pi_{1}^{*(1)}, \pi_{2}^{*(2)})$ are regular vectors in $\Omega^+$, then the following relations
Similar results hold true for the exterior domain. Assume that

\[
\text{Theorem 4.3.}
\]

\[
\text{Remark 4.1.}
\]

\[
\text{Theorem 4.2.}
\]

where the superposed bar denotes complex conjugate and

\[
C(\hat{U}, \hat{V}) = \mathcal{W}_1((a_{\alpha\beta}^{(1)}(v^*, \nu^*), a_{\alpha\beta}^{(2)}(v^*, \nu^*)), (a_{\alpha\beta}^{(1)}(u^*, \psi^*), a_{\alpha\beta}^{(2)}(u^*, \psi^*)))
\]

\[
+ \mathcal{W}_2((v_{\alpha}^{*(1)}, v_{\alpha}^{*(2)}), (\psi_{\alpha}^{*(1)}, \psi_{\alpha}^{*(2)}))
\]

\[
+ \xi(v_{\alpha}^{*(1)} - v_{\alpha}^{*(2)})(u_{\alpha}^{*(1)} - u_{\alpha}^{*(2)}) + \omega(\psi^{*(1)} - \psi^{*(2)}).
\]

Proof. The proof follows from the divergence theorem and the incompressibility conditions.

In view of

\[
C(\hat{U}, \hat{V}) = C(\hat{V}, \hat{U}),
\]

and (4.11) we obtain the following reciprocity theorem:

**Theorem 4.2.** If \( U \) and \( V \) are regular vectors in \( \Omega^+ \) then

\[
\int_{\Omega^+} \mathbf{V} \mathbf{D} \left( \frac{\partial}{\partial x} \right) \mathbf{U} dx = \int_{\partial\Omega} \hat{\mathbf{V}} \mathbf{H} \left( \frac{\partial}{\partial x}, \mathbf{n(x)} \right) \mathbf{U} dx - \int_{\Omega^+} C(\hat{U}, \hat{V}) dx
\]

\[
+ i\omega \int_{\Omega^+} \sum_{\alpha=1,2} \rho^{(a)} \left( v_{\alpha}^{*(a)} \nu^{*(a)} + j^{(a)} \nu^{*(a)} \psi^{*(a)} \right) dx;
\]

\[
\int_{\Omega^+} \mathbf{V} \mathbf{D} \left( \frac{\partial}{\partial x} \right) \mathbf{U} dx = \int_{\partial\Omega} \hat{\mathbf{V}} \mathbf{H} \left( \frac{\partial}{\partial x}, \mathbf{n(x)} \right) \mathbf{U} dx - \int_{\Omega^+} C(\hat{U}, \hat{V}) dx
\]

\[
- i\omega \int_{\Omega^+} \sum_{\alpha=1,2} \rho^{(a)} \left( v_{\alpha}^{*(a)} \nu^{*(a)} + j^{(a)} \nu^{*(a)} \psi^{*(a)} \right) dx
\]

and (4.11) we obtain the following reciprocity theorem:

\[
\int_{\Omega^+} \mathbf{V} \mathbf{D} \left( \frac{\partial}{\partial x} \right) \mathbf{U} - \mathbf{U} \mathbf{D} \left( \frac{\partial}{\partial x} \right) \mathbf{V} \right] dx = \int_{\partial\Omega} \hat{\mathbf{V}} \mathbf{H} \left( \frac{\partial}{\partial x}, \mathbf{n(x)} \right) \mathbf{U} - \hat{\mathbf{U}} \mathbf{H} \left( \frac{\partial}{\partial x}, \mathbf{n(x)} \right) \mathbf{V} \right] dx.
\]

**Remark 4.1.** Similar results hold true for the exterior domain \( \Omega^- \).

Concerning the problem \( (P_1) \), we give the following uniqueness result:

**Theorem 4.3.** Assume that \( \rho^{(a)} > 0 \) and the dissipation potential (2.13) is positive definite. Then two solutions \( \mathbf{V} \) and \( \mathbf{V} \) of the boundary value problem \( (P_1) \) corresponding to the same given data are so that \( \mathbf{V} = \mathbf{V} \), \( \mathbf{V}_7 - \mathbf{V}_7 = P^{(1)} \) and \( \mathbf{V}_8 - \mathbf{V}_8 = P^{(2)} \) where

\[
\text{grad } P^{(1)} = 0 \text{ and grad } P^{(2)} = 0.
\]

Proof. We denote by \( \mathbf{V} = (v_1^{*(1)}, v_2^{*(1)}, \nu^{*(1)}, v_1^{*(2)}, v_2^{*(2)}, \nu^{*(2)}, \psi^{*(1)}, \psi^{*(2)}) \) the difference between these two solutions. In view of the linearity of the problem, it is easy to see that \( \mathbf{V} \) is solution of the homogeneous equation corresponding to (4.5) and satisfy homogeneous boundary conditions.
Thus, using the relations (4.11) we can write
\[
\int_{\Omega^+} C(\tilde{V}, \tilde{V}) \, dx = i\omega \int_{\Omega^+} \sum_{a=1,2} \rho^{(a)}(v^{s(a)}_\alpha \overline{v}^{s(a)}_\alpha + j^{(a)} v^{s(a)}_\alpha \overline{v}^{s(a)}_\alpha) \, dx,
\]
(4.15)
\[
\int_{\Omega^+} C(\tilde{V}, \tilde{V}) \, dx = -i\omega \int_{\Omega^+} \sum_{a=1,2} \rho^{(a)}(\overline{v}^{s(a)}_\alpha v^{s(a)}_\alpha + j^{(a)} \overline{v}^{s(a)}_\alpha v^{s(a)}_\alpha) \, dx.
\]
In consequence, in view of (4.13), we have
\[
\int_{\Omega^+} C(\tilde{V}, \tilde{V}) dv = 0.
\]
(4.16)

But \( C(\tilde{V}, \tilde{V}) \) is a positive definite quadratic form, and thus, in view of the boundary conditions, we obtain
\[
\tilde{V}(x) = 0, \quad x \in \Omega^+.
\]
(4.17)
The form of the system (4.5) and the above relation complete the proof of the theorem.

We also have the following uniqueness result for the problem \( (P_2) \):

**Theorem 4.4.** Assume that \( \rho^{(a)} > 0 \) and the dissipation potential is positive definite (2.13). Then the boundary value problem \( (P_2) \) has a unique solution.

**Proof.** Let us note that two identities similar to (4.11) hold for every domain \( \Omega_{-R}, R > 0 \). Then, letting \( R \to \infty \) and using the asymptotic relations (4.6) and a similar argument as in the proof of Theorem 4.4 we deduce that the problem has at most one solution. The proof is complete.

### 4.2. Fundamental solution

In this subsection we consider \( \Omega \) to be the entire two-dimensional Euclidean space. We use the representation described in the previous section in order to determine a fundamental solution of equations of motion for the case of steady vibrations.

In the following, we denote by \( D_1^*, D_2^*, D_3^*, D_4^*, D_5^*, \Box_1^*, \Box_2^*, \Box_3^* \) and \( \Box \) the differential operations obtained from the expressions of \( D_1, D_2, D_3, D_4, D_5, \Box_1, \Box_2, \Box_3 \) and \( \Box \) by replacing \( Q_1, Q_2, P_1 \) and \( P_2 \) with the operators \( Q_1^*, Q_2^*, P_1^* \) and \( P_2^* \).

Let be \( y \in \mathbb{R}^2 \). According with \cite{24} and \cite{25}, the fundamental solution of the system (4.5) is a matrix \( \Gamma(x, y; \omega) = \|\Gamma_{ij}\|_{8 \times 6} \) which satisfies the condition
\[
D \left( \frac{\partial}{\partial x} \right) \Gamma(x, y; \omega) = Q, \quad x \in \mathbb{R}^2,
\]
(4.18)
where the matrix \( Q \) is given by
\[
Q_{r,s} = \delta(x - y)\delta_{rs}, \quad Q_{r,s} = Q_{8,s} = 0, \quad \text{for} \ r, s \in 1, 2, \ldots, 6.
\]
(4.19)

Using an appropriate procedure with that used in the previous section, we have the following theorem:
Theorem 4.5. Let

\[
\begin{align*}
v^{(1)}_\alpha &= \Delta D_1^* G^{(1)}_\alpha - D_1^* G^{(1)}_{\alpha,\beta} - \varepsilon_{\alpha\beta} D_{22}^* H^{(1)}_{\beta}, \\
v^{(1)}_\nu &= \varepsilon_{\alpha\beta} \Delta D_{22}^* G^{(1)}_{\alpha,\beta} + \Box^* H^{(1)} - \varepsilon_{\alpha\beta} \Delta D_{42}^* G^{(1)}_{\alpha,\beta} + \Box^* H^{(2)}, \\
v^{(2)}_\alpha &= \varepsilon_{\alpha\beta} \Delta D_{41}^* G^{(1)}_{\alpha,\beta} + \Delta D_{5}^* G^{(1)}_{\beta,\alpha} - \varepsilon_{\alpha\beta} D_{42}^* H^{(1)} + \\
&\quad + \Delta D_{5}^* G^{(1)}_{\beta,\alpha} - D_5^* G^{(1)}_{\beta,\alpha} - \varepsilon_{\alpha\beta} D_{21}^* H^{(2)} + P^*(\alpha), \\
\pi^{(1)} &= -\Box^* G^{(1)}_{\alpha,\alpha} + Q_1^* P^{(1)} + Q_3^* P^{(2)}, \\
\pi^{(2)} &= -\Box^* G^{(2)}_{\alpha,\alpha} + Q_3^* P^{(1)} + Q_2^* P^{(2)},
\end{align*}
\]

where \(G^{(a)}_\alpha\), \(H^{(a)}\) and \(P^{(a)}\) are solutions of the partial differential equations

\[
\Delta \Box^* G^{(a)}_\alpha = -F^{(a)}_\alpha, \quad \Box^* H^{(a)} = -L^{(a)} - \Delta P^{(a)} = 0.
\]

Then \(v^{(a)}_\alpha\), \(\pi^{(a)}\) and \(\pi^{(a)}\) satisfy the system (4.5).

We denote by \(q_n^2\), \(n = 1, 2, 3, 4\) the roots of the equation

\[
\Box^*(-q) = 0
\]

where by \(\Box^*(-q)\) we mean the algebraic expression obtained by replacing the operator \(\Delta\) with \(-q\) in the definition of the operator \(\Box^*\).

We assume that \(q_n^2 \neq q_m^2\) for \(n \neq m\), \(n, m = 1, 2, \ldots, 4\), and we choose the complex number \(q_n\) such that \(\text{Im}[q_n] \geq 0\) for \(n = 1, 2, \ldots, 4\).

With the help of these quantities, we can rewrite the equations (4.21) in the following form

\[
\begin{align*}
\Delta \prod_{n=1}^{4} (\Delta + q_n^2) G^{(a)}_\alpha &= -\chi F^{(a)}_\alpha, \\
\prod_{n=1}^{4} (\Delta + q_n^2) H^{(a)} &= -\chi L^{(a)}, \quad \Delta P^{(a)} = 0,
\end{align*}
\]

where

\[
\chi = [\mu_1 + k_1] \mu_2 + k_2 \sigma^2 - 1 [(\gamma_1 - \gamma_2)^2 + \gamma_1 \gamma_2 - \tau^2]^{-1}.
\]

The constant \(\chi\) is well defined because the dissipation potential is assumed to be positive defined. More exactly, from (2.8) and (2.12) we obtain \(\chi > 0\).

Proposition 4.1. Let assume that \(F^{(1)}_\alpha = \delta_{\alpha\beta} \delta(x - y), L^{(1)} = 0, F^{(2)} = 0\) and \(L^{(2)} = 0\). Then, the equations (4.21) have the solution \(G^{(1)}_\alpha = \delta_{\alpha\beta} E(x, y; \omega), \quad H^{(1)} = 0\), \(G^{(2)} = \)

where $H_0^{(1)}(\cdot)$ is the Hankel function.

**Proof.** First of all, we remark that
\[
\Delta (\Delta + q_n^2) E_n = -\chi \delta(x - y), \quad \text{for all } n = 1, 2, \ldots, 4.
\] (4.26)
Taking into account the relations
\[
\sum_{n=1}^{4} c_n = 0, \quad \sum_{m=n}^{4} \prod_{l=1}^{m-1} (q_l^2 - q_m^2) = 0 \quad \text{for } m = 2, 3
\] (4.27)
\[
\Delta (\Delta + q_n^2) E_m = -\chi \delta(x - y) + (q_n^2 - q_m^2) \Delta E_m \quad \text{for } n, m = 1, 2, \ldots, 4,
\]
and the method presented in the paper [13], we have
\[
\Delta \prod_{n=1}^{4} (\Delta + q_n^2) E(x, y; \omega) = -\chi \delta(x - y),
\] (4.78)
and the proof is complete.

We denote by $(\beta)\nu^{(1)}_\alpha$, $(\beta)\nu^{(2)}_\alpha$, $(\beta)\nu^{(1)}_\beta$, $(\beta)\nu^{(2)}_\beta$, $(\beta)\nu^{(1)}_\gamma$, $(\beta)\nu^{(2)}_\gamma$ the amplitudes caused by the concentrated loads $F^{(1)}_\alpha = \delta_{\alpha\beta}(x - y)$, $L^{(1)} = 0$, $F^{(2)}_\alpha = 0$ and $L^{(2)} = 0$. Thus, in view of the above proposition and of the representation (4.20) we obtain
\[
(\beta)\nu^{(1)}_\beta = -\Delta D_{22}^{\ast} E_{\beta\beta}, \quad (\beta)\nu^{(2)}_\beta = -\Delta D_{22}^{\ast} E_{\beta\beta}, \quad (\beta)\nu^{(1)}_\gamma = -\Delta D_{42}^{\ast} E_{\beta\beta}, \quad (\beta)\nu^{(2)}_\gamma = -\Delta D_{42}^{\ast} E_{\beta\beta};
\] (4.29)
Now, let assume that $F^{(1)}_\alpha = 0$, $L^{(1)} = \delta(x - y)$, $F^{(2)}_\alpha = 0$ and $L^{(2)} = 0$. Using the properties of the function $E$, corresponding to these loads we deduce that $G^{(1)}_\alpha = 0$, $H^{(1)} = \Delta E(x, y; \omega)$, $G^{(2)}_\alpha = 0$, $H^{(2)} = 0$, $P^{(a)} = 0$ is solution of the equations (4.21), and thus we obtain the amplitudes
\[
(3)\nu^{(1)}_\beta = -\varepsilon_{\alpha\beta\gamma}\Delta D_{22}^{\ast} E_{\beta\gamma}, \quad (3)\nu^{(2)}_\beta = -\varepsilon_{\alpha\beta\gamma}\Delta D_{42}^{\ast} E_{\beta\gamma}, \quad (3)\nu^{(1)}_\gamma = 0, \quad (3)\nu^{(2)}_\gamma = 0,
\] (4.30)
Similarly, we have the amplitudes

\[
(3+\beta)v_\alpha^{(1)} = -\Delta D_3^{\beta} \delta_{\alpha \beta} E + D_3^\ast E, \quad (3+\beta)v_\alpha^{(2)} = \varepsilon_{\beta \gamma 3} \Delta D_{42}^\ast E, \\
(3+\beta)\pi^{(1)} = 0, \quad (3+\beta)\pi^{(2)} = -\Box E, 
\]

and

\[
(6)v_\alpha^{(1)} = -\varepsilon_{\alpha \beta 3} D_{41}^\ast \Delta E, \quad (6)v_\alpha^{(2)} = \Delta \Box_5 E, \\
(6)\pi^{(1)} = 0, \quad (6)\pi^{(2)} = 0, 
\]

corresponding to the loads \(F_\alpha^{(1)} = 0, \quad L^{(2)} = 0\), \(F_\alpha^{(2)} = \delta_{\alpha \beta} (x - y)\) and \(L^{(2)} = \delta(x - y)\), respectively.

Using the above arguments and the Proposition 4.1 we can conclude:

**Theorem 4.6.** The matrix \(\Gamma(x, y; \omega)\) defined by

\[
\Gamma_{\alpha; \beta} = (3)v_\alpha^{(1)}, \quad \Gamma_{3; \beta} = (3)v_\alpha^{(1)}, \quad \Gamma_{3+\alpha; \beta} = (3)v_\alpha^{(2)}, \\
\Gamma_{6; \beta} = (3)v_\alpha^{(2)}, \quad \Gamma_{7; \beta} = (3)v_\alpha^{(2)}, \quad \Gamma_{8; \beta} = 0, \\
\Gamma_{\alpha; 3} = (3)v_\alpha^{(1)}, \quad \Gamma_{3; 3} = (3)v_\alpha^{(1)}, \quad \Gamma_{3+\alpha; 3} = (3)v_\alpha^{(2)}, \\
\Gamma_{6; 3} = (3)v_\alpha^{(2)}, \quad \Gamma_{7; 3} = 0, \quad \Gamma_{8; 3} = 0, \\
\Gamma_{\alpha; 3+\beta} = (3)v_\alpha^{(1)}, \quad \Gamma_{3; 3+\beta} = (3)v_\alpha^{(1)}, \quad \Gamma_{3+\alpha; 3+\beta} = (3)v_\alpha^{(2)}, \\
\Gamma_{6; 3+\beta} = (3)v_\alpha^{(2)}, \quad \Gamma_{7; 3+\beta} = 0, \quad \Gamma_{8; 3+\beta} = (3)v_\alpha^{(2)}, \\
\Gamma_{\alpha; 6} = (6)v_\alpha^{(1)}, \quad \Gamma_{3; 6} = (6)v_\alpha^{(1)}, \quad \Gamma_{3+\alpha; 6} = (6)v_\alpha^{(2)}, \\
\Gamma_{6; 6} = (6)v_\alpha^{(2)}, \quad \Gamma_{7; 6} = 0, \quad \Gamma_{8; 6} = 0, 
\]

is a fundamental solution of the system (4.5).

According to [26], we stress that a fundamental solution is unique up to a matrix which has as columns solutions of the homogeneous system

\[
\mathbf{D} \left( \frac{\partial}{\partial x} \right) \mathbf{U} = 0.
\]

But, in view of Theorem 4.4, we note that the unique regular solution in \(\mathbb{R}^2\) of the homogeneous system (4.35) is \(\mathbf{U} = 0\).

From the properties of the Hankel function and of the function \(E\), we can say that all the columns of the matrix \(\Gamma(x, y; \omega)\) are regular vectors in \(\mathbb{R}^2\). Thus, the matrix \(\Gamma(x, y; \omega)\) is the unique fundamental solution, up to a rearrangement of the columns, for which the columns are regular vectors.
References


